

METRICALLY STATIONARY, AXIALLY SYMMETRIC,
ISOLATED SYSTEMS IN QUASI-METRIC GRAVITY

DAG ØSTVANG

Department of Physics, Norwegian University of Science and Technology (NTNU)
7491 Trondheim, Norway*(Received June 15, 2007; revised version received October 26, 2007;
final version received March 17, 2008)*

The gravitational field exterior respectively interior to an axially symmetric, metrically stationary, isolated spinning source made of perfect fluid is examined within the quasi-metric framework. (A metrically stationary system is defined as a system which is stationary except for the direct effects of the global cosmic expansion on the space-time geometry.) Field equations are set up and solved approximately for the exterior part. To lowest order in small quantities, the gravitomagnetic part of the found metric family corresponds with the Kerr metric in the metric approximation. On the other hand, the gravitoelectric part of the found metric family includes a tidal term characterized by the free quadrupole-moment parameter J_2 describing the effect of source deformation due to the rotation. This term has no counterpart in the Kerr metric. Finally, the geodetic effect for a gyroscope in orbit is calculated. There is a correction term, unfortunately barely too small to be detectable by Gravity Probe B, to the standard expression.

PACS numbers: 04.50+h, 04.80.Cc

1. Introduction

Sources of gravitation do as a rule rotate. The rotation should in itself gravitate and thus affect the associated gravitational fields. In General Relativity (GR) this is well illustrated by the Kerr metric describing the gravitational field outside a spinning black hole. More generally, for cases where no exact solutions exist, it is possible to find numerical solutions of the full Einstein equations, both interior and exterior to a stationary spinning source made of perfect fluid with a prescribed equation of state. Besides, on a more analytical level, weak field and slow angular velocity approximations are useful to show the dominant effects of rotation on gravitational fields.

Such approximations may be checked for accuracy against numerical calculations of the full Einstein equations. See, *e.g.*, [1] for a recent review of rotating bodies in GR.

Similarly, in any realistic alternative theory of gravity it must be possible to calculate the effects of rotation on gravitational fields. In particular this applies to quasi-metric gravity (the quasi-metric framework is described in detail elsewhere [2, 3]). In this paper, equations relevant for a metrically stationary, axially symmetric, isolated system in quasi-metric gravity are set up and a first attempt is made to find solutions. The paper is organized as follows. Sec. 2 contains a brief survey of the quasi-metric framework. In Sec. 3, the relevant equations are set up and the gravitational field outside a metrically stationary, axially symmetric, isolated spinning source in quasi-metric gravity is calculated approximately and compared to the Kerr metric. In the limit of no rotation, one gets back the spherically symmetric, metrically static case treated in [4] if the source cannot support shear forces. In Sec. 4 we calculate the geodetic effect for a gyroscope in orbit within the quasi-metric framework. Sec. 5 contains some concluding remarks.

2. Quasi-metric gravity described succinctly

Quasi-metric relativity (QMR), including its current observational status, is described in detail elsewhere [2–4]. Here we give a very brief survey and include only the formulae needed for calculations.

The basic idea which acts as a motivation for postulating the quasi-metric geometrical framework is that the cosmic expansion should be described as an inherent geometric property of quasi-metric space-time itself and not as a kinematical phenomenon subject to dynamical laws. That is, in QMR, the cosmic expansion is described as a general phenomenon that does not have a cause. This means that its description should not depend on the causal structure associated with any semi-Riemannian manifold. Such an idea is attractive since in this way, one should be able to avoid the in principle enormous multitude of potential possibilities, regarding cosmic genesis, initial conditions and evolution, present if space-time is modeled as a semi-Riemannian manifold. Therefore, one expects that any theory of gravity compatible with the quasi-metric framework is more predictive than any metric theory of gravity when it comes to cosmology.

The geometric basis of the quasi-metric framework consists of a 5-dimensional differentiable product manifold $\mathcal{M} \times \mathbf{R}_1$, where $\mathcal{M} = \mathcal{S} \times \mathbf{R}_2$ is a (globally hyperbolic) Lorentzian space-time manifold, $\mathbf{R}_1$ and $\mathbf{R}_2$ are two copies of the real line and $\mathcal{S}$ is a compact Riemannian 3-dimensional manifold (without boundaries). The *global time function* t is then introduced as a co-ordinate on $\mathbf{R}_1$. The product topology of $\mathcal{M}$ implies that once t is given, there must exist a “preferred” ordinary time coordinate x^0 on $\mathbf{R}_2$ such that

x^0 scales like ct . A coordinate system on $\mathcal{M}$ with a global time coordinate of this type we call a *global time coordinate system* (GTCS). Hence, expressed in a GTCS $\{x^\mu\}$ (where μ can take any value $0-3$), x^0 is interpreted as a global time coordinate on $\mathbf{R}_2$ and $\{x^j\}$ (where j can take any value $1-3$) as spatial coordinates on $\mathcal{S}$. The class of GTCSs is a set of preferred coordinate systems inasmuch as the equations of QMR take special forms in a GTCS.

The manifold $\mathcal{M} \times \mathbf{R}_1$ is equipped with two degenerate 5-dimensional metrics $\bar{\mathbf{g}}_t$ and $\mathbf{g}_t$. By definition the metric $\bar{\mathbf{g}}_t$ represents a solution of field equations, and from $\bar{\mathbf{g}}_t$ one can construct the “physical” metric $\mathbf{g}_t$ which is used when comparing predictions to experiments. To reduce the 5-dimensional space-time $\mathcal{M} \times \mathbf{R}_1$ to a 4-dimensional space-time, we just slice the 4-dimensional sub-manifold $\mathcal{N}$ determined by the equation $x^0 = ct$ (using a GTCS) out of $\mathcal{M} \times \mathbf{R}_1$. It is essential that there is no arbitrariness in this choice of slicing. That is, the identification of x^0 with ct must be unique since the two global time coordinates should be physically equivalent; the only reason to separate between them is that they are designed to parameterize fundamentally different physical phenomena. Note that $\mathcal{S}$ is defined as a compact 3-dimensional manifold to avoid ambiguities in the slicing of $\mathcal{M} \times \mathbf{R}_1$.

Moreover, in $\mathcal{N}$, $\bar{\mathbf{g}}_t$ and $\mathbf{g}_t$ are interpreted as one-parameter metric families. Thus by construction, $\mathcal{N}$ is a 4-dimensional space-time manifold equipped with two one-parameter families of Lorentzian 4-metrics parameterized by the global time function t . This is the general form of the quasi-metric space-time framework. We will call $\mathcal{N}$ a *quasi-metric space-time manifold*. The reason why $\mathcal{N}$ cannot be represented by a semi-Riemannian manifold is that the affine connection compatible with any metric family is non-metric; see [2, 3] for more details.

From the definition of quasi-metric space-time we see that it is constructed as consisting of two mutually orthogonal foliations: on the one hand space-time can be sliced up globally into a family of 3-dimensional space-like hypersurfaces (called the fundamental hypersurfaces (FHSs)) by the global time function t , on the other hand space-time can be foliated into a family of time-like curves everywhere orthogonal to the FHSs. These curves represent the world lines of a family of hypothetical observers called the fundamental observers (FOs), and the FHSs together with t represent a preferred notion of space and time. That is, the equations of any theory of gravity based on quasi-metric geometry should depend on quantities obtained from this preferred way of splitting up space-time into space and time.

The metric families $\bar{\mathbf{g}}_t$ and $\mathbf{g}_t$ may be decomposed into parts respectively normal to and intrinsic to the FHSs. The normal parts involve the unit normal vector field families $\bar{\mathbf{n}}_t$ and $\mathbf{n}_t$ of the FHSs, with the property

$\bar{\mathbf{g}}_t(\bar{\mathbf{n}}_t, \bar{\mathbf{n}}_t) = \mathbf{g}_t(\mathbf{n}_t, \mathbf{n}_t) = -1$. The parts intrinsic to the FHSs are the spatial metric families $\bar{\mathbf{h}}_t$ and $\mathbf{h}_t$, respectively. We may then write

$$\bar{\mathbf{g}}_t = -\bar{\mathbf{g}}_t(\bar{\mathbf{n}}_t, \cdot) \otimes \bar{\mathbf{g}}_t(\bar{\mathbf{n}}_t, \cdot) + \bar{\mathbf{h}}_t, \quad (1)$$

and similarly for the decomposition of $\mathbf{g}_t$. Moreover, expressed in a GTCS, $\bar{\mathbf{n}}_t$ may be written as (using the Einstein summation convention)

$$\bar{\mathbf{n}}_t \equiv \bar{n}_{(t)}^\mu \frac{\partial}{\partial x^\mu} = \bar{N}_t^{-1} \left(\frac{\partial}{\partial x^0} - \frac{t_0}{t} \bar{N}^k \frac{\partial}{\partial x^k} \right), \quad (2)$$

where t_0 is an arbitrary reference epoch, $\bar{N}_t$ is the family of lapse functions of the FOs and $(t_0/t)\bar{N}^k$ are the components of the shift vector field family of the FOs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. (A similar formula is valid for $\mathbf{n}_t$.) Note that in the rest of this paper we will use the symbol ‘ $\bar{\cdot}$ ’ to mean a scalar product with $-\bar{\mathbf{n}}_t$. A useful quantity derived from $\bar{N}_t$ is the 4-acceleration field $\bar{\mathbf{a}}_{\mathcal{F}}$ of the FOs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$ which is a quantity intrinsic to the FHSs. Expressed in a GTCS, $\bar{\mathbf{a}}_{\mathcal{F}}$ is defined by its components

$$c^{-2} \bar{a}_{\mathcal{F}j} \equiv \frac{\bar{N}_{t,j}}{\bar{N}_t}, \quad \bar{y}_t \equiv c^{-1} \sqrt{\bar{a}_{\mathcal{F}k} \bar{a}_{\mathcal{F}}^k}, \quad (3)$$

where its norm is given by $c\bar{y}_t$ and where a comma denotes partial derivation.

For reasons explained in [2,3], the form of $\bar{\mathbf{g}}_t$ is restricted such that it contains only one dynamical degree of freedom. That is, gravity is required to be essentially *scalar* in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. Besides, the geometry of the FHSs is defined to represent a gravitational scale as measured in atomic units. Therefore $\bar{\mathbf{g}}_t$ takes an even more restricted form. And expressed in a GTCS (where the spatial coordinates do not depend on t), the most general form allowed for the family $\bar{\mathbf{g}}_t$ is represented by the family of line elements (this may be taken as a definition)

$$\bar{ds}_t^2 = \left[\bar{N}_s \bar{N}^s - \bar{N}_t^2 \right] (dx^0)^2 + 2 \frac{t}{t_0} \bar{N}_i dx^i dx^0 + \frac{t^2}{t_0^2} \bar{N}_t^2 S_{ik} dx^i dx^k, \quad (4)$$

where $S_{ik} dx^i dx^k$ is the metric of the 3-sphere $\mathbf{S}^3$ (with radius equal to ct_0) and $\bar{N}_i \equiv \bar{N}_t^2 S_{ik} \bar{N}^k$. Note that the form of $\bar{\mathbf{g}}_t$ is strictly preserved only under coordinate transformations between GTCSs where the spatial coordinates do not depend on t . (However, if one transforms to a GTCS where the new spatial coordinates x'^j do depend on t , but also fulfill the scaling conditions $\frac{\partial}{\partial t} \frac{\partial x'^j}{\partial x^i} = 0$, $\frac{\partial}{\partial t} \frac{\partial x'^j}{\partial x^0} = -\frac{1}{t} \frac{\partial x'^j}{\partial x^0}$, the line element family expressed in such a GTCS may still be taken to be of the form (4), since the extra terms induced in the line element family and in the affine connection influence

neither the field equations nor the equations of motion.) Also note that, by formally setting $t/t_0 = 1$ and replacing the metric of the 3-sphere with an Euclidean 3-metric in equation (4), we get a single metric which represents the correspondence with metric gravity. By definition this so-called *metric approximation* is possible whenever a correspondence with metric theory can be found by approximating tensor field families by single tensor fields not depending on t .

Field equations from which $\bar{N}_t$ and $\bar{N}^j$ can be determined are given as couplings between projections of the Ricci tensor family $\bar{\mathbf{R}}_t$ and the active stress-energy tensor $\mathbf{T}_t$. The field equations read (using a GTCS)

$$\begin{aligned} 2\bar{R}_{(t)\perp\perp} &= 2(c^{-2}\bar{a}_{\mathcal{F}|i}^i + c^{-4}\bar{a}_{\mathcal{F}i}\bar{a}_{\mathcal{F}}^i - \bar{K}_{(t)ik}\bar{K}_{(t)}^{ik} + \mathcal{L}_{\bar{\mathbf{n}}_t}\bar{K}_t) \\ &= \kappa(T_{(t)\perp\perp} + \hat{T}_{(t)i}^i), \end{aligned} \quad (5)$$

$$\bar{R}_{(t)j\perp} = \bar{K}_{(t)j|i}^i - \bar{K}_{t,j} = \kappa T_{(t)j\perp}, \quad (6)$$

where $\mathcal{L}_{\bar{\mathbf{n}}_t}$ denotes Lie derivation in the direction normal to the FHS and $\bar{\mathbf{K}}_t$ is the extrinsic curvature tensor family (with trace $\bar{K}_t$) of the FHSs. Moreover, $\kappa \equiv 8\pi G/c^4$, a “hat” denotes an object projected into the FHSs and the symbol ‘|’ denotes spatial covariant derivation. The value of G is by convention chosen as that measured in a (hypothetical) local gravitational experiment in an empty universe at epoch t_0 . Note that all quantities correspond to the metric family $\bar{\mathbf{g}}_t$.

It is useful to have an explicit expression for $\bar{\mathbf{K}}_t$, which may be calculated from equation (4). Using a GTCS we find

$$\bar{K}_{(t)ij} = \frac{t}{2t_0\bar{N}_t}(\bar{N}_{i|j} + \bar{N}_{j|i}) + \left(\frac{\bar{N}_{t,\perp}}{\bar{N}_t} - \frac{t_0}{t}c^{-2}\bar{a}_{\mathcal{F}k}\frac{\bar{N}^k}{\bar{N}_t}\right)\bar{h}_{(t)ij}, \quad (7)$$

$$\bar{K}_t = \frac{t_0}{t}\frac{\bar{N}^i_{|i}}{\bar{N}_t} + 3\left(\frac{\bar{N}_{t,\perp}}{\bar{N}_t} - \frac{t_0}{t}c^{-2}\bar{a}_{\mathcal{F}k}\frac{\bar{N}^k}{\bar{N}_t}\right). \quad (8)$$

It is also convenient to have explicit expressions for the curvature intrinsic to the FHSs. From equation (4) one easily calculates

$$\bar{H}_{(t)ij} = c^{-2}\left(\bar{a}_{\mathcal{F}|k}^k - \frac{1}{\bar{N}_t^2 t^2}\right)\bar{h}_{(t)ij} - c^{-4}\bar{a}_{\mathcal{F}i}\bar{a}_{\mathcal{F}j} - c^{-2}\bar{a}_{\mathcal{F}i|j}, \quad (9)$$

$$\bar{P}_t = \frac{6}{(\bar{N}_t c t)^2} + 2c^{-4}\bar{a}_{\mathcal{F}k}\bar{a}_{\mathcal{F}}^k - 4c^{-2}\bar{a}_{\mathcal{F}|k}^k, \quad (10)$$

where $\bar{\mathbf{H}}_t$ is the Einstein tensor family intrinsic to the FHSs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$.

The coordinate expression for the covariant divergence of $\mathbf{T}_t$, i.e., $\hat{\nabla} \cdot \mathbf{T}_t$, reads

$$T_{(t)\mu\bar{*}\nu}^\nu \equiv T_{(t)\mu;\nu}^\nu + c^{-1}T_{(t)\mu\bar{*}t}^0, \quad (11)$$

where the symbol ‘ $\bar{*}$ ’ denotes degenerate covariant derivation compatible with the family $\bar{\mathbf{g}}_t$ and a semicolon denotes metric covariant derivation in component notation. Moreover, assuming that there is no local matter creation, we have

$$T_{(t)\mu\bar{*}t}^0 = -\frac{2}{\bar{N}_t} \left(\frac{1}{t} + \frac{\bar{N}_{t,t}}{\bar{N}_t} \right) T_{(t)\bar{\perp}\mu}. \quad (12)$$

We also have

$$T_{(t)\mu;\nu}^\nu = 2 \frac{\bar{N}_{t,\nu}}{\bar{N}_t} T_{(t)\mu}^\nu = 2c^{-2} \bar{a}_{\mathcal{F}i} \hat{T}_{(t)\mu}^i - 2 \frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} T_{(t)\bar{\perp}\mu}, \quad (13)$$

which, together with equations (11) and (12), constitute the local conservation laws in QMR. Note that even the metric approximation of equation (13) is different from its counterpart in metric gravity.

To construct $\mathbf{g}_t$ from $\bar{\mathbf{g}}_t$ we need the 3-vector field family $\mathbf{v}_t$. Expressed in a GTCS $\mathbf{v}_t$ by definition has the components [2, 3]

$$v_{(t)}^j \equiv \bar{y}_t b_{\mathcal{F}}^j, \quad v = \bar{y}_t \sqrt{\bar{h}_{(t)ik} b_{\mathcal{F}}^i b_{\mathcal{F}}^k}, \quad (14)$$

where v is the norm of $\mathbf{v}_t$ and $\mathbf{b}_{\mathcal{F}}$ is a 3-vector field found from the equations

$$\left[\bar{a}_{\mathcal{F}|k}^k + c^{-2} \bar{a}_{\mathcal{F}k} \bar{a}_{\mathcal{F}}^k \right] b_{\mathcal{F}}^j - \left[\bar{a}_{\mathcal{F}|k}^j + c^{-2} \bar{a}_{\mathcal{F}k} \bar{a}_{\mathcal{F}}^j \right] b_{\mathcal{F}}^k - 2\bar{a}_{\mathcal{F}}^j = 0. \quad (15)$$

We now define the unit vector field $\bar{\mathbf{e}}_b \equiv \frac{t_0}{t} \bar{e}_b^i \frac{\partial}{\partial x^i}$ and the corresponding co-vector field $\bar{\mathbf{e}}^b \equiv \frac{t}{t_0} \bar{e}_i^b dx^i$ along $\mathbf{b}_{\mathcal{F}}$. Then we have [2,3]

$$g_{(t)00} = \left(1 - \frac{v^2}{c^2} \right)^2 \bar{g}_{(t)00}, \quad (16)$$

$$g_{(t)0j} = \left(1 - \frac{v^2}{c^2} \right) \left[\bar{g}_{(t)0j} + \frac{t}{t_0} \frac{2v}{1 - \frac{v}{c}} (\bar{e}_b^i \bar{N}_i) \bar{e}_j^b \right], \quad (17)$$

$$g_{(t)ij} = \bar{g}_{(t)ij} + \frac{t^2}{t_0^2} \frac{4\frac{v}{c}}{(1 - \frac{v}{c})^2} \bar{e}_i^b \bar{e}_j^b. \quad (18)$$

These formulae define the transformation $\bar{\mathbf{g}}_t \rightarrow \mathbf{g}_t$. Notice that we have eliminated any possible t -dependence of $\bar{N}_t$ in equations (16)–(18) by setting $t = x^0/c$ where it occurs. This implies that N does not depend explicitly on t .

3. Metrically stationary, axially symmetric systems

In this section we examine the gravitational field interior respectively exterior to an isolated, axially symmetric, spinning source made of perfect fluid. We require that the rotation of the source should have no time dependence apart from the effects coming from the global cosmic expansion. Besides, as for the spherically symmetric, metrically static case treated in [4], we require that the only time dependence of the gravitational field is via the cosmic scale factor. (See the appendix where it is shown that we can neglect any net translatory motion of the source with respect to the cosmic rest frame without loss of generality.) But contrary to the metrically static case, there is a non-zero shift vector field present due to the rotation of the source. However, we require that a GTCS can be found where $\bar{N}_t$ and $\bar{N}_j$ are independent of x^0 and t . We call this a *metrically stationary* case.

The axial symmetry can be directly imposed on equation (4). Introducing a spherical GTCS $\{x^0, \rho, \theta, \phi\}$ where ρ is an isotropic radial coordinate and where the shift vector field points in the negative ϕ -direction, $\bar{N}_t$ and $\bar{N}_\phi$ do not depend on ϕ and equation (4) takes the form

$$\overline{ds}_t^2 = [\bar{N}_\phi \bar{N}^\phi - \bar{N}_t^2] (dx^0)^2 + 2 \frac{t}{t_0} \bar{N}_\phi d\phi dx^0 + \frac{t^2}{t_0^2} \bar{N}_t^2 \left[\frac{d\rho^2}{1 - \frac{\rho^2}{\Xi_0^2}} + \rho^2 d\Omega^2 \right], \quad (19)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, $\Xi_0 \equiv ct_0$ and $0 \leq \bar{N}_t \rho < \Xi_0$. The range of ρ is limited for both physical and mathematical reasons; truly isolated systems cannot exist in quasi-metric gravity [4]. That is, nontrivial global solutions of the field equations on $\mathbf{S}^3$ for isolated systems do not exist, according to the maximum principle applied to closed Riemannian manifolds. However, for some isolated systems QMR allows exact “semiglobal” solutions on *half* of $\mathbf{S}^3$ (with reasonable boundary conditions). Since the half of $\mathbf{S}^3$ is an open manifold (with boundary), the maximum principle does not disallow such solutions.

Using the definition $\bar{B} \equiv \bar{N}_t^2$, equation (19) may conveniently be rewritten in the form

$$\begin{aligned} \overline{ds}_t^2 = \bar{B} & \left[- (1 - \bar{V}^2 \rho^2 \sin^2\theta) (dx^0)^2 + 2 \frac{t}{t_0} \bar{V} \rho^2 \sin^2\theta d\phi dx^0 \right. \\ & \left. + \frac{t^2}{t_0^2} \left(\frac{d\rho^2}{1 - \frac{\rho^2}{\Xi_0^2}} + \rho^2 d\Omega^2 \right) \right], \end{aligned} \quad (20)$$

where

$$\bar{V} \equiv \frac{\bar{N}_\phi}{\bar{B} \rho^2 \sin^2\theta}. \quad (21)$$

Simple expressions for the non-vanishing components of the extrinsic curvature tensor family may be found from equations (7), (20) and (21). They read (note that the trace $\bar{K}_t$ vanishes)

$$\begin{aligned}\bar{K}_{(t)\rho\phi} &= \bar{K}_{(t)\phi\rho} = \frac{t}{2t_0} \sqrt{\bar{B}} \rho^2 \sin^2 \theta \bar{V}_{,\rho} , \\ \bar{K}_{(t)\theta\phi} &= \bar{K}_{(t)\phi\theta} = \frac{t}{2t_0} \sqrt{\bar{B}} \rho^2 \sin^2 \theta \bar{V}_{,\theta} .\end{aligned}\quad (22)$$

The unknown quantities $\bar{B}(\rho, \theta)$ and $\bar{V}(\rho, \theta)$ may now in principle be calculated from the field equations and local conservation laws.

3.1. The interior field

We will now set up the general field equations interior to the source, which is modelled as a perfect fluid. However, no attempt will be made to find a solution. To begin with we consider $\mathbf{T}_t$ for a perfect fluid

$$\mathbf{T}_t = (\tilde{\rho}_m + c^{-2} \tilde{p}) \bar{\mathbf{u}}_t \otimes \bar{\mathbf{u}}_t + \tilde{p} \bar{\mathbf{g}}_t , \quad (23)$$

where $\tilde{\rho}_m$ is the active mass-energy density in the local rest frame of the fluid and $\tilde{p}$ is the active pressure. Furthermore, $\bar{\mathbf{u}}_t$ is the 4-velocity vector family in $(\mathcal{N}, \bar{\mathbf{g}}_t)$ of observers co-moving with the fluid. It is useful to set up the general formula for the split-up of $\bar{\mathbf{u}}_t$ into pieces respectively normal to and intrinsic to the FHSs:

$$\bar{\mathbf{u}}_t = \bar{\gamma}^* (c \bar{\mathbf{n}}_t + \bar{\mathbf{w}}_t) , \quad \bar{\gamma}^* \equiv \left(1 - \frac{\bar{w}^2}{c^2} \right)^{-\frac{1}{2}} , \quad (24)$$

where $\bar{\mathbf{w}}_t$ (with norm $\bar{w}$) is the 3-velocity family with respect to the FOs. Note that due to the axial symmetry, $\bar{\mathbf{w}}_t$ points in the $\pm\phi$ -direction. Moreover, by definition the quantity ρ_m is the passive mass-energy density as measured in the local rest frame of the fluid and p is the passive pressure. The relationship between $\tilde{\rho}_m$ and ρ_m is given by

$$\rho_m = \begin{cases} \frac{t_0}{t} \bar{N}_t^{-1} \tilde{\rho}_m & \text{for a fluid of material particles,} \\ \frac{t_0^2}{t^2} \bar{N}_t^{-2} \tilde{\rho}_m & \text{for the electromagnetic field,} \end{cases} \quad (25)$$

and a similar relationship exists between $\tilde{p}$ and p . The reason why the relationship between $\tilde{\rho}_m$ and ρ_m is different for a null fluid than for other perfect fluid sources is that gravitational or cosmological spectral shifts of null particles influence their passive mass-energy but not their active mass-energy.

The next step is to use equations (23) and (24) to find suitable expressions for the source terms of the field equations (5) and (6). We find

$$\begin{aligned} T_{(t)\perp\perp} + \hat{T}_{(t)i}^i &= \hat{\gamma}^2 \left(1 + \frac{\bar{w}^2}{c^2}\right) (\bar{\rho}_m c^2 + \bar{p}) + 2\bar{p} \\ &\equiv \frac{t_0^2}{t^2 \bar{B}} \left[\hat{\gamma}^2 \left(1 + \frac{\bar{w}^2}{c^2}\right) (\bar{\rho}_m c^2 + \bar{p}) + 2\bar{p} \right], \end{aligned} \quad (26)$$

where $\bar{\rho}_m$ is the coordinate volume density of active mass and $\bar{p}$ is the associated pressure. Moreover, we find (assuming that the source rotates in the positive ϕ -direction)

$$T_{(t)\perp\phi} = \hat{\gamma}^2 \frac{\bar{w}_{(t)\phi}}{c} (\bar{\rho}_m c^2 + \bar{p}) = \frac{t_0}{t\sqrt{\bar{B}}} \hat{\gamma}^2 \rho \sin\theta \frac{\bar{w}}{c} (\bar{\rho}_m c^2 + \bar{p}). \quad (27)$$

The nontrivial parts of the local conservation laws (13) yield

$$\begin{aligned} \bar{p}_{,\rho} &= - \left[\bar{\rho}_m c^2 - 3\bar{p} - \hat{\gamma}^2 \bar{V} \rho \sin\theta \frac{\bar{w}}{c} (\bar{\rho}_m c^2 + \bar{p}) \right] \frac{\bar{B}_{,\rho}}{2\bar{B}} + \hat{\gamma}^2 \rho^{-1} \frac{\bar{w}^2}{c^2} (\bar{\rho}_m c^2 + \bar{p}), \\ \bar{p}_{,\theta} &= - \left[\bar{\rho}_m c^2 - 3\bar{p} - \hat{\gamma}^2 \bar{V} \rho \sin\theta \frac{\bar{w}}{c} (\bar{\rho}_m c^2 + \bar{p}) \right] \frac{\bar{B}_{,\theta}}{2\bar{B}} + \hat{\gamma}^2 \cot\theta \frac{\bar{w}^2}{c^2} (\bar{\rho}_m c^2 + \bar{p}). \end{aligned} \quad (28)$$

Note that the metrically stationary condition implies that one must have an equation of state of the form $p \propto \rho_m$ since otherwise $\bar{\rho}_m$ and $\bar{p}$ cannot both be independent of t .

We are now in position to set up the field equations (5), (6) for the system in as simple as possible form. After calculating the necessary derivatives and doing some simple algebra we find the two coupled partial differential equations

$$\begin{aligned} &\left(1 - \frac{\rho^2}{\Xi_0^2}\right) \bar{B}_{,\rho\rho} + \frac{1}{\rho^2} \bar{B}_{,\theta\theta} + \frac{2}{\rho} \left(1 - \frac{3\rho^2}{2\Xi_0^2}\right) \bar{B}_{,\rho} + \frac{\cot\theta}{\rho^2} \bar{B}_{,\theta} \\ &= \bar{B} \left\{ \rho^2 \sin^2\theta \left[\left(1 - \frac{\rho^2}{\Xi_0^2}\right) (\bar{V}_{,\rho})^2 + \frac{1}{\rho^2} (\bar{V}_{,\theta})^2 \right] \right. \\ &\quad \left. + \kappa \left[\hat{\gamma}^2 \left(1 + \frac{\bar{w}^2}{c^2}\right) (\bar{\rho}_m c^2 + \bar{p}) + 2\bar{p} \right] \right\}, \end{aligned} \quad (29)$$

$$\begin{aligned} &\left(1 - \frac{\rho^2}{\Xi_0^2}\right) \bar{V}_{,\rho\rho} + \frac{1}{\rho^2} \bar{V}_{,\theta\theta} + \left[\frac{4}{\rho} - \frac{5\rho}{\Xi_0^2} + \left(1 - \frac{\rho^2}{\Xi_0^2}\right) \frac{\bar{B}_{,\rho}}{\bar{B}} \right] \bar{V}_{,\rho} \\ &+ \left[3\cot\theta + \frac{\bar{B}_{,\theta}}{\bar{B}} \right] \frac{1}{\rho^2} \bar{V}_{,\theta} = 2\kappa \hat{\gamma}^2 \frac{\bar{w}}{c} \frac{(\bar{\rho}_m c^2 + \bar{p})}{\rho \sin\theta}. \end{aligned} \quad (30)$$

Notice that the left hand side of equation (29) is equal to $\nabla_s^2 \bar{B}$, where ∇_s^2 is the Laplacian compatible with the standard metric on $\mathbf{S}^3$ (with radius equal to Ξ_0).

To avoid problems with coordinate pathologies along the axis of rotation it would probably be convenient to express equations (28)–(30) in Cartesian coordinates rather than trying to solve them numerically as they stand. However, due to their complexity no attempt to find an interior solution will be made in this paper.

3.2. The exterior field

For illustrative purposes, let us first consider exact solutions of equation (29) without source terms for the *metrically static*, axisymmetric case; we may then set $\bar{V} = 0$. If we also insist that the solution $\bar{B}^{(\text{ms})}(\rho, \theta)$ fulfils the boundary condition $\bar{B}^{(\text{ms})}(\Xi_0, \theta) = 1$ (which should not be taken to be a realistic *physical* constraint, since true isolated systems do not exist in QMR [4]), it turns out that the solution (on the half of $\mathbf{S}^3$) must take the form

$$\bar{B}^{(\text{ms})}(\rho, \theta) = 1 - \frac{r_{\text{s}0}}{\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \left[1 - J_2 \frac{\bar{\mathcal{R}}^2}{2\rho^2} (3\cos^2\theta - 1) \right], \quad (31)$$

where J_2 is the (static) quadrupole-moment parameter and the other quantities are as in equation (32) below. It is clear that the source corresponding to this solution is a body which has a nonspherical shape (oblate spheroid) in absence of any rotation. Thus this body cannot be made of perfect fluid, since the source material must be able to support shear forces. From the solution (31) one is in principle able to construct the counterpart exact “physical” metric family $\mathbf{g}_t^{(\text{ms})}$ using equations (14)–(18). However, the expressions thus obtained are extremely complicated so we will not include them here. A series expansion can be obtained from equation (41) below in the limit of no rotation with a non-zero static quadrupole-moment parameter.

Returning to the metrically stationary, axisymmetric case; so far we have not been able to find any exact exterior solution (where $\bar{V} \neq \text{const.}$) of equations (29) and (30) (without sources). On the other hand, it is straightforward to find the first few terms of a series solution. Whether or not this series converges to a real solution on the half of $\mathbf{S}^3$ is not known. It is plausible that it does, however, since in the limit of no rotation, the first terms of the series are identical to those obtained by writing the solution (31) as a series expansion.

The first few terms of the series solution is (for the case where the source spins in the positive ϕ -direction):

$$\begin{aligned}\bar{B}(\rho, \theta) = & 1 - \frac{r_{s0}}{\rho} + \frac{r_{s0}\rho}{2\Xi_0^2} + J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{2\rho^3} (3\cos^2\theta - 1) \\ & + \frac{r_{s0}^2 a_0^2}{2\rho^4} (3\sin^2\theta - 1) + \dots,\end{aligned}\quad (32)$$

$$\bar{V}(\rho, \theta) = -\frac{r_{s0}a_0}{\rho^3} \left(1 + \frac{3r_{s0}}{4\rho} + \dots\right), \quad (33)$$

where J_2 now is the rotationally induced quadrupole-moment parameter and $\bar{\mathcal{R}}$ is the mean coordinate radius of the source. Furthermore, r_{s0} is the Schwarzschild radius at epoch t_0 defined from the Komar mass M_{t_0} [5, 6], *i.e.*, $r_{s0} \equiv (2M_{t_0}G)/c^2$ where

$$M_{t_0} \equiv c^{-2} \int \int \int \left[\bar{N}_{t_0} (T_{(t_0)\bar{\imath}\bar{\imath}} + \hat{T}_{(t_0)i}^i) - 2\bar{N}^\phi T_{(t_0)\bar{\imath}\phi} \right] \sqrt{\bar{h}_{t_0}} d^3x, \quad (34)$$

and a_0 is a length at epoch t_0 defined from the angular momentum integral J_{t_0} [6] (the integrals are in principle taken over the half of $\mathbf{S}^3$, but with no contributions exterior to the source)

$$a_0 \equiv \frac{J_{t_0}}{cM_{t_0}}, \quad J_{t_0} \equiv c^{-1} \int \int \int \psi^\phi T_{(t_0)\bar{\imath}\phi} \sqrt{\bar{h}_{t_0}} d^3x. \quad (35)$$

Here $\bar{h}_{t_0}$ is the determinant of $\bar{\mathbf{h}}_{t_0}$ and $\psi \equiv \frac{\partial}{\partial\phi}$ is a Killing vector field associated with the axial symmetry. Note that $J_t = t^2/t_0^2 J_{t_0}$ is the active angular momentum of the source at epoch t . (The corresponding passive angular momentum of the source is given by $L_t = t/t_0 L_{t_0}$, where $L_{t_0} = c^{-1} \int \int \int \psi^\phi \bar{\mathcal{T}}_{(t_0)\bar{\imath}\phi} \sqrt{\bar{h}_{t_0}} d^3x$, and where $\bar{\mathcal{T}}_t$ is the passive stress-energy tensor in $(\mathcal{N}, \bar{\mathbf{g}}_t)$.) We now insert equations (32) and (33) into equation (20). Taking into account a relevant number of terms we get

$$\begin{aligned}\overline{ds_t^2} = & - \left(1 - \frac{r_{s0}}{\rho} + \frac{r_{s0}\rho}{2\Xi_0^2} + J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{2\rho^3} (3\cos^2\theta - 1) - \frac{r_{s0}^2 a_0^2}{2\rho^4} \cos^2\theta + \dots \right) (dx^0)^2 \\ & - 2 \frac{t}{t_0} \left(1 - \frac{r_{s0}}{4\rho} + \dots \right) \frac{r_{s0}a_0}{\rho} \sin^2\theta d\phi dx^0 \\ & + \frac{t^2}{t_0^2} \left(1 - \frac{r_{s0}}{\rho} + \dots \right) \left(\frac{d\rho^2}{1 - \frac{\rho^2}{\Xi_0^2}} + \rho^2 d\Omega^2 \right).\end{aligned}\quad (36)$$

To construct $\mathbf{g}_t$ from $\bar{\mathbf{g}}_t$ we need to calculate the vector field $\mathbf{b}_{\mathcal{F}}$ from equation (15) ($\bar{\mathbf{a}}_{\mathcal{F}}$ and its derivatives may be found from equations (3) and (31)).

These calculations get quite complicated so it is convenient to do them by computer. The result is

$$b_{\mathcal{F}}^{\rho} = \rho \left(1 - \frac{r_{s0}}{2\rho} - \frac{r_{s0}^2}{4\rho^2} + J_2 \frac{3\bar{\mathcal{R}}^2}{2\rho^2} (3\cos^2\theta - 1) - \frac{r_{s0}\rho}{4\Xi_0^2} - \frac{r_{s0}^3}{8\rho^3} \right. \\ \left. + \frac{3r_{s0}a_0^2}{2\rho^3} (3\sin^2\theta - 2) - J_2 \frac{3\bar{\mathcal{R}}^2 r_{s0}}{4\rho^3} (3\cos^2\theta - 1) + \dots \right), \quad (37)$$

$$\rho b_{\mathcal{F}}^{\theta} = -\frac{3\sin(2\theta)}{\rho} \left(J_2 \bar{\mathcal{R}}^2 \left(1 - \frac{3r_{s0}}{4\rho} + \dots \right) - \frac{3r_{s0}a_0^2}{2\rho} + \dots \right). \quad (38)$$

Furthermore, we need the quantity v defined in equation (14). This may be expressed by $\bar{B}$ and its derivatives together with the components of $\mathbf{b}_{\mathcal{F}}$. We find

$$v = \frac{c}{2\bar{B}} \sqrt{\left(1 - \frac{\rho^2}{\Xi_0^2}\right) (\bar{B}_{,\rho})^2 + \rho^{-2} (\bar{B}_{,\theta})^2} \sqrt{\left(1 - \frac{\rho^2}{\Xi_0^2}\right)^{-1} (b_{\mathcal{F}}^{\rho})^2 + (\rho b_{\mathcal{F}}^{\theta})^2} \\ = \frac{r_{s0}c}{2\rho} \left(1 + \frac{r_{s0}}{2\rho} + \frac{r_{s0}^2}{4\rho^2} + \frac{\rho^2}{2\Xi_0^2} + \frac{r_{s0}^3}{8\rho^3} - \frac{r_{s0}\rho}{2\Xi_0^2} - J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{2\rho^3} (3\cos^2\theta - 1) \right. \\ \left. - \frac{r_{s0}a_0^2}{2\rho^3} (3\sin^2\theta + 2) + \dots \right). \quad (39)$$

Finally, to do the transformation shown in equation (18) we need the quantities $\bar{e}_{\rho}^b$ and $\bar{e}_{\theta}^b$. Since $\bar{e}_{\theta}^b$ is equal to $\rho b_{\mathcal{F}}^{\theta}$ to the accuracy calculated here, it is sufficient to write down the expression for $\bar{e}_{\rho}^b$. A straightforward calculation yields

$$\bar{e}_{\rho}^b = 1 - \frac{r_{s0}}{2\rho} - \frac{r_{s0}^2}{8\rho^2} + \frac{\rho^2}{2\Xi_0^2} + \frac{r_{s0}\rho}{4\Xi_0^2} - \frac{r_{s0}^3}{16\rho^3} + J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{4\rho^3} (3\cos^2\theta - 1) + \dots, \quad (40)$$

and the transformations (16)–(18) then yield, to desired accuracy

$$ds_t^2 = -\left(1 - \frac{r_{s0}}{\rho} - \frac{r_{s0}^2}{2\rho^2} + \frac{r_{s0}\rho}{2\Xi_0^2} + J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{2\rho^3} (3\cos^2\theta - 1) - \frac{r_{s0}^2 a_0^2}{2\rho^4} \cos^2\theta \right. \\ \left. + \frac{3r_{s0}^4}{16\rho^4} - \frac{r_{s0}^2}{2\Xi_0^2} + \dots \right) (dx^0)^2 - 2\frac{t}{t_0} \left(1 - \frac{r_{s0}}{4\rho} + \dots \right) \frac{r_{s0}a_0}{\rho} \sin^2\theta d\phi dx^0 \\ \left. + \frac{t^2}{t_0^2} \left(\left(1 + \frac{r_{s0}}{\rho} + \frac{r_{s0}^2}{\rho^2} + \dots \right) \frac{d\rho^2}{1 - \frac{\rho^2}{\Xi_0^2}} + \left(1 - \frac{r_{s0}}{\rho} + O\left(\frac{r_{s0}^3}{\rho^3}\right)\right) \rho^2 d\Omega^2 \right). \quad (41)$$

Note that to $O(r_{s0}^3/\rho^3)$ or higher, the spatial metric family $\mathbf{h}_t$ is not diagonal in these coordinates.

It may be convenient to express the metric family (41) in an “almost Schwarzschild” radial coordinate r defined by

$$\begin{aligned} r &\equiv \rho \sqrt{1 - \frac{r_{s0}}{\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0^2}}} = \left(1 - \frac{r_{s0}}{2\rho} - \frac{r_{s0}^2}{8\rho^2} - \frac{r_{s0}^3}{16\rho^3} + \frac{r_{s0}\rho}{4\Xi_0^2} + \dots\right)\rho, \\ \rho &= \left(1 + \frac{r_{s0}}{2r} + \frac{r_{s0}^2}{8r^2} - \frac{r_{s0}r}{4\Xi_0^2} + \dots\right)r. \end{aligned} \quad (42)$$

That is, to the order in small quantities considered in equation (41), at epoch t_0 the surface area of spheres centered on the origin is equal to $4\pi r^2$. Expressed in the new radial coordinate (41) reads

$$\begin{aligned} ds_t^2 &= -\left(1 - \frac{r_{s0}}{r} + \frac{3r_{s0}^3}{8r^3} + \frac{r_{s0}r}{2\Xi_0^2} + J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{2r^3} (3\cos^2\theta - 1) \left(1 - \frac{3r_{s0}}{2r}\right) \right. \\ &\quad \left. - \frac{r_{s0}^2 a_0^2}{2r^4} \cos^2\theta - \frac{r_{s0}^4}{16r^4} - \frac{r_{s0}^2}{2\Xi_0^2} + \dots\right) (dx^0)^2 \\ &\quad - 2\frac{t}{t_0} \left(1 - \frac{3r_{s0}}{4r} + \dots\right) \frac{r_{s0}a_0}{r} \sin^2\theta d\phi dx^0 \\ &\quad + \frac{t^2}{t_0^2} \left(\left(1 + \frac{r_{s0}}{r} + \frac{r_{s0}^2}{4r^2} + \frac{r^2}{\Xi_0^2} + \dots\right) dr^2 + r^2 \left(1 + O\left(\frac{r_{s0}^3}{r^3}\right)\right) d\Omega^2 \right). \end{aligned} \quad (43)$$

This expression represents the gravitational field outside an isolated, metrically stationary, axially symmetric spinning source made of perfect fluid (obeying an equation of state of the form $p \propto \rho_m$) to the given accuracy in small quantities. Note in particular the presence of a tidal term containing the free parameter J_2 describing the effect of source deformation due to the rotation. This tidal term has a counterpart in Newtonian gravitation. (However, to higher order there is also a term due to the rotation itself, and this term has no counterpart in Newtonian gravitation.) The existence of a tidal term means that the metric family (43) has built into itself the necessary flexibility to represent the gravitational field exterior to a variety of sources. That is, since the exact equation of state describing the source is not specified, the effect of the rotation on the source and thus its quadrupole-moment should not be exactly known either, since this effect depends on material properties of the source. On the other hand, which is well-known, no such flexibility is present in the Kerr metric, meaning that the Kerr metric can only represent the gravitational field outside a source which material properties are of no concern; *e.g.*, a spinning black hole.

4. The geodetic and Lense–Thirring effects

To calculate the predicted geodetic and Lense–Thirring effects within the quasi-metric framework we can use the metric family (43) with some extra simplifications.

We thus examine the behaviour of a small gyroscope in orbit around a metrically stationary, axially symmetric, isolated source. We may assume that the source is so small that any dependence on the global curvature of space can be neglected. Furthermore, we assume that the exterior gravitational field of the source is so weak that it can be adequately represented by equation (43) with the highest order terms cut out, *i.e.*,

$$ds_t^2 = -\left(1 - \frac{r_{s0}}{r} + J_2 \frac{\bar{\mathcal{R}}^2 r_{s0}}{2r^3} (3\cos^2\theta - 1) + \dots\right)(dx^0)^2 - 2\frac{t}{t_0} \frac{r_{s0}a_0}{r} \sin^2\theta d\phi dx^0 + \frac{t^2}{t_0^2} \left(\left(1 + \frac{r_{s0}}{r} + \dots\right) dr^2 + r^2 d\Omega^2 \right). \quad (44)$$

In this section we calculate the predicted geodetic effect. In this case equation (44) can be simplified even further by assuming that the gravitational field is spherically symmetric, *i.e.*, that the spin of the source can be neglected. We then set $J_2 = a_0 = 0$ in equation (44) and it takes the form

$$ds_t^2 = -B(r)(dx^0)^2 + \frac{t^2}{t_0^2} \left(A(r) dr^2 + r^2 d\Omega^2 \right), \quad (45)$$

where $A(r)$ and $B(r)$ are given as series expansions from equation (44) (with spin parameters neglected).

Our derivation of the geodetic effect in QMR will be a counterpart to a similar calculation valid for GR and presented in [7]. To simplify calculations we assume that the gyroscope orbits in the equatorial plane and that the orbit is a circle with constant radial coordinate $r = R$. Furthermore, the gyroscope has spin $\mathbf{S}_t$ and 4-velocity $\mathbf{u}_t$. Then the norm $S_* \equiv \sqrt{\mathbf{S}_t \cdot \mathbf{S}_t}$ is constant along its world line and, moreover, normal to $\mathbf{u}_t$, *i.e.*

$$\mathbf{S}_t \cdot \mathbf{u}_t = 0, \quad \Rightarrow \quad S_{(t)0} = -S_{(t)i} \frac{dx^i}{dx^0}. \quad (46)$$

The equation of motion for the spin $\mathbf{S}_t$ is the equation of parallel transport along its world line in quasi-metric space-time, *i.e.*

$$\nabla_{\mathbf{u}_t}^* \mathbf{S}_t = 0, \quad \Rightarrow \quad \frac{dS_{(t)}^\mu}{d\tau_t} = -\tilde{\Gamma}_{\lambda\nu}^{\mu} S_{(t)}^\lambda u_{(t)}^\nu - \tilde{\Gamma}_{\lambda t}^{\mu} S_{(t)}^\lambda \frac{dt}{d\tau_t}. \quad (47)$$

Next we define the angular velocity of the gyroscope. This is given by $\Omega_t \equiv \frac{d\phi}{dt} = c \frac{d\phi}{dx^0}$. Thus $u_{(t)}^\phi \equiv \frac{d\phi}{d\tau_t} = c^{-1} \Omega_t \frac{dx^0}{d\tau_t}$. Now a constant of motion J

for the orbit is given by the equation [3, 4] (using the notation $' \equiv \frac{\partial}{\partial r}$)

$$\frac{t}{t_0} R^2 \Omega_t = B(R) J c, \quad J = \frac{B'(R) R^3}{2B^2(R)}, \quad (48)$$

where the last expression is shown in [8]. Equations (48) then yield

$$\Omega_t = \frac{t_0}{t} \sqrt{\frac{B'(R)}{2R}} c. \quad (49)$$

Also, from the fact that $\mathbf{u}_t \cdot \mathbf{u}_t = -c^2$, we find

$$u_{(t)}^0 = \frac{dx^0}{d\tau_t} = \frac{c}{\sqrt{B(R) - \frac{1}{2}B'(R)R}}, \quad (50)$$

and using (50), equation (46) yields

$$S_{(t)}^0 = \frac{t}{t_0} B^{-1}(R) \sqrt{\frac{1}{2}B'(R)R^3} S_{(t)}^\phi. \quad (51)$$

We now insert the expressions found above into equation (47). (The relevant connection coefficients can be found in [3] or [4].) Equation (47) then yields a set of 2 coupled, first order ordinary differential equations of the form

$$\frac{d}{dt} \left[\frac{t}{t_0} S_{(t)}^r \right] = f(R) S_{(t)}^\phi, \quad \frac{d}{dt} \left[\frac{t}{t_0} S_{(t)}^\phi \right] = -g(R) S_{(t)}^r, \quad (52)$$

where the functions $f(R)$, $g(R)$ are given by

$$\begin{aligned} f(R) &\equiv \frac{c}{A(R)} \left(\sqrt{\frac{1}{2}B'(R)R} - B^{-1}(R) \sqrt{\frac{1}{8}B'^3(R)R^3} \right), \\ g(R) &\equiv \sqrt{\frac{B'(R)}{2R^3}} c. \end{aligned} \quad (53)$$

A solution of the system (52) can be found by computer. Assuming that $\mathbf{S}_t$ points in the (positive) radial direction at epoch t_0 , the solution of (52) reads

$$S_{(t)}^r = \frac{t_0}{t} S_* A^{-1/2}(R) \cos \left[\omega_S t_0 \ln \left(\frac{t}{t_0} \right) \right], \quad (54)$$

$$S_{(t)}^\phi = -\frac{t_0}{t} \frac{S_* \Omega_{t_0}}{\sqrt{A(R)R\omega_S}} \sin \left[\omega_S t_0 \ln \left(\frac{t}{t_0} \right) \right], \quad (55)$$

where

$$\omega_S \equiv \sqrt{f(R)g(R)} = A^{-1/2}(R) \sqrt{1 - \frac{B'(R)R}{2B(R)}} \Omega_{t_0}. \quad (56)$$

After one complete orbit of the gyroscope $t = t_0 + (2\pi)/\Omega_t$, and the angle between $\mathbf{S}_t$ and a unit vector $\mathbf{e}_r$ in the radial direction is given by

$$\alpha = \arccos \left(\frac{\mathbf{S}_t \cdot \mathbf{e}_r}{S_*} \right) = \omega_S t_0 \ln \left[1 + \frac{2\pi}{\Omega_t t_0} \right]. \quad (57)$$

The difference $\Delta\phi$ between a complete circle of 2π radians and the angular advancement of $\mathbf{S}_t$ will then be

$$\begin{aligned} \Delta\phi &= 2\pi - \alpha = 2\pi \left[\frac{3r_{s0}}{4R} - \frac{\pi R}{\Xi_0} \sqrt{\frac{2R}{r_{s0}}} + \dots \right] \\ &= \frac{3\pi M_{t_0} G}{c^2 R} - \frac{2\pi^2}{t_0} \sqrt{\frac{R^3}{M_{t_0} G}} + \dots \end{aligned} \quad (58)$$

We see that there is a quasi-metric correction term in addition to the usual GR result. Unfortunately, the difference amounts only to about $-5 \times 10^{-5}''$ per year for a satellite orbiting the Earth, *i.e.* the predicted correction is too small by a factor about ten to be detectable by Gravity Probe B.

One may also calculate the Lense–Thirring effect for a gyroscope in polar orbit, using equation (44). But except from the variable scale factor, the off-diagonal term in (44) is the same as for the Kerr metric. Any correction term to the Lense–Thirring effect should therefore depend on the inverse age of the Universe and thus be far too small to be detectable. But notice that, for a gyroscope in orbit around the Earth, there is also an extra contribution term of the type shown in equation (58), to the geodetic effect coming from the Earth's orbit around the Sun. Numerically this correction term is similar to the correction term found above.

5. Conclusion

In GR, very many exterior and interior solutions of Einstein's equations are possible in principle for axisymmetric stationary systems. The problem is to find physically reasonable solutions where the exterior and interior solutions join smoothly to form an asymptotically flat, global solution. Such solutions would be candidates for modelling isolated spinning stars. And although no exact solution having the desired properties has been found so far, accurate analytical approximations exist, and also numerical solutions of the full Einstein equations, see, *e.g.*, [9].

The quasi-metric counterpart to axially symmetric, stationary systems in GR, is metrically stationary, axially symmetric systems. This research subject is largely unexplored. However, this paper contains some basic results for such systems. That is, we have set up the relevant equations for a metrically stationary, axially symmetric isolated source within the quasi-metric framework, both interior and exterior to the source. A series solution was found for the exterior part. The biggest difference between the found solution and the Kerr metric is the presence of a term containing the free parameter J_2 representing the quadrupole-moment of the source. Such a free parameter is necessary to ensure sufficient flexibility so that the solution does not unduly constrain the nature of the source. On the other hand, the multipole moments of the Kerr metric are fixed. This means that, unlike the Kerr metric, the metric family found in this paper may represent the gravitational field exterior to a variety of sources. Besides, for the case when the source is made of perfect fluid, in the limit where the rotation of the source vanishes, the found metrically stationary, axially symmetric solution becomes identical to the spherically symmetric, metrically static exterior solution found in [4] (to the given accuracy). This means that, for a source made of perfect fluid, its quadrupole-moment vanishes in the limit of no rotation and that the source should be unable to support shear forces. Thus for such a source, its quadrupole-moment is purely due to rotational deformation. It is also possible that part of the source's quadrupole-moment is static, in which case the source cannot be made of perfect fluid. One possible limit of no rotation is then given from equation (31).

Since the field equations (5), (6) in some sense represent only a subset of the full Einstein equations [2, 3], one would expect that the number of possible solutions of equations (29) and (30) (with and without sources) are considerably smaller than for the GR counterpart. In particular, one would expect that the number of unphysical solutions are much smaller, and since the metric family $\bar{g}_t$ is constrained to take the form (4), that the problem of smooth matching between interior and exterior solutions would be more or less absent. However, if this is correct will only be known from further research.

Appendix A

Preferred frame effects are absent in QMR

Both in this paper and in [4] it is assumed that the gravitating source has no net translatory motion with respect to the FOs. However, a given object will usually have a non-zero velocity with respect to the cosmic rest frame (*i.e.*, a non-zero dipole moment in the cosmic microwave background will in general be observed for observers being at rest with respect to the

object). Since the FOs do not move on average with respect to the cosmic rest frame it might be natural to assume that an isolated, gravitating source could have a net translatory motion with respect to the FOs. This would represent a “preferred frame” effect which might possibly be measurable. However, as we shall see, an isolated source can never have a net translatory motion with respect to the FOs in QMR, so there will be no preferred-frame effects.

Let us consider an isolated, metrically stationary, gravitating source moving with constant speed $\bar{U}$ with respect to a GTCS where the cosmic substratum is at rest on average. That is, the cosmic rest frame may be represented by a cylindrical GTCS (x^0, ξ', z', ϕ') oriented such that the source moves in the positive z' -direction with coordinate velocity $\frac{dz'}{dx^0} = \frac{t_0}{t} \frac{\bar{U}}{c}$, where we for simplicity have neglected the global curvature of the FHSs. This system has axial symmetry and the shift vector field has a z' -component only, so from equation (24) we find

$$\bar{w}_{(t)}^{z'} = \frac{c}{\bar{N}_t} \left(\frac{dz'}{dx^0} + \frac{t_0}{t} \bar{N}^{z'} \right), \quad \Rightarrow \quad \bar{N}^{z'} = -\frac{\bar{U} - \bar{w}}{c}. \quad (\text{A.1})$$

The line element family (4) then takes the form ($\bar{B} \equiv \bar{N}_t^2$)

$$\begin{aligned} \bar{ds}_t^2 = \bar{B} \Big[& - \left(1 - \frac{(\bar{U} - \bar{w})^2}{c^2} \right) (dx^0)^2 - 2 \frac{t}{t_0} \frac{\bar{U} - \bar{w}}{c} dz' dx^0 \\ & + \frac{t^2}{t_0^2} \left(d\xi'^2 + dz'^2 + \xi'^2 d\phi'^2 \right) \Big]. \end{aligned} \quad (\text{A.2})$$

Here $\bar{B}$ and $\bar{w}$ are functions of x^0 , ξ' and z' but not of ϕ' . We now assume that the stationary nature of the system makes it possible to eliminate the dependence on x^0 by making the coordinate transformation

$$\xi = \xi', \quad z = z' - \frac{t_0}{t} \frac{\bar{U}}{c} (x^0 - x_0^0), \quad \phi = \phi', \quad (\text{A.3})$$

where x_0^0 is a constant. Holding t constant we find the coordinate differential

$$dz = dz' - \frac{t_0}{t} \frac{\bar{U}}{c} dx^0 \quad \Rightarrow \quad \frac{dz}{dx^0} = \frac{dz'}{dx^0} - \frac{t_0}{t} \frac{\bar{U}}{c} = 0, \quad (\text{A.4})$$

confirming that the source has no net velocity with respect to the new GTCS. Using equation (A.4) we now write the metric family (A.2) on the form (since the coordinate transformation (A.3) satisfies the scaling properties mentioned in section 2, it makes no difference that t is held constant)

$$\bar{ds}_t^2 = \bar{B} \Big[- \left(1 - \frac{\bar{w}^2}{c^2} \right) (dx^0)^2 + 2 \frac{t}{t_0} \frac{\bar{w}}{c} dz dx^0 + \frac{t^2}{t_0^2} \left(d\xi^2 + dz^2 + \xi^2 d\phi^2 \right) \Big], \quad (\text{A.5})$$

where $\bar{B}$ and $\bar{w}$ are functions of ξ and z only. Note that the performed coordinate transformation yields a new shift vector field pointing in the z -direction with magnitude $\bar{w}/c$. From equations (7), (8) and (A.5) we may calculate the non-vanishing components of the extrinsic curvature tensor. We find

$$\bar{K}_{(t)zz} = \frac{t}{t_0} \left(\sqrt{\bar{B}} \frac{\bar{w}_{,z}}{c} + \frac{\bar{w}}{2c} \frac{\bar{B}_{,z}}{\sqrt{\bar{B}}} \right), \quad \bar{K}_{(t)\xi z} = \bar{K}_{(t)z\xi} = \frac{t}{2t_0} \sqrt{\bar{B}} \frac{\bar{w}_{,\xi}}{c}, \quad (\text{A.6})$$

$$\bar{K}_{(t)\xi\xi} = \frac{t}{2t_0} \frac{\bar{w}}{c} \frac{\bar{B}_{,z}}{\sqrt{\bar{B}}}, \quad \bar{K}_{(t)\phi\phi} = \frac{t}{2t_0} \frac{\bar{w}}{c} \frac{\xi^2 \bar{B}_{,z}}{\sqrt{\bar{B}}}, \quad (\text{A.7})$$

$$\bar{K}_t = \frac{t_0}{t} \bar{B}^{-1/2} \left(\frac{\bar{w}_{,z}}{c} + \frac{3\bar{w}}{2c} \frac{\bar{B}_{,z}}{\bar{B}} \right). \quad (\text{A.8})$$

Using equations (A.6), (A.7) and (A.8) we may now compute the necessary quantities entering into the field equations (5), (6). After some calculations the field equations yield

$$\begin{aligned} & \frac{\bar{B}_{,\xi\xi}}{\bar{B}} + \left(1 - 3 \frac{\bar{w}^2}{c^2} \right) \frac{\bar{B}_{,zz}}{\bar{B}} + \frac{3\bar{w}^2}{2c^2} \left(\frac{\bar{B}_{,z}}{\bar{B}} \right)^2 + \frac{\bar{B}_{,\xi}}{\xi \bar{B}} \\ & - 2 \frac{\bar{w} \bar{w}_{,zz}}{c^2} - 2 \left(\frac{\bar{w}_{,z}}{c} \right)^2 - 2 \left(\frac{\bar{w}_{,\xi}}{c} \right)^2 - 4 \frac{\bar{B}_{,z}}{\bar{B}} \frac{\bar{w} \bar{w}_{,z}}{c^2} = 0, \end{aligned} \quad (\text{A.9})$$

$$\bar{w}_{,\xi\xi} + \left[\frac{1}{\xi} + \frac{3}{2} \frac{\bar{B}_{,\xi}}{\bar{B}} \right] \bar{w}_{,\xi} - \frac{\bar{B}_{,z}}{\bar{B}} \bar{w}_{,z} - \left[\frac{2\bar{B}_{,zz}}{\bar{B}} - 3 \left(\frac{\bar{B}_{,z}}{\bar{B}} \right)^2 \right] \bar{w} = 0, \quad (\text{A.10})$$

$$\bar{w}_{,\xi z} + \frac{\bar{B}_{,z}}{\bar{B}} \bar{w}_{,\xi} + \left[\frac{2\bar{B}_{,\xi z}}{\bar{B}} - 3 \frac{\bar{B}_{,\xi} \bar{B}_{,z}}{\bar{B}^2} \right] \bar{w} = 0. \quad (\text{A.11})$$

Equations (A.9)–(A.11) are far too complicated to have any hope of finding an exact solution. But what we can do is to look for a weak field solution. That is, for weak field we can set (since by hypothesis, the FOs are at rest with respect to the cosmic rest frame far away from the object)

$$\bar{B} \approx 1, \quad \bar{w} = \bar{U} - \bar{\epsilon}, \quad |\bar{\epsilon}| \ll 1, \quad (\text{A.12})$$

and neglect all non-linear terms in equations (A.9)–(A.11). Using equation (A.12) the weak field versions of equations (A.9)–(A.11) then read

$$\bar{B}_{,\xi\xi} + \left(1 - 3 \frac{\bar{U}^2}{c^2} \right) \bar{B}_{,zz} + \frac{1}{\xi} \bar{B}_{,\xi} + 2 \frac{\bar{U}}{c^2} \bar{\epsilon}_{,zz} = 0, \quad (\text{A.13})$$

$$\frac{1}{\bar{U}} \bar{\epsilon}_{,\xi\xi} + 2 \bar{B}_{,zz} + \frac{1}{\bar{U} \xi} \bar{\epsilon}_{,\xi} = 0, \quad \bar{\epsilon}_{,\xi z} = 2 \bar{U} \bar{B}_{,\xi z}. \quad (\text{A.14})$$

Since $\bar{\epsilon}$ is required to vanish far from the source, (A.14) yields

$$\bar{\epsilon} = 2 \bar{U} (\bar{B} - 1), \quad \bar{B}_{,\xi\xi} + \bar{B}_{,zz} + \frac{1}{\xi} \bar{B}_{,\xi} = 0, \quad (\text{A.15})$$

but this expression for $\bar{\epsilon}$ inserted into equation (A.13) yields

$$\bar{B}_{,\xi\xi} + \left(1 + \frac{\bar{U}^2}{c^2}\right) \bar{B}_{,zz} + \frac{1}{\xi} \bar{B}_{,\xi} = 0, \quad (\text{A.16})$$

which is inconsistent unless $\bar{U} = 0$. Equations (A.9)–(A.11) thus have no solution unless $\bar{w} = 0$ in which case the solution is equivalent to that of the metrically static, axially symmetric case. That is, the gravitational field of an isolated source does not depend on the source's net motion with respect to the cosmic rest frame; such motions may be neglected without loss of generality.

REFERENCES

- [1] N. Stergioulas, *Living Reviews in Relativity* **6**, 3 (2003).
- [2] D. Østvang, *Grav. Cosmol.* **11**, 205 (2005) [gr-qc/0112025].
- [3] D. Østvang, Doctoral Thesis, 2001 [gr-qc/0111110].
- [4] D. Østvang, *Grav. Cosmol.* **13**, 1 (2007) [gr-qc/0201097].
- [5] A. Komar, *Phys. Rev.* **113**, 934 (1959).
- [6] R.M. Wald, *General Relativity*, The University of Chicago Press, 1984.
- [7] J.B. Hartle, *Gravity: an Introduction to Einstein's General Relativity*, Addison-Wesley, 2003.
- [8] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons Inc., 1972.
- [9] E. Berti, F. White, A. Manipoulou, M. Bruni, *MNRAS* **358**, 923 (2005).