RENYI ENTROPIES OF A BLACK HOLE

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The Renyi entropies, H_l, of Hawking radiation contained in a thin shell surrounding the black hole are evaluated. When the width of the shell is adjusted to the energy content corresponding to the mass defect, the Bekenstein–Hawking formula for the Shannon ($S = H_1$) entropy of a black hole is reproduced. This result does not depend on the distance of the shell from the horizon. The Renyi entropies of higher order, however, are sensitive to it.

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1. Introduction

We have shown recently [1] that the the Bekenstein–Hawking formula for the entropy of a black hole [2, 3]

$$S_M = 4\pi M^2 \tag{1}$$

can be obtained by evaluation of the entropy content of the Hawking radiation in an adequately selected region (an infinitesimal spherical shell close to the horizon) and integration of the result over the whole history of the black hole. S_M gives an important information about the probabilities p_i of the states ψ_i forming the black hole[1]. Indeed, when p_i's are introduced into the general formula for entropy of a statistical system

$$S = -\sum_i p_i \log p_i \tag{2}$$

they have to satisfy (1).

[1] Throughout this paper we take $G = \hbar = 1$.

To obtain more information about p_i it is necessary to investigate other quantities of similar nature. In [1] we discussed the Renyi entropies [4] defined as

$$H_l = -\frac{1}{1-l}\log C_l\,, \tag{3}$$

where C_l are moments of p_i (coincidence probabilities),

$$C_l = \sum_{i=1}^{\Gamma}[p_i]^l\,, \tag{4}$$

where Γ is the total number of states of the system were evaluated.

It is not difficult to show that H_l's are related to the standard (Shannon) entropy S

$$S = H_1\,, \tag{5}$$

where the r.h.s. must be understood as a limiting value of H_l when $l \to 1$.

It was shown [1] that when the procedure leading to (1) is applied to evaluation the Renyi entropies, one obtains

$$H_l = \left(1 + \frac{1}{l} + \frac{1}{l^2} + \frac{1}{l^3}\right)\frac{S_M}{4}\,. \tag{6}$$

This relation is identical to that satisfied by the Renyi entropies of the ideal photon gas at equilibrium in the flat space [6].

The results (1) and (6) of [1] were obtained by evaluating the properties of Hawking radiation close to the horizon of the black hole. In the present paper we investigate the question how much these results change when one considers the Hawking radiation at a certain distance from the horizon. To this end we evaluated the Renyi entropies of Hawking radiation emitted into a thin shell around the black hole. The radius r of the shell (in Schwarzschild coordinates) is kept proportional to the radius ρ of the black hole

$$r = \rho/\alpha\,, \tag{7}$$

with α being a constant over the whole history of the black hole. With this new assumption the argument of [1] is repeated: The width of the shell is expressed in terms of the change of the mass of the black hole, as determined from the amount of the energy emitted into the shell; The total Renyi entropies are obtained by summing the contributions from the whole history of the black hole.

It is found that this procedure provides the correct Bekenstein–Hawking formula (1) for H_1, *i.e.* the entropy of the black hole [*c.f.* (5)]. This shows that, indeed, the entropy of the black hole is encoded in the Hawking radiation. This result does not depend on the parameter α, indicating that this information is not necessarily attached to the horizon but it is contained in the Hawking radiation anywhere around the black hole. It reflects the simple fact that the entropy emitted by the black hole is conserved during evolution of the Hawking radiation.

When the same procedure is applied to evaluate the Renyi entropies, the result turns out not independent of α. As one moves out of the horizon of the black hole (*i.e.* when α decreases from 1 towards 0), the Renyi entropies H_l decrease substantially (by a factor of about 10). Thus the Renyi entropies are not conserved when the Hawking radiation expands into the space out of the black hole.

If one wants to identify the Renyi entropies of the Hawking radiation with those of the black hole itself, it seems natural to take the values obtained by considering the region close to the horizon. It is remarkable that in this case our calculation gives the result which is identical to that of the free photon gas in the flat space.

To summarize, we evaluate the Renyi entropies of the Hawking radiation in a certain region of configuration space. This region is selected in such a way that the standard Bekenstein–Hawking formula for entropy is recovered. The same procedure is then employed for evaluation of other Renyi entropies.

In the next section the general formulae for entropies of the photon gas are given. The boundary conditions for the black hole are discussed in Section 3. In Section 4 the volume left by the shrinking black hole is estimated and expressed in terms of mass defect. The final formulae for the Shannon and Renyi entropies are obtained in Section 5. Our conclusions are listed in the last section. Evaluation of some relevant integrals is described in the Appendix.

2. A general formula for Renyi entropies of the photon gas

The probability of having n_1 photons in a state with energy ϵ_1, n_2 photons with energy ϵ_2,... is given by

$$P(n_1, n_2, ...n_M) = \prod_{m=1}^{M} \left(1 - e^{-\beta\epsilon_m}\right) e^{-\beta n_m \epsilon_m}\,, \tag{8}$$

where $\beta = 1/T$ and we have put the chemical potential to zero.

The coincidence probabilities are

$$C_l = \sum_{n_1,n_2,...} [P(n_1,n_2,...n_M)]^l = \prod_{m=1}^{M} \frac{\left(1-e^{-\beta\epsilon_m}\right)^l}{\left(1-e^{-\beta l\epsilon_m}\right)} \,. \tag{9}$$

This gives for the Renyi entropies

$$H_l = \frac{1}{1-l}\log C_l = -\sum_{m=1}^{M}\log\left(1-e^{-\beta\epsilon_m}\right) + \frac{1}{1-l}\sum_{m=1}^{M}\log\left(\frac{1-e^{-\beta\epsilon_m}}{1-e^{-\beta l\epsilon_m}}\right) . \tag{10}$$

Finally, when the sum over photon states is replaced by an integral we have

$$H_l = \int \frac{d^3p dV}{(2\pi\hbar)^3} W_l(\beta\epsilon) \,, \tag{11}$$

where dV is the volume element in configuration space, $\epsilon = \epsilon(p)$ is the energy of the photon of momentum p and

$$W_l(z) = -\log(1-e^{-z}) + \frac{1}{1-l}\log\left(\frac{1-e^{-z}}{1-e^{-lz}}\right) . \tag{12}$$

This formula (with the relation $\epsilon = p$, valid in flat space) was derived in [6].

3. Boundary conditions for the black hole

Our problem now is to evaluate H_l for the radiation emitted by the black hole of radius ρ into an infinitesimal layer around its surface. To this end we first have to specify the physical meaning of the energy ϵ and of the temperature $T = 1/\beta$ in the case of the black hole. Assuming that the emitted radiation is in equilibrium, we conclude that the temperature T is a constant, provided ϵ is the energy *conserved* in the process[2].

This identification implies that T should be taken as the Hawking temperature

$$T = T_{\rm H} = \frac{1}{4\pi\rho} = \frac{1}{8\pi M} \,. \tag{13}$$

To perform the integration in (11) we have to determine the phase-space volume.

[2] Since the gravitational field outside of the black hole is static, the conserved energy can be defined, *c.f.* [7], Chapter 88, and [8], Chapter 27.

Consider first the configuration space. We begin by evaluation of the contribution from a thin layer around the surface of the black hole. In presence of the gravitational field the 3-dimensional volume element can be expressed as

$$dV = \epsilon_{0ijk} dx_1^i dx_2^j dx_3^k \ \rightarrow \ \sqrt{|g|} u^\mu \epsilon_{\mu\nu\lambda\kappa} dx_1^\nu dx_2^\lambda dx_3^\kappa \,, \tag{14}$$

where $u^\mu = dx^\mu/ds$ is the 4-velocity of the volume element and g is the determinant of the metric tensor. For the volume element at rest we have

$$1 = u^\mu u_\mu = (u^0)^2 g_{00} \ \rightarrow \ u^0 = 1/\sqrt{g_{00}} \,. \tag{15}$$

In terms of the Schwarzschild coordinates we thus obtain

$$dV = r^2 dr \sin\theta d\theta d\phi / \sqrt{g_{00}} \,, \tag{16}$$

with

$$g_{00} = g_{00}(r;\rho) = 1 - \rho/r = 1 - \alpha \,, \tag{17}$$

with α defined in (7).

Integration over the angular variables implies

$$dV = 4\pi r^2 dr / \sqrt{g_{00}} \,. \tag{18}$$

At this point we observe that, as seen from (17), $g_{00}(\rho;\rho) = 0$, *i.e.* the formula (18) exhibits a singularity at $r = \rho$. Consequently, (17) can only be applied at $r > \rho$. We shall show below that the final result does not depend on this limitation.

Next step is to perform integration over momentum space. To this end we observe that the wave length of the radiation emitted from a sphere of radius ρ must be smaller than the diameter of the sphere [2, 9].

The condition $\lambda = h/p \leq 2\rho$ [2, 9], where λ is the wavelength, gives the lower limit of integration

$$p_{\min} = \frac{h}{2\rho} = \frac{\pi\hbar}{\rho} \,. \tag{19}$$

Since p is the true momentum of the photon (and not just a parametrization), the volume element in momentum space, after integration over angles, is simply

$$d^3p \rightarrow 4\pi p^2 dp \,. \tag{20}$$

Introducing all this into (11) we obtain for the Renyi entropy contained in the layer between r and $r+dr$

$$dH_l = \frac{2r^2 dr}{\pi\sqrt{g_{00}}} \int\limits_{p_{\min}}^{\infty} p^2 dp W_l(\beta\epsilon(p))\,. \tag{21}$$

To perform the integration it is necessary to express the energy ϵ in terms of the momentum p.

Putting

$$\epsilon = \gamma p\,, \tag{22}$$

where γ represents possible modifications of the black-body spectrum in presence of the black hole, we can rewrite (21) as

$$dH_l = \frac{2}{\pi\sqrt{g_{00}}}\frac{dr}{r}\left(\frac{r}{\gamma\beta\hbar}\right)^3 \Phi_l\,, \tag{23}$$

where Φ_l are numerical constants defined as

$$\Phi_l = \int\limits_{\epsilon_0}^{\infty} z^2 dz W_l(z)\,, \tag{24}$$

with W_l defined in (12), and

$$\epsilon_0 = p_{\min}\gamma\beta = \frac{\pi\gamma}{\rho T} = 4\pi^2\gamma\,, \tag{25}$$

where T is the Hawking temperature T_{H}, given by (13). One sees that the lower limit of integration in (24) depends on the factor γ expressing the modification of the photon energy in presence of the black hole.

If the only effect of the black hole on the spectrum of the photon (conserved) energy ϵ is the presence of the gravitational field, then [7])

$$\gamma = \sqrt{g_{00}} \tag{26}$$

implying that $\epsilon \approx 0$ close to the surface of the black hole. On the other hand, if the gravitational filed is neglected we have $\gamma = 1$, *i.e.* $\epsilon_0 = 4\pi^2$.

From (23), using (13) we have

$$dH_l = \frac{1}{32\pi^4\sqrt{g_{00}}}\frac{dr}{r}\left(\frac{1}{\alpha\gamma}\right)^3 \Phi_l\,. \tag{27}$$

This formula gives the Renyi entropy of the black-body radiation contained inside the layer between ρ and $\rho + dr$.

4. Relation to the mass defect

We have evaluated the contribution to the Renyi entropies from the Hawking radiation emitted into an infinitesimal layer of width dr outside of a black hole. At this point the width dr is still arbitrary. To connect it to the physical properties of the black hole we observe that it can be related to the change of the black hole mass, dM.

To relate dr to dM we observe that the emission of radiation causes the decrease of the mass of the black hole by the amount of emitted energy reduced by the amount of free energy used in the process of emission and shrinking of the black hole:

$$dM = dE - dF\,, \tag{28}$$

where F is the free energy of the photon gas.

The amount of emitted energy dE can be evaluated from the well-known formula for the photon gas:

$$dE = \frac{d^3p dV}{(2\pi\hbar)^3}\,\epsilon\,\frac{e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}} = \frac{d^3p dV}{(2\pi\hbar)^3}\,\gamma p\,\frac{e^{-\beta\gamma p}}{1-e^{-\beta\gamma p}}\,. \tag{29}$$

Similarly, using the relation between the free energy and the statistical sum Z we have

$$dF = -\frac{d^3p dV}{(2\pi\hbar)^3} T\log Z = \frac{d^3p dV}{(2\pi\hbar)^3} T\log(1-e^{-\beta\gamma p})\,. \tag{30}$$

The integration over dVd^3p goes exactly as in the previous section and we obtain, using the same boundary conditions for momentum integration

$$dE = \frac{1}{32\pi^4\sqrt{g_{00}}}\frac{dr}{r}\left(\frac{1}{\alpha\gamma}\right)^3 T\Omega\,, \tag{31}$$

$$dF = \frac{1}{32\pi^4\sqrt{g_{00}}}\frac{dr}{r}\left(\frac{1}{\alpha\gamma}\right)^3 T\omega\,, \tag{32}$$

where

$$\Omega = \int\limits_{\epsilon_0}^{\infty} z^3 dz\frac{e^{-z}}{1-e^{-z}}\,, \qquad \omega = \int\limits_{\epsilon_0}^{\infty} z^2 dz\log(1-e^{-z})\,. \tag{33}$$

Introducing (31) and (32) into (28) we obtain

$$dM = \frac{1}{32\pi^4\sqrt{g_{00}}}\frac{dr}{r}\left(\frac{1}{\alpha\gamma}\right)^3 T(\Omega-\omega)\,. \tag{34}$$

Consequently,

$$\frac{dr}{r} = \frac{32\pi^4\sqrt{g_{00}}\alpha^3\gamma^3}{(\Omega-\omega)}\frac{dM}{T_H}\,. \tag{35}$$

5. Renyi entropies of the black hole

We are now in the position to determine the Renyi entropies. To this end we introduce dr given by the formula (35) into (27). One sees that the factors $\sqrt{g_{00}}(\alpha\gamma)^3$ in the numerator and denominator cancel exactly and one obtains

$$dH_l = \frac{\Phi_l}{\Omega-\omega}8\pi M dM\,, \tag{36}$$

where we have used (13). Thus we have expressed the change of the Renyi entropy in terms of the change in the black hole mass, essentially repeating the original procedure of Bekenstein [2,9].

After integration of (36) from 0 to M we thus have

$$H_l = \frac{\Phi_l}{\Omega-\omega}\frac{4\pi G M^2}{\hbar} = \frac{\Phi_l}{\Omega-\omega}S\,, \tag{37}$$

where S is the Bekenstein–Hawking entropy of the black hole [2,3], given by (1). It is demonstrated in the Appendix that $\Phi_1 = \Omega-\omega$ and thus (5) implies the correct formula (1) for the Shannon entropy.

When the numerical constants Φ_l, Ω and ω (evaluated in the Appendix) are introduced into (37) one obtains

$$H_l = \frac{\Phi_l}{\Phi_1}S\,, \tag{38}$$

where

$$\Phi_1 = 8F_4(u) + 8\epsilon_0 F_3(u) + 4\epsilon_0^2 F_2(u) + \epsilon_0^3 F_1(u)\,, \tag{39}$$

and for $l \geq 2$

$$\begin{aligned}\Phi_l &= \frac{l}{l-1}\left[2F_4(u) + 2\epsilon_0 F_3(u) + \epsilon^2 F_2(u)\right] \\ &\quad - \frac{1}{l-1}\left[\frac{2F_4(u^l)}{l^3} + \frac{2\epsilon_0 F_3(u^l)}{l^2} + \frac{\epsilon_0^2 F_2(u^l)}{l}\right]\,,\end{aligned} \tag{40}$$

with $u = e^{-\epsilon_0}$ and

$$F_k(x) = \sum_{n=1}^{\infty} \frac{x^k}{n^k} \,. \tag{41}$$

A simple algebra shows that for $\epsilon_0 = 0$ one recovers (6). On the other hand, when ϵ_0 is large we have for $l \geq 2$

$$H_l = \frac{l}{l-1} \frac{1 + 2/\epsilon_0 + 2/\epsilon_0^2}{\epsilon_0(1 + 2/\epsilon_0 + 4/\epsilon_0^2 + 4/\epsilon_0^3)} S \,. \tag{42}$$

One sees that in this case the second Renyi entropy is by factor of about 20 times smaller than the Shannon entropy.

The dependence of the ratio $H_l/S = \Phi_l/\Phi_1$, evaluated from (40), is plotted in Fig. 1 *versus* g_{00}.

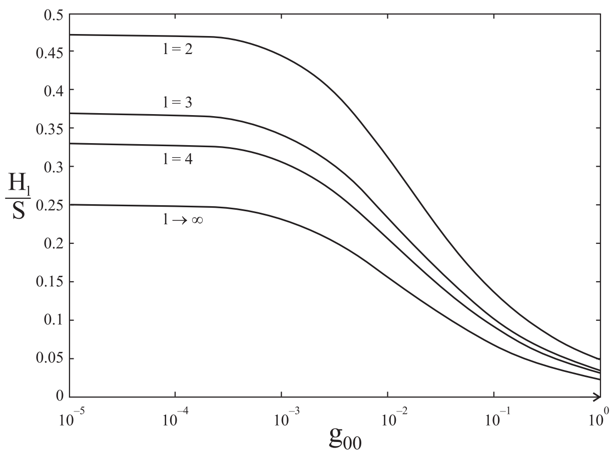

Fig. 1. The ratio H_l/S plotted *versus* $g_{00} = 1 - \rho/r$.

6. Conclusion and comments

Our results can be summarized as follows.

(i) Starting from the assumption that statistical properties of a Schwarzschild black hole of radius ρ are encoded in the Hawking radiation, we have evaluated its Renyi entropies by counting the states of the Hawking radiation which fills an infinitely thin spherical shell of radius $r > \rho$, where $\rho = 2M$ is the radius of the black hole. The width of the shell is determined from the condition that the energy of the radiation contained there corresponds to the mass defect which the black hole

suffers during the emission process. This procedure leads to the correct formula for the Shannon entropy of the black hole, independently of the radius r of the shell. The results for Renyi entropies H_l at $l \geq 2$ are sensitive, however, to the chosen value of the ratio r/ρ. Thus additional condition is needed to obtain a unique result.

(ii) As shown in [1], our argument can be also applied to the shell infinitely close to the horizon. In spite of the singularity of the Schwarzschild metric at the horizon, the result turns out finite and no regularization procedure is necessary. This limit can thus be considered as a realization of the idea that the statistical properties of the black hole are determined by its surface [2, 3, 9]. In this case the Renyi entropies are uniquely determined and their relation to the Shannon entropy is identical to that obtained for the free photon gas in the flat space.

We would like to add two more comments.

(a) Since the Renyi entropies provide additional information about the statistical properties of the black hole, we feel that our calculation may be useful in the search for its internal structure.

(b) Although in our argument we have used the description of Hawking radiation in terms of plain waves, an analogous calculation can be performed using the spherical waves [1], as formulated, *e.g.*, in [5]. We have checked that both calculations give identical results.

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Appendix

Evaluation of numerical constants

To find Φ_l we have to evaluate integral

$$\Theta_l = \int_{\epsilon_0}^{\infty} z^2 dz \log\left(1 - e^{-lz}\right) . \tag{43}$$

This can be done by expansion in series of e^{-lz}

$$\Theta_l = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{\epsilon_0}^{\infty} z^2 dz e^{-nlz} \tag{44}$$

giving

$$\begin{aligned} \Theta_l &= -\epsilon_0^2 \sum_{n=1}^{\infty} \frac{l^2 n^2 + 2ln/\epsilon_0 + 2/e_0^2}{l^3 n^4} e^{-nl\epsilon_0} \\ &= -\epsilon_0^2 \left[\frac{F_2\left(u^l\right)}{l} + \frac{2F_3\left(u^l\right)}{l^2 \epsilon_0} + \frac{2F_4\left(u^l\right)}{l^3 \epsilon_0^2} \right], \end{aligned} \tag{45}$$

where $u = e^{-\epsilon_0}$ and

$$F_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} . \tag{46}$$

Using this we have for $l \geq 2$

$$\Phi_l = -\frac{l}{l-1}\Theta_1 + \frac{1}{l-1}\Theta_l \approx \frac{l}{l-1}\epsilon_0^2 \left(\frac{1+2}{\epsilon_0} + \frac{2}{\epsilon_0^2} \right) e^{-\epsilon_0} , \tag{47}$$

where in the second equality we have kept only lowest order terms in $e^{-\epsilon_0}$.

To obtain limit $l \to 1$ we observe that

$$\begin{aligned} F_k\left(u^l\right) &= F_k(u\,(1-(l-1)\epsilon_0) \approx F_k(u) - (l-1)\epsilon_0 u F_k'(u) \\ &= F_k(u) - (l-1)\epsilon_0 F_{k-1}(u) . \end{aligned} \tag{48}$$

Consequently,

$$\begin{aligned} \Theta_l - \Theta_1 &\approx 2F_4(u)\left(1 - 1/l^3\right) + 2\epsilon_0 F_3(u)\left(1 - 1/l^2\right) + \epsilon_0^2 F_2(u)\left(1 - 1/l\right) \\ &\quad +(l-1)[2\epsilon_0 F_3(u) + 2\epsilon_0^2 F_2(u) + \epsilon_0^2 F_1(u)] \\ &\approx (l-1)[6F_4(u) + 6\epsilon_0 F_3(u) + 3\epsilon_0^2 F_2(u) + \epsilon_0^3 F_1(u)] , \end{aligned} \tag{49}$$

and thus

$$\Phi_1 = -\Theta_1 + \frac{\Theta_l - \Theta_1}{l-1} = 8F_4(u) + 8\epsilon_0 F_3(u) + 4\epsilon_0^2 F_2(u) + \epsilon_0^3 F_1(u) . \tag{50}$$

To evaluate Ω we also expand the integrand in powers of e^{-z}

$$\Omega = \sum_{n=1}^{\infty} \int_{\epsilon_0}^{\infty} z^3 dz e^{-nz} = 6F_4(u) + 6\epsilon_0 F_3(u) + 3\epsilon_0^2 F_2(u) + \epsilon_0^3 F_1(u) . \tag{51}$$

To obtain ω we simply observe that

$$\omega = \int\limits_{\epsilon_0}^{\infty} z^2 dz \log(1 - e^{-z}) = \Theta_1 \tag{52}$$

and, consequently,

$$\Phi_1 = \Omega - \omega\,. \tag{53}$$

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