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THE CASIMIR ENERGY OF DIRAC FIELD UNDER A GENERAL BOUNDARY CONDITION USING THE ZETA FUNCTION METHOD

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Using the generalized zeta-function method we discuss the influence of a uniform magnetic field on the Casimir energy of a fermionic field submitted to a general boundary condition which interpolates continuously periodic and antiperiodic ones. After computing the relevant fermionic determinant we show that the Casimir effect can be enhanced by the external magnetic field, in agreement to the known results established in the literature. We also compute the corresponding Casimir pressure and present the result graphically by sketching its behavior as a function of the magnetic field and the parameter θ which defines the boundary condition. This analytical result is a new one in the known literature.

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1. Introduction

The modification in the vacuum energy for quantum fields due to the presence of boundary conditions (which constitute a mathematical model to the actual physical interaction with some surrounding medium) is a problem that has received growing attention since its discovery by Casimir in 1948 [1].

In 1975 Johnson [2] calculated for the first time the Casimir fermionic effect in the MIT-bag model. The interest was related to the possibility of giving description of the behavior of quarks in a hadron as well as the processes which involved electrons in QED. By the end of the 90thies a better description of this behavior was obtained when the boundary conditions were imposed on a spherical surface [3].

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According to [4], we can think of the Casimir effect as the effect of non-trivial space topology on vacuum fluctuations for any relativistic quantum field. The non-trivial topology has as sources background fields, constraints and boundary conditions. As it is also discussed in Ref. [4], vacuum fluctuations of a charged quantum field are also affected by external fields.

The perturbation of Casimir energy by an external magnetic field was firstly considered by Elizalde *et al.* [5] and Cougo-Pinto *et al.* [6]. The influence on the Casimir energy of the Dirac field was firstly studied for an antiperiodic boundary condition by Cougo-Pinto *et al.* [7]. These authors, by using Schwinger's proper-time representation for the effective action, showed that the Casimir effect can be enhanced by a magnetic field. The results for a Dirichlet boundary condition as well as for a periodic boundary condition can be seen in Ref. [4] and for a general condition interpolating periodic and antiperiodic conditions through the summation of modes can be found in Ref. [8].

On the other hand, Tort *et al.* present the modification of Casimir effect submitted to a magnetic field and considering the MIT boundary conditions [9, 10]. More recently the influence of an external magnetic field on the complex scalar field as well as on the Casimir energy of the Dirac field was considered by Ostrowski [11] who calculates these by direct solving the field equations using the mode summation method and comparing the numerical results with those obtained in [4]. Further discussions on Casimir effect can be seen in [12, 13].

The modification on Casimir energy of the Dirac field due to a general boundary condition which interpolates between the periodic and antiperiodic conditions will be considered through an appropriately mathematical function with a parameter θ and calculated by using the zeta function method. This procedure, through a mathematical point of view, is an interesting one because it can be applied to a variety of situations and so, we can get a wider approach of the particular mathematical technique used in this particular case, namely the zeta function method.

Our objective is to obtain the results for the fermionic part in [4] in an analytic way by using the generalized zeta function method. The relevance of the zeta function method was considered in many references especially in [14–16]. Here, we reconsider the boundary conditions assumed in [8], but instead of computing the Casimir energy for the Dirac field as the energy of the Dirac sea, we shall compute the corresponding fermionic determinant through the generalized zeta function method (see Ref. [17] and for a review of this method see Ref. [14] and references therein). The main advantage of this method is to get the answer with a minimum of spurious terms. From the Casimir energy we also calculate the Casimir pressure and we present its graphical behavior. We present an analytical calculation as well as an analysis of the weak and strong field limits for the Casimir pressure. This last calculation for the generalized boundary condition used here is a new one in the literature.

The generalized boundary condition to be considered here is such that the Dirac field suffers a phase shift, which we call θ, each time the variable z is changed by an amount a:

$$\psi\left(x,y,z+a\right) = e^{\imath\theta} \quad \psi\left(x,y,z\right) \,, \tag{1}$$

$$\bar{\psi}\left(x,y,z+a\right) = e^{-\imath\theta} \quad \bar{\psi}\left(x,y,z\right) \,. \tag{2}$$

For $\theta=\pi$ we recover the original antiperiodic boundary condition and with $\theta=0$ we have a periodic boundary condition. For other values of θ, between 0 and π, we obtain interpolations from the usual periodic to antiperiodic boundary conditions.

2. The Casimir energy via generalized zeta function method

Methods based on analytical extension, like the generalized zeta function method, usually provide the final answer with a minimum of spurious terms and as such they can be used in order to check results obtained by others methods. In this sense, the purpose of this work is to re-obtain the results quoted in Refs. [4, 7, 8] in the context of the zeta function method. Our starting point is the Lagrangian density for Dirac field in the presence of an external field $A^e_\mu\left(x\right)$, namely:

$$\mathcal{L} = -\bar{\psi}\left(\gamma\cdot\mathit{\Pi}+m\right)\psi\,, \tag{3}$$

where $\mathit{\Pi}_\mu$ is defined by $\mathit{\Pi}_\mu = \imath\partial_\mu - eA_\mu$. The vacuum energy for the Dirac field we are interested in can be obtained from the vacuum-to-vacuum transition amplitude under the influence of a uniform magnetic field as well as the boundary conditions under consideration, which is given by

$$\begin{aligned}\langle 0_+|0_-\rangle^A_\theta &= N\int\limits_{C_\theta}\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left\{-\imath\int d^4x\bar{\psi}\left(\gamma\cdot\mathit{\Pi}+m-\imath\epsilon\right)\psi\right\}\\ &= N'\det(\gamma\cdot\mathit{\Pi}+m-\imath\epsilon)\,,\end{aligned} \tag{4}$$

where N and N' are normalization constants such that $\langle 0_+|0_-\rangle^{A=0}_\theta = 1$ and the subscript C_θ in the integral means that the functional integration must be over the Grassmann variables that satisfy the boundary conditions. The convergence of integral (4) is guaranteed by the change $m\to m-\imath\epsilon$, with $\epsilon>0$. Defining $W_\theta(A)$ by $\langle 0_+|0_-\rangle_\theta = e^{iW_\theta(A)}$ we can write:

$$W_\theta(A) = -\imath \mathrm{Tr} \ln \left[(\gamma \cdot \mathit{\Pi} + m) \right] + \imath \mathrm{Tr} \ln \left[(\gamma \cdot P + m) \right] \,, \tag{5}$$

since we are using the normalization condition $W_\theta(A = 0) = 0$.

The connection between $W_\theta(A)$ and the vacuum energy is the following:

$$\mathcal{E}(a, B) = -\frac{W_\theta(A)}{T} \,, \tag{6}$$

where we are assured that the vacuum energy will depend on the parameter a and the external magnetic field B. Now, we carry out the evaluation of $W_\theta(A)$. We know that $\det(m \pm \gamma \cdot \mathit{\Pi})$ is a Lorentz scalar and hence it is independent of the sign of $\gamma \cdot \mathit{\Pi}$, so that we can write [18]:

$$\ln \det (m + \gamma \cdot \mathit{\Pi}) = \tfrac{1}{2} \mathrm{Tr} \ln \left(m^2 - (\gamma \cdot \mathit{\Pi})^2 \right) . \tag{7}$$

This formula allows us to work with a second order formalism. On the other hand, for a constant magnetic field it can be shown that:

$$-(\gamma \cdot \mathit{\Pi})^2 \;=\; \mathit{\Pi}^2 - \imath \sigma^{\mu\nu} \mathit{\Pi}_\mu \mathit{\Pi}_\nu \;=\; \mathit{\Pi}^2 - eB\sigma^3 \,, \tag{8}$$

where

$$\sigma^3 = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} ,$$

and σ_z is the Pauli matrix in the z direction. Consequently, we get:

$$\tfrac{1}{2} \mathrm{Tr} \ln \left(m^2 - (\gamma \cdot \mathit{\Pi})^2 \right) \;=\; 2 \mathrm{tr}_x \left[\ln \left(m^2 + \mathit{\Pi}^2 - eB \right) + \ln \left(m^2 + \mathit{\Pi}^2 + eB \right) \right] ,$$

where Tr means total trace (functional and over the Dirac indices) and tr_x means only functional trace. Analogously, we have:

$$\tfrac{1}{2} \mathrm{Tr} \ln (m^2 + P^2) = 2 \mathrm{tr}_x \ln \left(m^2 + P^2 \right) = 2 \ln \det{}_x \left(m^2 + P^2 \right) . \tag{9}$$

Combining the last results, we can obtain

$$\begin{aligned} W_\theta(A) \;=\; & -\imath \left[\ln \det{}_x \left(m^2 + \mathit{\Pi}^2 - eB \right) + \ln \det{}_x \left(m^2 + \mathit{\Pi}^2 + eB \right) \right] \\ & + 2\imath \ln \det{}_x \left(m^2 + P^2 \right) . \end{aligned} \tag{10}$$

Recalling that the zeta function prescription for the determinant of an operator H is given by [14]:

$$\det H = \exp \left\{ -\frac{\partial \zeta(s = 0; H)}{\partial s} \right\} \,,$$

with $\zeta(s;H) := \mathrm{Tr}\, H^{-s}$, we have:

$$\begin{aligned} W_\theta\left[A^e\right] &= \imath\zeta'\left(s=0, m^2 - \imath\epsilon + \Pi^2 - eB\right) + \imath\zeta'\left(s=0, m^2 - \imath\epsilon + \Pi^2 + eB\right) \\ &\quad -2\imath\zeta'\left(s=0, m^2 - \imath\epsilon + P^2\right). \end{aligned} \tag{11}$$

It is convenient to work in the Euclidean spacetime. Performing the euclideanization of Π^2 leads to:

$$\Sigma^2 := \quad m^2 + \Pi_{\mathrm{E}}^2 = m^2 - \partial_\tau^2 - \partial_3^2 + \left(\vec{p} - e\vec{A}\right)_\perp^2, \tag{12}$$

where subscript $\perp$ means coordinates 1 and 2 (orthogonal to 3), while the potential, $A_\mu = (0, -By, 0, 0)$ generates the uniform constant magnetic field,

$$F_{\mu\nu} = \begin{cases} 0, & \text{if} \quad \mu,\nu = 0\,, \\ 0, & \text{if} \quad \mu,\nu = 3\,, \\ B, & \text{if} \quad \mu = 1, \nu = 2\,. \end{cases}$$

The eigenvalues of $m^2 + \Pi_{\mathrm{E}}^2$ (let us call them β) that satisfy

$$\left(m^2 - \partial_\tau^2 - \partial_3^2 + \left(\vec{p} - e\vec{A}\right)_\perp^2\right)\chi = \beta\chi\,, \tag{13}$$

with the boundary conditions under consideration, are given by

$$\beta_{k_0,l,n} = m^2 + k_0^2 + \left(\frac{\theta + 2\pi l}{a}\right)^2 + (2n+1)\, eB\,, \tag{14}$$

where $n = 0, 1, \ldots$ and ℓ is an integer. Then, the relevant zeta functions are:

$$\zeta\left(s, H_1\right) = \frac{eBAT_{\mathrm{E}}}{2\pi} \sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left[f(\theta) + 2neB\right]^{-s}, \tag{15}$$

$$\zeta\left(s, H_2\right) = \frac{eBAT_{\mathrm{E}}}{2\pi} \sum_{l=-\infty}^{\infty}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left[f(\theta) + 2\left(n+1\right)eB\right]^{-s}, \tag{16}$$

$$2\zeta\left(s, H_0\right) = 2\frac{AaT_{\mathrm{E}}}{(2\pi)^4}\int_{-\infty}^{\infty} dk_0 d^3k\left[m^2 + k_0^2 + k_1^2 + k_2^2 + k_3^2\right]^{-s}, \tag{17}$$

where $f(\theta)$ is

$$f(\theta) = m^2 + k_0^2 + \left(\frac{\theta + 2\pi l}{a}\right)^2. \tag{18}$$

The factor $eBA/(2\pi)$ present in Eqs. (15) and (16) takes into account the Landau levels' degeneracy. Using the definition of the Euler Gamma function (see formula (8.312–2) in [19]) we get:

$$\zeta_{\text{reg}}(s) := \zeta(s, H_1) + \zeta(s, H_2) - 2\zeta(s, H_0)\,, \tag{19}$$

where

$$\begin{aligned}\zeta(s, H_1) &= \frac{1}{\Gamma(s)} T_{\text{E}} \frac{eBA}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d\xi \xi^{s-1} \\ &\quad \times \{\exp[-\xi(f(\theta) + 2neB)]\}\,, \end{aligned}\tag{20}$$

$$\begin{aligned}\zeta(s, H_2) &= \frac{1}{\Gamma(s)} T_{\text{E}} \frac{eBA}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d\xi \xi^{s-1} \\ &\quad \times \{\exp[-\xi(f(\theta) + 2(n+1)eB)]\}\,, \end{aligned}\tag{21}$$

and

$$\begin{aligned}-2\zeta(s, H_0) &= -\frac{1}{\Gamma(s)} \frac{2T_{\text{E}}Aa}{(2\pi)^4} \int_{-\infty}^{\infty} dk_0 d^3k \int_0^{\infty} d\xi \xi^{s-1} \\ &\quad \times \exp\left[-\xi\left(m^2 + k_0^2 + k_1^2 + k_2^2 + k_3^2\right)\right]. \end{aligned}\tag{22}$$

Next, we evaluate only the two first terms in (19)

$$\begin{aligned}&\frac{T_{\text{E}}}{\Gamma(s)} \frac{eBA}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d\xi \xi^{s-1} \exp[-\xi f(\theta)] \\ &\times \sum_{n=0}^{\infty} \{\exp(-2neB\xi) + \exp[-2(n+1)eB\xi]\}\,. \end{aligned}\tag{23}$$

Taking into account the formula for geometrical series, we rewrite

$$\sum_{n=0}^{\infty} [\exp(-2neB\xi) + \exp(-2(n+1)eB\xi)] = \coth(eB\xi)\,. \tag{24}$$

With the help of Poisson summation rule, we recast the sum in ℓ as

$$\sum_{l=-\infty}^{\infty} \exp\left[-\xi\left(\frac{\theta + 2\pi l}{a}\right)^2\right] = \sum_{r=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[2\pi i r x - \xi\left(\frac{\theta + 2\pi x}{a}\right)^2\right] dx\,. \tag{25}$$

Using the last two equations and integrating over k_0, Eq. (23) becomes:

$$\frac{1}{\Gamma(s)}\frac{eBAT_{\mathrm{E}}}{(2\pi)^2}\sqrt{\pi}\sum_{r=-\infty}^{\infty}\int_0^{\infty}d\xi\xi^{s-\frac{3}{2}}\coth(eB\xi)$$
$$\times\int_{-\infty}^{\infty}dx\exp\left[2\pi irx-\xi\left(m^2+\left(\frac{\theta+2\pi x}{a}\right)^2\right)\right]. \tag{26}$$

Now, using that

$$\int_{-\infty}^{\infty}dx\exp\left[2\pi irx-\xi\left(\frac{\theta+2\pi x}{a}\right)^2\right]=\frac{a}{2\sqrt{\pi\xi}}\exp\left(-ir\theta-\left(\frac{ra}{2}\right)^2\frac{1}{\xi}\right),$$

the Eq. (26) becomes

$$\frac{1}{\Gamma(s)}\frac{T_{\mathrm{E}}Aa}{2(2\pi)^2}\sum_{r=-\infty}^{\infty}\exp(-ir\theta)\int_0^{\infty}d\xi\xi^{s-3}\exp\left(-m^2\xi-\left(\frac{ra}{2}\right)^2\frac{1}{\xi}\right)$$
$$+\frac{1}{\Gamma(s)}\frac{eBT_{\mathrm{E}}Aa}{2(2\pi)^2}\sum_{r=-\infty}^{\infty}\exp(-ir\theta)\int_0^{\infty}d\xi\xi^{s-2}L(eB\xi)$$
$$\times\exp\left(-m^2\xi-\left(\frac{ra}{2}\right)^2\frac{1}{\xi}\right), \tag{27}$$

where $L(x)=\coth(x)-1/x$ is the Langevin function. The $r=0$ term in the first sum of expression (27) exactly cancels the last term in Eq. (17). Using the integral representation for the modified Bessel function of second kind (see formula (8.432–6) in [19]) and performing the integration over ξ and making the change of variable $\xi=a^2/\sigma \quad \Rightarrow \quad d\xi=-(a/\sigma)^2\,d\sigma$ in the second integral in the Eq. (27), we obtain

$$\zeta_{\mathrm{reg}}(s) = \frac{1}{\Gamma(s)}\frac{T_{\mathrm{E}}Aa}{(2\pi)^2}\left\{2\left(\frac{2m}{a}\right)^{2-s}\sum_{r=1}^{\infty}\frac{1}{r^{2-s}}K_{2-s}(mra)\cos(r\theta)\right.$$
$$+\frac{eB}{2a^{2-2s}}\sum_{r=-\infty}^{\infty}\exp(-ir\theta)\int_0^{\infty}\frac{d\sigma}{\sigma^{-s}}L\left(\frac{eBa^2}{\sigma}\right)$$
$$\left.\times\exp\left(-\left(\frac{r}{2}\right)^2\sigma-\frac{(am)^2}{\sigma}\right)\right\}. \tag{28}$$

In order to evaluate $\zeta'_{\rm reg}(0)$ we use the fact that if $\zeta_{\rm reg}(s) = F(s)/\Gamma(s)$, with F being a function which is analytic at $s=0$, then $\zeta'_{\rm reg}(0) = F(0)$. Grouping all the previous results, or more precisely, using equations (6), (11), (19) and taking the result for $\zeta'_{\rm reg}(0)$, we finally obtain the Casimir energy density:

$$\begin{aligned}\frac{\mathcal{E}(a,B)}{A} \equiv \frac{1}{A\,T_{\rm E}}\zeta'_{\rm reg}(0) &= \frac{2(am)^2}{\pi^2 a^3}\sum_{r=1}^{\infty}\frac{1}{r^2}K_2(amr)\cos(r\theta)\\ &+\frac{eB}{8\pi^2 a}\sum_{r=-\infty}^{\infty}\exp(-ir\theta)\int\limits_0^{\infty}d\sigma L\left(\frac{eBa^2}{\sigma}\right)\\ &\times\exp\left(-\frac{r^2\sigma}{4}-\frac{(am)^2}{\sigma}\right). \qquad (29)\end{aligned}$$

Now, observe that the contribution coming from the last term on the right hand side of Eq. (29) if we put $r=0$, namely,

$$\frac{eBA}{8\pi^2 a}\int\limits_0^{\infty}d\sigma L\left(\frac{eBa^2}{\sigma}\right)\exp\left(-\frac{(am)^2}{\sigma}\right),$$

which may be written as

$$(Aa)\,\frac{eB}{8\pi^2}\int\limits_0^{\infty}d\xi L(eB\xi)\exp\left(-m^2\xi\right),$$

then it is proportional to the volume Aa. Hence, it gives rise to a uniform energy density per unit volume (independent of parameter a) and does not have to be taken into account in the Casimir energy. Thus the final result is

$$\begin{aligned}\frac{\mathcal{E}(a,B)}{A} &= \frac{2(am)^2}{\pi^2 a^3}\sum_{r=1}^{\infty}\frac{1}{r^2}K_2(amr)\cos(r\theta)+\frac{eB}{4\pi^2 a}\sum_{r=1}^{\infty}\cos(r\theta)\\ &\times\int\limits_0^{\infty}d\sigma L\left(\frac{eBa^2}{\sigma}\right)\exp\left(-\frac{r^2\sigma}{4}-\frac{(am)^2}{\sigma}\right). \qquad (30)\end{aligned}$$

This is our main result, since it gives the Casimir energy density for a variety of situations interpolating continuously periodic and antiperiodic boundary conditions for the Dirac field in the presence of a uniform magnetic field.

To verify the consistence of our expression, equation (30), we now take the limits for $\theta = 0$ and $\theta = \pi$, *i.e.*, the periodic and antiperiodic cases, respectively.

For the periodic situation the result is achieved when we put $\theta = 0$ in the above equation:

$$\frac{\mathcal{E}_{\text{pdc}}\left(a,B\right)}{A} = \frac{2\left(am\right)^{2}}{\pi^{2}a^{3}}\sum_{r=1}^{\infty}\frac{1}{r^{2}}K_{2}\left(amr\right)+\frac{eB}{4\pi^{2}a}\sum_{r=1}^{\infty}\int_{0}^{\infty}d\sigma L\left(\frac{eBa^{2}}{\sigma}\right) \times\exp\left(-\left(\frac{r}{2}\right)^{2}\sigma-\frac{\left(am\right)^{2}}{\sigma}\right), \tag{31}$$

while the antiperiodic one is given by $(\theta = \pi)$

$$\frac{\mathcal{E}_{\text{apd}}\left(a,B\right)}{A} = -\frac{2\left(am\right)^{2}}{\pi^{2}a^{3}}\sum_{r=1}^{\infty}\frac{\left(-1\right)^{r+1}}{r^{2}}K_{2}\left(amr\right)-\frac{eB}{4\pi^{2}a}\sum_{r=1}^{\infty}\left(-1\right)^{r+1} \times\int_{0}^{\infty}d\sigma L\left(\frac{eBa^{2}}{\sigma}\right)\exp\left(-\left(\frac{r}{2}\right)^{2}\sigma-\frac{\left(am\right)^{2}}{\sigma}\right). \tag{32}$$

We call attention to the fact that the part independent of external magnetic field in (32) gives the expected result in the limit of zero mass, *i.e.*,

$$\frac{\mathcal{E}}{A} = -\xi\frac{\pi^{2}}{720a^{3}}\,, \tag{33}$$

with $\xi = 7 \times 4$. For the second term we follow the analysis established in [4] which shows that the external magnetic field increases the fermionic Casimir energy due to the presence of a quadrature which is strictly positive, decreases monotonically as r increases and goes to zero as $r \to \infty$.

Our results given by equations (30), (31) and (32) are in perfect agreement with Refs. [4, 7, 8]. Particularly, in Ref. [7] a numerical analysis for the antiperiodic condition shows that for the electron field constrained in a typical Casimir cavity, the enhancement in the Casimir energy is far from being detectable in laboratory. However, for different geometries and field configurations the situation can change. Besides, an analogous effect will occur in QCD and despite of the fact that quarks are much heavier than electrons, the dimensions involved are much smaller so that the effect may become significant.

In the next section, we are interested in the derivation of the Casimir pressure as well as in analyzing its behavior, so in order to achieve this, we get to starting point Eq. (30) for the Casimir density energy.

3. The Casimir pressure

Now, from the expression of density energy in (30) we can obtain the corresponding Casimir pressure

$$P = -\frac{\partial}{\partial a}\left(\frac{\mathcal{E}\left(a,B\right)}{A}\right), \tag{34}$$

taking into account that

$$L\left(\frac{eBa^2}{\sigma}\right) = \coth\left(\frac{eBa^2}{\sigma}\right) - \left(\frac{eBa^2}{\sigma}\right)^{-1}, \tag{35}$$

$$K_2\left(amr\right) = \frac{1}{2}\left(\frac{amr}{2}\right)^2 \int_0^\infty d\eta\eta^{-3} \exp\left(-\eta - \frac{\left(amr\right)^2}{4\eta}\right). \tag{36}$$

Carrying out the derivations we have

$$\frac{\partial}{\partial a}L\left(\frac{eBa^2}{\sigma}\right) = -\left(\frac{2eBa^2}{\sigma}\right)\operatorname{csch}^2\left(\frac{eBa^2}{\sigma}\right) + \left(\frac{eBa^2}{2\sigma}\right)^{-1}, \tag{37}$$

and observing the relation (8.486.12) in Ref. [19] we get

$$a\frac{\partial}{\partial a}K_2\left(amr\right) \to z\frac{\partial}{\partial z}K_2\left(z\right) = -2K_2\left(z\right) - zK_1\left(z\right), \tag{38}$$

where $z = amr$, and so we can write the Casimir pressure as

$$\begin{aligned} P\left(a,m,B\right) &= \frac{2\left(am\right)^2}{\pi^2 a^4}\sum_{r=1}^{\infty}\frac{\cos\left(r\theta\right)}{r^2}\left(3K_2\left(amr\right) + \left(amr\right)K_1\left(amr\right)\right) \\ &\quad + \frac{eB}{4\pi^2 a^2}\sum_{r=1}^{\infty}\cos\left(r\theta\right)\int_0^\infty d\sigma\left[\left(1 + \frac{2\left(am\right)^2}{\sigma}\right)L\left(\frac{eBa^2}{\sigma}\right)\right. \\ &\quad \left. + \left(\frac{2eBa^2}{\sigma}\right)\operatorname{csch}^2\left(\frac{eBa^2}{\sigma}\right) - \left(\frac{eBa^2}{2\sigma}\right)^{-1}\right] \\ &\quad \times\exp\left(-\frac{r^2\sigma}{4} - \frac{\left(am\right)^2}{\sigma}\right). \end{aligned} \tag{39}$$

The Eq. (39) gives our final result for the Casimir pressure of the Dirac field in the presence of the external magnetic field $\boldsymbol{B}$. The r.h.s. of the equation (39) has two terms, the first one is independent of the magnetic field $\boldsymbol{B}$ and comes from the contribution of the first term in the r.h.s. of

(30) while the second one is magnetic field dependent. The behavior of the Casimir pressure in (39) is determined by the same quadrature as in Eq. (30).

Figures 1 and 2 give a comparison of graphical plots for the external magnetic dependent condition in the Casimir pressure taken for two values of θ according to the Eq. (39). All the figures were obtained by using the MAPLE® program.

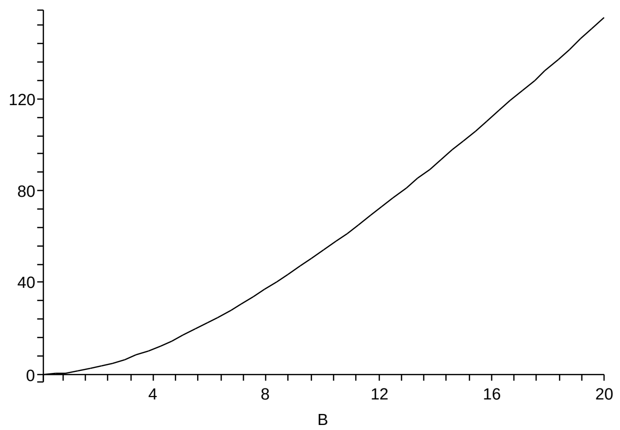

Fig. 1. Casimir pressure *versus* magnetic field (in gauss) for the periodic case given by the Eq. (39). The parameter a is taken as $1\mu m$.

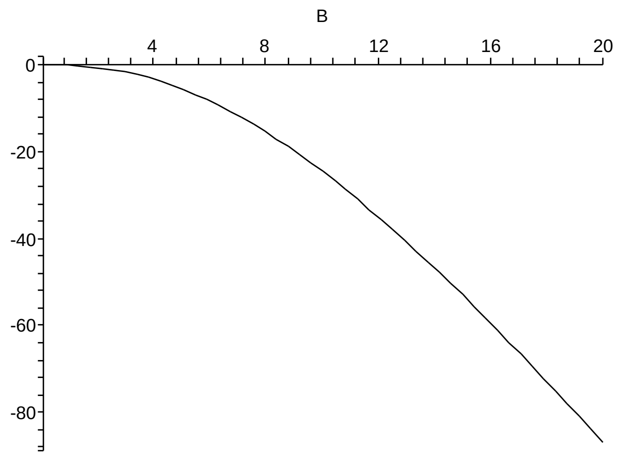

Fig. 2. Casimir pressure *versus* magnetic field (in gauss) for the antiperiodic case given by the Eq. (39). The parameter a is taken as $1\mu m$.

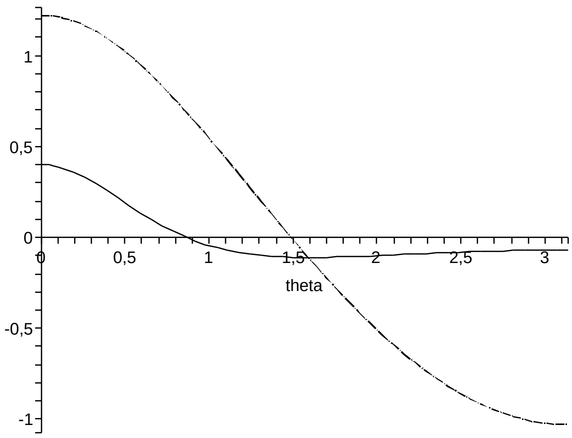

Fig. 3. The Casimir pressure *versus* θ. A graphical plot compares B-dependent and B-independent parts in Eq. (39). The magnetic field B is taken to be 1 gauss and we are assuming $am = 1$.

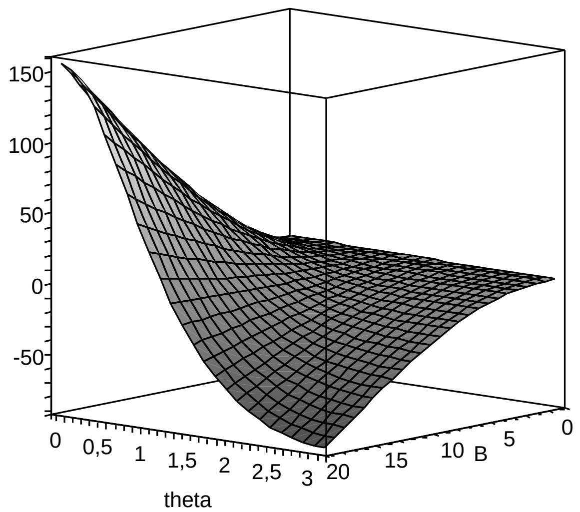

Fig. 4. Casimir pressure *versus* the magnetic field B and the θ parameter as given by Eq. (39). B varying from 0.01 to 20 gauss and the θ parameter varying from 0 to π. The separation between the plates was taken as $a = 1\mu m$.

Starting from expression (39) we can study the weak field and zero mass limits as well as the opposite situation, that is, the strong field limit for $am \ll 1$ and $am \gg 1$. We start with the weak field limit, where

$$
\begin{aligned}
&\frac{eB}{4\pi^2 a^2}\left(1+\frac{2\,(am)^2}{\sigma}\right) L\left(\frac{eBa^2}{\sigma}\right)\\
&=\left(1+\frac{2\,(am)^2}{\sigma}\right)\frac{\sigma}{4\pi^2 a^4}\left(\frac{\cosh\left(\frac{eBa^2}{\sigma}\right)}{\sinh\left(\frac{eBa^2}{\sigma}\right)\Big/\left(\frac{eBa^2}{\sigma}\right)}-1\right)\to 0\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{eB}{4\pi^2 a^2}\left(\left(\frac{2eBa^2}{\sigma}\right)\operatorname{csch}^2\left(\frac{eBa^2}{\sigma}\right)-\left(\frac{eBa^2}{2\sigma}\right)^{-1}\right)\\
&=\frac{\sigma}{2\pi^2 a^5}\left\{\frac{1}{\sinh^2\left(\frac{eBa^2}{\sigma}\right)\Big/\left(\frac{eBa^2}{\sigma}\right)^2}-1\right\}\to 0\,.
\end{aligned}
$$

In this limit the Casimir pressure becomes

$$
P\,(a,m) \;=\; \frac{2\,(am)^2}{\pi^2 a^4}\sum_{r=1}^{\infty}\frac{\cos\,(r\theta)}{r^2}\left(3K_2\,(amr)+(amr)\,K_1\,(amr)\right). \tag{40}
$$

We can immediately write the form for Eq. (40) when $m\to\infty$

$$
\begin{aligned}
P\,(a,m\to\infty) \;=\; &\frac{6(am)^2}{\pi^2 a^4}\sum_{r=1}^{\infty}\sqrt{\frac{\pi}{2amr}}\cos\,(r\theta)\left[\frac{1}{r^2}+\frac{45}{8amr^3}+\frac{am}{3r}\right]\\
&\times\exp\,(-amr)\,,
\end{aligned} \tag{41}
$$

where we take the asymptotic form for Bessel functions given by [19]

$$
\lim_{z\to\infty} K_\nu\,(z)\sim\sqrt{\frac{\pi}{2z}}\exp\,(-z)\left[\sum_{k=0}^{n-1}(2z)^{-k}\,\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\Gamma\left(\nu-k+\frac{1}{2}\right)}\right]. \tag{42}
$$

Now, we can obtain the expression for P in the zero mass limit. To reach this we need the expression of $K_n\,(z)$ for small arguments [19]

$$
K_n\,(z)|_{z\to 0}\sim\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\,\frac{(n-k-1)!}{k!\,(z/2)^{n-2k}}\,.
$$

So we get

$$
K_2\,(amr)|_{m\to 0}\sim\frac{1}{2}\left(\frac{1}{\left(\frac{amr}{2}\right)^2}-\frac{1}{\left(\frac{amr}{2}\right)}\right)\quad\text{and}\quad K_1\,(amr)|_{m\to 0}\sim\frac{1}{amr}\,.
$$

Using these results we obtain the equation (40) in the form

$$P\left(a,m\right) \;=\; \sum_{r=1}^{\infty}\left[\frac{12}{\pi^{2}a^{4}}\frac{\cos\left(r\theta\right)}{r^{4}}-\frac{6am}{\pi^{2}a^{4}}\frac{\cos\left(r\theta\right)}{r^{3}}+\frac{2\left(am\right)^{2}}{\pi^{2}a^{4}}\frac{\cos\left(r\theta\right)}{r^{2}}\right]\,, \tag{43}$$

and taking the limit $m \to 0$ we have

$$\lim_{m\to 0} P\left(a,m\right) \;\equiv\; P\left(a\right) = \frac{12}{\pi^{2}a^{4}}\sum_{r=1}^{\infty}\frac{\cos\left(r\theta\right)}{r^{4}}\,. \tag{44}$$

In the particular case for $\theta = \pi$, *i.e.*, $\cos\left(r\theta\right) = \left(-1\right)^{r}$, the Eq. (44) leads to

$$P\left(a\right) = \frac{12}{\pi^{2}a^{4}}\sum_{r=1}^{\infty}\frac{\left(-1\right)^{r}}{r^{4}} = \frac{12}{\pi^{2}a^{4}}\left(1-\frac{1}{8}\right)\zeta\left(4\right) = \xi\frac{\pi^{2}}{240a^{4}}\,,$$

which is the usual result [4] with the factor $\xi = (7\times 4)$. Now we take again the Eq. (39) to analyze the strong field limit. In this situation the second term in (39) is dominated by the exponential function with its maximum value $\exp\left(-amr\right)$ at $\sigma = 2m/r$. Further, we obtain

$$\lim_{B\to\infty} L\left(\frac{eBa^{2}}{\sigma}\right) \;=\; \lim_{B\to\infty}\left[\coth\left(\frac{eBa^{2}}{\sigma}\right)-\left(\frac{eBa^{2}}{\sigma}\right)^{-1}\right] = 1-\left(\frac{eBa^{2}}{\sigma}\right)^{-1},$$

and

$$\lim_{B\to\infty}\left(\frac{2\left(eBa\right)^{2}}{\sigma}\right)\operatorname{csch}^{2}\left(\frac{eBa^{2}}{\sigma}\right) = 0\,,$$

and carrying out the change of variable $\eta = r^{2}\sigma/4$ the second term becomes

$$\frac{eB}{\pi^{2}a^{2}}\sum_{r=1}^{\infty}\frac{\cos\left(r\theta\right)}{r^{2}}\int_{0}^{\infty}d\eta\left(1+\frac{\left(amr\right)^{2}}{2\eta}\right)\exp\left(-\eta-\frac{\left(amr\right)^{2}}{4\eta}\right)$$
$$-\frac{1}{\pi^{2}a^{4}}\sum_{r=1}^{\infty}\frac{\cos\left(r\theta\right)}{r^{2}}\int_{0}^{\infty}d\eta\left(2\left(am\right)^{2}+\frac{12\eta}{r^{2}}\right)\exp\left(-\eta-\frac{\left(amr\right)^{2}}{4\eta}\right)\,. \tag{45}$$

Taking into account the integral representation for the modified Bessel function of second kind (see formula (8.432−6) in [19]) and noting that $K_{-\nu}\left(z\right) = K_{\nu}\left(z\right)$, we can rewrite (45) as

$$\frac{eB}{\pi^{2}a^{2}}\sum_{r=1}^{\infty}\frac{\cos\left(r\theta\right)}{r^{2}}\Big(\left(amr\right)K_{1}\left(amr\right)+\left(amr\right)^{2}K_{0}\left(amr\right)\Big)$$
$$-\frac{2\left(am\right)^{2}}{\pi^{2}a^{4}}\sum_{r=1}^{\infty}\frac{\cos\left(r\theta\right)}{r^{2}}\left(3K_{2}\left(amr\right)+\left(am\right)K_{1}\left(amr\right)\right)\,. \tag{46}$$

The second term of Eq. (46) cancels the first terms of (39), so we are left with the following expression in the strong field limit

$$P(a,m,B)=\frac{eBm}{\pi^2 a}\sum_{r=1}^{\infty}\frac{\cos(r\theta)}{r}\left(K_1(amr)+(amr)K_0(amr)\right). \tag{47}$$

If we consider the relation (8.486.17) in [19], the Eq. (47) takes the form

$$P(a,m,B)=\frac{eBm}{\pi^2 a}\sum_{r=1}^{\infty}\frac{\cos(r\theta)}{r}\left((amr)K_2(amr)-K_1(amr)\right). \tag{48}$$

For $\theta=\pi$ (antiperiodic case), Eq. (48) becomes

$$P(a,m,B) = \frac{eBm}{\pi^2 a}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\left((amr)K_2(amr)-K_1(amr)\right). \tag{49}$$

We come back to Eq. (48) to evaluate its limits for $am\ll 1$ and $am\gg 1$ in the situation of strong magnetic fields. Following [4], the strong magnetic field regime in these cases is described, respectively, by $|B|\gg\left(\phi_0/a^2\right)$ and $|B|\ll\left(\phi_0/a^2\right)(2\pi a/\lambda_c)$. Firstly, we consider the limit $am\ll 1$ into the Eq. (48), so the Bessel functions reduce to

$$K_1(amr)\sim(amr)^{-1}\quad\text{and}\quad K_2(amr)\sim 2(amr)^{-2}-(amr)^{-1},$$

and using the relation (42) we obtain

$$P(a,m,B)=\frac{eB}{\pi^2 a^2}\sum_{r=1}^{\infty}\frac{\cos(r\theta)}{r^2}-\frac{eBm}{\pi^2 a}\sum_{r=1}^{\infty}\frac{\cos(r\theta)}{r}. \tag{50}$$

Now we analyze the second limit ($am\gg 1$) and by taking

$$K_1(amr)=K_2(amr)\sim\sqrt{\frac{\pi}{2amr}}\exp(-amr),$$

the Casimir pressure in this situation becomes

$$\begin{aligned}P(a,m,B) \sim\ & eB\sqrt{\frac{m^3}{2a\pi^3}}\sum_{r=1}^{\infty}\frac{\cos(r\theta)}{r^{1/2}}\exp(-amr)\\ & -eB\left(\frac{m}{2\pi^3 a^3}\right)^{1/2}\sum_{r=1}^{\infty}\frac{\cos(r\theta)}{r^{3/2}}\exp(-amr). \end{aligned}\tag{51}$$

The particular forms for equations (50) and (51), when $\theta = \pi$, are given by

$$P\left(a,m,B\right) \;=\; -\frac{eB}{2\pi^2 a^2}\zeta\left(2\right) - \frac{eBm}{\pi^2 a}\zeta\left(1\right) \,, \tag{52}$$

and

$$P\left(a,m,B\right) \;\sim\; -2eBa\left(\frac{m}{2\pi a}\right)^{3/2}\exp\left(-amr\right) \,, \tag{53}$$

respectively. So, we can conclude that the uniform magnetic field leads to the enhancement both of the Casimir density energy and the Casimir pressure. Nevertheless, the results are still far from detectable in the laboratory for typical values of the order of parameter am.

4. Conclusions

We have calculated the Casimir energy for a fermionic field under the influence of an uniform magnetic field using the generalized zeta function method and considering a generalized boundary condition which interpolates continuously periodic and antiperiodic conditions. This procedure allows us to get results with a minimum of spurious terms. Also, the use of a method other than the mode summation and Schwinger's proper time is another way to confirm the results found in the literature [4, 7, 8].

Next, we consider the Casimir energy expression in order to evaluate the corresponding pressure and by doing this we determined the weak field as well as strong field limits. Using the MAPLE® program, we have been able to show a graphical sketch of the Casimir pressure behavior in Eq. (39) as a function of the magnetic field B and the θ parameter. In particular, we exhibited the behavior for the part dependent of magnetic field when the θ parameter takes the values 0 and π, respectively. In particular, the graphical sketch of the antiperiodic case shows that the Casimir pressure is enhanced by the presence of the external magnetic field.

As a next step in this development we consider, the possibility of re-obtaining the results in [9] for the Casimir energy for the Dirac field under MIT boundary conditions by using the generalized zeta function method.

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