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ACCELERATING QUANTUM UNIVERSE

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The origin of negative pressure fluid (the dark energy) is investigated in the quantum model of the homogeneous, isotropic and closed universe filled with a uniform scalar field and a perfect fluid which defines a reference frame. The equations of the model are reduced to the form which allows a direct comparison between them and the equations of the Einsteinian classical theory of gravity. It is shown that quantized scalar field has a form of a condensate which behaves as an anti-gravitating medium. The theory predicts an accelerating expansion of the universe even if the vacuum energy density vanishes. An anti-gravitating effect of a condensate has a purely quantum nature. It is shown that the universe with the parameters close to the Planck ones can go through the period of exponential expansion. The conditions under which in semi-classical approximation the universe looks effectively like spatially flat with negative deceleration parameter are determined. The reduction to the standard model of classical cosmology is discussed.

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1. Introduction

An accelerating expansion of the present-day Universe which was discovered in the late 90th [1] and confirmed in subsequent type Ia supernova observations [2,3] gives an evidence that some mysterious component of the matter-energy (called the dark energy [4]) exists in the Universe. It behaves as an anti-gravitating medium (fluid). In standard ΛCDM model [5–7] this fluid is introduced from the beginning in the form of the cosmological constant which is identified with the vacuum energy density. Such a phenomenological approach gives a good description of the observational data, but it generates the cosmological constant problem [7]. Nowadays there are several models of the dark energy based on different reasonable concepts [8–12]. It is quite possible that the origin of the dark energy is connected with quantum processes in the early Universe.

As is well known quantum theory adequately describes properties of various physical systems. Its universal validity demands that the Universe as a whole must obey quantum laws as well. Since quantum effects are not *a priori* restricted to certain scales [13], then one should not conclude in advance, without research into the properties of the Universe in the theory more general than classical cosmology, that they cannot have any impact on processes at large scales (the motivation to develop quantum cosmology see *e.g.* in [14–16]). One can expect that in semi-classical limit negative pressure fluid ought to arise as the remnant of the early quantum epoch.

Under construction of quantum theory of gravity one can proceed from the idea that properties of any quantum system can be described on the basis of the solution of some partial differential equation. If a Lagrangian is given then equations themselves can be derived from the principle of least action. Passing from the Lagrangian formalism to the Hamiltonian one it is possible, in principle, to construct consecutive quantum theory of gravity using the method of canonical quantization. The first problem on this way is to choose generalized variables. Following *e.g.* Refs. [17–19], it is convenient to choose metric tensor components and matter fields as such variables. But the functional equations [18, 19] obtained in this approach prove to be insufficiently suitable for specific problems of quantum theory of gravity and cosmology. These equations do not contain a time variable in an explicit form. This, in turn, gives rise to the problem of interpretation of the wave function of the universe (see *e.g.* discussion in Ref. [5] and references therein). A cause of the failure can be easily understood with the help of Dirac's constraint system theory [20]. It has been found that the structure of constraints in general relativity is such that variables which correspond to true dynamical degrees of freedom cannot be singled out from canonical variables of geometrodynamics. This difficulty is stipulated by an absence of predetermined way to identify spacetime events in generally covariant theory [21].

One way of solving the conceptual problems of theory of gravity is connected with the introduction of material reference systems [21–23]. A perfect fluid can determine a reference frame during consideration of different model systems in classical theory [24–27] and may appear useful after the passage to the quantum case as well. Choosing the relativistic matter as the simplest physical realization of a perfect fluid one can obtain the appropriate equations of quantum geometrodynamics for the minisuperspace model [28] which allow to study the properties of the early Universe and to pass to general relativity in the semi-classical limit.

In this paper the problem of an accelerating expansion of the universe is analyzed in the framework of the exactly solvable quantum model. We consider the homogeneous, isotropic and closed universe filled with a uniform scalar field and the relativistic matter which defines a reference frame.

In Sec. 2 the basic concepts and the equations of the quantum model of the universe obtained in Ref. [28] are introduced. In Sec. 3 the canonical equations which determine the time change of the scale factor, scalar field and the operators of the momenta canonically conjugate with these two variables are obtained. In Sec. 4 the quantum analogues of the Einstein–Friedmann equations are given and the conclusion is drawn about an anti-gravitating effect of a condensate of excitation quanta of oscillations of a primordial scalar field about an equilibrium state. In Sec. 5 the semi-classical limit is considered and the Einstein–Friedmann equations are obtained. In Sec. 6 the properties of the universe with large masses of a condensate in the state of the true vacuum of scalar field are studied. The conditions under which the universe can undergo the phase of exponential expansion (inflation) and look effectively like spatially flat with negative deceleration parameter are established. Section 7 presents some concluding remarks.

In this paper we use the modified Planck system of units in which $l_{\mathrm{P}} = \sqrt{2G\hbar/(3\pi c^3)}$ is taken as a unit of length, $\rho_{\mathrm{P}} = 3c^4/(8\pi G l_{\mathrm{P}}^2)$ is a unit of energy density and so on. All relations are written for dimensionless values.

2. Equations of quantum model of the universe

Let us consider the homogeneous, isotropic and closed universe which is described by the Robertson–Walker metric

$$ds^2 = a^2(\eta)\,[N^2(\eta) d\eta^2 - d\Omega^2]\,, \tag{1}$$

where a is the cosmic scale factor, N is the lapse function that specifies the time reference scale, $d\Omega^2$ is an interval element on a unit three-sphere, η is the time variable (conformal time at $N = 1$). Let us suppose that the universe is filled with the uniform scalar field ϕ with the potential energy density (potential) $V(\phi)$ and a perfect fluid which defines material reference frame. The Hamiltonian H of such a system has the form of a linear combination of constraints [28] and weakly vanishes (in Dirac's since [20]),

$$\begin{aligned} H \;=\; & \frac{N}{2}\left\{-\,\pi_a^2 - a^2 + \frac{2}{a^2}\,\pi_\phi^2 + a^4[V(\phi) + \rho]\right\} \\ & + \lambda_1\left\{\pi_\Theta - \frac{1}{2}\,a^3\rho_0 s\right\} + \lambda_2\left\{\pi_{\tilde\lambda} + \frac{1}{2}\,a^3\rho_0\right\} \approx 0\,, \end{aligned} \tag{2}$$

where $\rho = \rho(\rho_0, s)$ is the energy density of a perfect fluid, ρ_0 is the density of the rest mass, s is the specific entropy. The Θ is the thermasy (potential for the temperature, $T = \Theta_{,\nu}U^\nu$). The $\tilde\lambda$ is the potential for the specific Gibbs free energy f taken with an inverse sign, $f = -\,\tilde\lambda_{,\nu}U^\nu$. The U^ν is the four-velocity. The π_a, π_ϕ, π_Θ, $\pi_{\tilde\lambda}$ are the momenta canonically conjugate with the variables a, ϕ, Θ, $\tilde{\lambda}$, respectively. The momenta π_{ρ_0} and π_s conjugate with the variables ρ_0 and s vanish identically. The N, λ_1, and λ_2 are Lagrange multipliers.

From the conservation of primary constraints in time it follows the conservation laws

$$E_0 \equiv a^3 \rho_0 = \text{const.}\,, \qquad s = \text{const.}\,, \tag{3}$$

where the first relation describes the conservation law of a macroscopic value which characterizes the number of particles. For example, if a perfect fluid is composed of baryons, then this condition reflects the conservation of baryon number. The second equation in (3) represents the conservation of the specific entropy.

Taking into account these conservation laws and vanishing of the momenta π_{ρ_0} and π_s, one can discard degrees of freedom corresponding to the variables ρ_0 and s, and convert the second-class constraints into first-class constraints in accordance with Dirac's proposal [20]. We have used the same approach in Ref. [28]. In quantum theory first-class constraint equations become constraints on the wave function Ψ.

It is convenient to pass from the generalized variables Θ and $\widetilde{\lambda}$ to the non-coordinate co-frame

$$\begin{aligned} h d\tau &= s\, d\Theta - d\widetilde{\lambda}\,, \\ h dy &= s\, d\Theta + d\widetilde{\lambda}\,, \end{aligned} \tag{4}$$

where $h = (\rho + p)/\rho_0$ is the specific enthalpy which plays the role of inertial mass [24], p is the pressure, τ is proper time in every point of space. The corresponding derivatives commute between themselves,

$$[\partial_\tau, \partial_y] = 0\,.$$

Thus the quantum universe in which a perfect fluid is taken in the form of relativistic matter is described by the equations [28]

$$\left\{ - i\, \partial_{\tau_c} - \frac{1}{2}\, E_0 \right\} \Psi = 0\,, \tag{5}$$

$$\left\{ - \partial_a^2 + \frac{2}{a^2}\, \partial_\phi^2 + a^2 - a^4 V(\phi) - E \right\} \Psi = 0\,, \tag{6}$$

where Ψ does not depend on the variable y, $E \equiv a^4 \rho = \text{const}$, ρ is the energy density of relativistic matter. The value τ_c is the time variable connected with the proper time τ by the differential relation, $d\tau_c = h\, d\tau$. Eq. (5) has a particular solution in the form

$$\Psi = e^{iE\bar{\tau}} |\psi\rangle\,, \tag{7}$$

where we have changed the time scale using the relation $E\bar{\tau} = \frac{1}{2}E_0\tau_c$. The vector $|\psi\rangle$ is defined in the space of two variables a and ϕ, and determined by Eq. (6) which we rewrite in the form

$$\left(\hat{\pi}_a^2 + a^2 - 2\,a\,\hat{H}_\phi - E\right)|\psi\rangle = 0\,, \tag{8}$$

where $\hat{\pi}_a = -\,i\,\partial_a$ is the operator of the momentum canonically conjugate with the variable a,

$$\hat{H}_\phi = \frac{1}{2}\,a^3\hat{\rho}_\phi \tag{9}$$

is the operator of mass-energy of a scalar field in a comoving volume $\frac{1}{2}a^3$. Here

$$\hat{\rho}_\phi = \frac{2}{a^6}\,\hat{\pi}_\phi^2 + V(\phi) \tag{10}$$

is the operator of the energy density of a scalar field, $\hat{\pi}_\phi = -i\,\partial_\phi$ is the operator of the momentum canonically conjugate with the variable ϕ.

Eqs. (7) and (8) are equivalent[1] to the Schrödinger-type equation which was obtained in Refs. [29,30] within the bounds of the scheme [21] for incorporating of a reference systems in general relativity through the imposition of coordinate conditions before variation of the action.

Eq. (8) can be integrated with respect to ϕ, if one determines the form of the potential $V(\phi)$. As in Ref. [28] we consider the solution of Eq. (8) when the field ϕ is near its minimum at the point $\phi = \sigma$. Then $V(\phi)$ can be approximated by the expression

$$V(\phi) = \rho_\sigma + \frac{m_\sigma^2}{2}\,(\phi - \sigma)^2, \tag{11}$$

where $\rho_\sigma = V(\sigma)$, $m_\sigma^2 = [d^2V(\phi)/d\phi^2]_\sigma > 0$. If $\phi = \sigma$ is the point of absolute minimum, then $\rho_\sigma = 0$ and the state σ corresponds to the true vacuum of a primordial scalar field, while the state with $\rho_\sigma \neq 0$ matches with the false vacuum [31].

Introducing the new variable

$$x = \left(\frac{m_\sigma a^3}{2}\right)^{1/2}(\phi - \sigma)\,, \tag{12}$$

which describes a deviation of the field ϕ from its equilibrium state, we find that

$$\hat{H}_\phi|u_k\rangle = \left(M_k + \frac{1}{2}\,a^3\rho_\sigma\right)|u_k\rangle\,, \tag{13}$$

[1] These equations can be obtained even without an introduction of proper time τ by building a time variable from the matter variables (*cf.* [25–27]), *e.g.* considering the thermasy Θ as a time variable.

where $\langle x|u_k\rangle$ are the functions of harmonic oscillator with $k = 0, 1, 2, \ldots,$ and

$$M_k = m_\sigma \left(k + \tfrac{1}{2}\right) . \tag{14}$$

The quantity M_k describes an amount of matter (mass) in the universe related to a scalar field. This mass is represented in the form of a sum of masses of excitation quanta of the spatially coherent oscillations of the field ϕ about the equilibrium state σ, k is the number of quanta. The mentioned oscillations correspond to a condensate of zero-momentum ϕ quanta with the mass m_σ. If $\rho_\sigma \neq 0$, then the value $\frac{1}{2}\, a^3 \rho_\sigma$ is the energy of the false vacuum in the universe with the scale factor a.

We shall look for the solution of Eq. (8) in the form of the superposition of the states with different masses M_k

$$|\psi\rangle = \sum_k |f_k\rangle |u_k\rangle . \tag{15}$$

Using orthonormality of the states $|u_k\rangle$ we obtain the equation for the vector $|f_k\rangle$

$$\left(\hat{\pi}_a^2 + a^2 - 2aM_k - a^4\rho_\sigma - E\right) |f_k\rangle = 0 . \tag{16}$$

In the case $\rho_\sigma = 0$ this equation is exactly integrable [28]. At the same time the state Ψ has a finite norm, $\langle\Psi|\Psi\rangle < \infty$ (*cf. e.g.* Ref. [21]).

3. Canonical equations

The equation of motion for an arbitrary function $\mathcal{O}$ of the variables a, ϕ, π_a and π_ϕ has the form

$$\frac{d\mathcal{O}}{d\eta} \approx \{\mathcal{O}, H\} , \tag{17}$$

where H is the Hamiltonian (2), the sign $\approx$ means that Poisson brackets $\{\ldots , \ldots\}$ must all be worked out before the use of the constraint equations. In quantum theory the equation of motion for an operator $\hat{\mathcal{O}}$ in the Heisenberg representation takes the form

$$\frac{d\hat{\mathcal{O}}}{d\eta} \approx \frac{1}{i}[\hat{\mathcal{O}}, \hat{H}] , \tag{18}$$

where $[\ldots , \ldots]$ is a commutator, and $\hat{H}$ is determined by the expression (2), in which all dynamical variables are substituted with operators.

Let $\hat{\mathcal{O}} = a$, then from Eq. (18) we obtain

$$\frac{d\hat{a}}{d\eta} = -\, N\, \hat{\pi}_a . \tag{19}$$

This relation can be considered as a definition of the momentum operator $\hat{\pi}_a$. Let $\hat{\mathcal{O}} = \hat{\pi}_a$. Then from Eq. (18) it follows that

$$\frac{1}{N}\frac{d\hat{\pi}_a}{d\eta} = \frac{2}{a^3}\hat{\pi}_\phi^2 + a - 2\,a^3 V(\phi)\,. \tag{20}$$

If $\hat{\mathcal{O}} = \phi$, then

$$\frac{1}{N}\frac{d\phi}{d\eta} = \frac{2}{a^2}\hat{\pi}_\phi\,. \tag{21}$$

If $\hat{\mathcal{O}} = \hat{\pi}_\phi$, then

$$\frac{1}{N}\frac{d\hat{\pi}_\phi}{d\eta} = -\frac{a^4}{2}\frac{dV(\phi)}{d\phi}\,. \tag{22}$$

4. Quantum analogues of the Einstein–Friedmann equations

Eq. (16) can be written in the form

$$\left\{\left(\frac{1}{a^2}\,\hat{\pi}_a\right)^2 - \rho_{\rm tot} + \frac{1}{a^2}\right\}|f_k\rangle = 0\,, \tag{23}$$

where the quantity

$$\rho_{\rm tot} = \rho_k\, +\, \rho\, +\, \rho_\sigma \tag{24}$$

with the components

$$\rho_k \equiv \frac{2M_k}{a^3}\,, \qquad \rho \equiv \frac{E}{a^4} \tag{25}$$

can be interpreted as a total energy density. It represents itself the sum of the energy densities of a condensate of ϕ quanta ρ_k, relativistic matter ρ and the false vacuum ρ_σ.

In order to obtain the second equation of quantum model we use the operator equation (20) rewritten in the equivalent form

$$\frac{1}{a^3 N}\frac{d\hat{\pi}_a}{d\eta} - \hat{\rho}_\phi + 3V(\phi) - \frac{1}{a^2} = 0\,. \tag{26}$$

Acting with this operator equation on $|\psi\rangle$, taking into account Eqs. (9), (11), (13), (15) and

$$\begin{aligned} V(\phi)|u_k\rangle \;=\; & \left(\frac{1}{a^3}M_k + \rho_\sigma\right)|u_k\rangle + \frac{1}{2a^3}\sqrt{\left(M_k - \frac{m_\sigma}{2}\right)\left(M_k - \frac{3m_\sigma}{2}\right)}|u_{k-2}\rangle \\ & + \frac{1}{2a^3}\sqrt{\left(M_k + \frac{m_\sigma}{2}\right)\left(M_k + \frac{3m_\sigma}{2}\right)}|u_{k+2}\rangle\,, \end{aligned} \tag{27}$$

we obtain

$$\left\{\frac{1}{a^3 N}\frac{d\hat{\pi}_a}{d\eta} + \frac{1}{2}\rho_k + 2\rho_\sigma - \frac{1}{a^2}\right\}|f_k\rangle + \sum_{k'}\mathcal{P}_{kk'}|f_{k'}\rangle = 0\,, \tag{28}$$

where the operator

$$\begin{aligned}\mathcal{P}_{kk'} &= \frac{3}{2a^3}\left\{\sqrt{\left(M_k + \frac{m_\sigma}{2}\right)\left(M_k + \frac{3m_\sigma}{2}\right)}\,\delta_{k',k+2}\right.\\ &\left.+\sqrt{\left(M_k - \frac{m_\sigma}{2}\right)\left(M_k - \frac{3m_\sigma}{2}\right)}\,\delta_{k',k-2}\right\}\end{aligned} \tag{29}$$

takes into account the quantum effects stipulated by the character of the wavefunction (15) which is the superposition of $|f_k\rangle$ corresponding to different masses of a condensate.

Eqs. (23) and (28) are the quantum analogues of the Einstein–Friedmann equations in general relativity. In the approximation $\mathcal{P}_{kk'} = 0$ Eq. (28) can be represented in the "standard" form for the perfect fluid source

$$\left\{\frac{1}{a^3 N}\frac{d\hat{\pi}_a}{d\eta} + \frac{1}{2}\left(\rho_{\text{tot}} - 3\,p'_{\text{tot}}\right) - \frac{1}{a^2}\right\}|f_k\rangle = 0\,, \tag{30}$$

where ρ_{tot} is the density (24), while

$$p'_{\text{tot}} = p'_k + p + p_\sigma \tag{31}$$

with the components

$$p'_k = 0\,, \qquad p = \frac{1}{3}\rho\,, \qquad p_\sigma = -\rho_\sigma\,, \tag{32}$$

is the isotropic pressure (here a dash marks the fact that the sum over k' in Eq. (28) is not taken into account). From Eq. (32) one can conclude that the constant component ρ_σ of the energy density (24) is described by the vacuum-type equation of state, while a condensate of ϕ quanta in the case under consideration is a perfect fluid with zero pressure. But taking into account the presence of operator $\mathcal{P}_{kk'}$ in Eq. (28) leads to the considerable modification of physical properties of a condensate.

Let us note the result (32) is non-trivial. Starting from a uniform scalar field, after quantization we obtain the pressure-less matter component (dust).

We shall study the role of $\mathcal{P}_{kk'} \neq 0$ in Eq. (28). In the limit of large values of k we have $M_k \gg \frac{3}{2}\, m_\sigma$, and the operator $\mathcal{P}_{kk'}$ takes the form

$$\mathcal{P}_{kk'} = \frac{3}{2}\rho_k \delta_{kk'} \qquad \text{at} \qquad k \gg 1\,. \tag{33}$$

Substituting Eq. (33) into (28) we obtain the equation

$$\left\{ \frac{1}{a^3 N} \frac{d\hat{\pi}_a}{d\eta} + \frac{1}{2}\left(\rho_{\text{tot}} - 3\, p_{\text{tot}}\right) - \frac{1}{a^2} \right\} |f_k\rangle = 0\,, \tag{34}$$

where the pressure is equal to

$$p_{\text{tot}} = p_k + p + p_\sigma\,, \tag{35}$$

and the equations of state for relativistic matter and vacuum are the same as in Eq. (32), but the equation of state of a condensate takes the form

$$p_k = -\rho_k\,. \tag{36}$$

This means that taking into account the quantum effects caused by the non-zero operator $\mathcal{P}_{kk'}$ at large values of masses M_k leads to the fact that a condensate of ϕ quanta obtains the property of an anti-gravitating medium. As a result the quantum universe will expand in an accelerating mode at $\rho_k > \rho$ even in the state of true vacuum of the field ϕ ($\rho_\sigma = 0$).

5. Semi-classical limit

We shall take the wave function $\langle a|f_k\rangle$ in the form

$$\langle a|f_k\rangle = A(a) e^{iS(a)}\,, \tag{37}$$

where A and S are some real functions of a and the index k is omitted for simplicity. Substituting Eq. (37) into (23) we obtain the equivalent equation

$$\frac{1}{a^4}\left(\partial_a S\right)^2 - \rho_{\text{tot}} - \frac{1}{a^4}\frac{\partial_a^2 A}{A} + \frac{1}{a^2} - \frac{i}{a^4 A^2}\,\partial_a\left(A^2 \partial_a S\right) = 0\,. \tag{38}$$

The imaginary part gives the continuity equation. Eq. (38) written in the ordinary units takes the form

$$\frac{1}{a^4}\left(\partial_a S\right)^2 - \frac{8\pi G}{3c^4}\rho_{\text{tot}} - \left(\frac{2G\hbar}{3\pi c^3}\right)^2 \frac{1}{a^4}\frac{\partial_a^2 A}{A} + \frac{1}{a^2} - i\frac{2G\hbar}{3\pi c^3}\frac{1}{a^4 A^2}\partial_a\left(A^2 \partial_a S\right) = 0\,. \tag{39}$$

Neglecting the terms which are proportional to $\hbar$ and $\hbar^2$ we get the classical Hamilton–Jacobi equation for the action S. The classical momentum π_a is equal to

$$\pi_a = \partial_a S = -a\,\frac{da}{d\tau}\,, \tag{40}$$

where $d\tau = aN d\eta$, τ is proper time.

Substituting Eq. (37) into (34) and taking into account the rule for the derivative of an operator $\mathcal{O}$,

$$\left\langle f_k \left| \frac{1}{N}\,\frac{d\mathcal{O}}{d\eta} \right| f_k \right\rangle = \frac{d}{N d\eta}\,\langle f_k | \mathcal{O} | f_k \rangle\,,$$

we obtain

$$\frac{1}{a^3 N}\,\frac{d}{d\eta}\,(\partial_a S) + \frac{1}{2}\,(\rho_{\text{tot}} - 3p_{\text{tot}}) - \frac{1}{a^2} - \frac{i}{a^3 N}\,\frac{d}{d\eta}\left(\frac{\partial_a A}{A}\right) = 0\,. \tag{41}$$

The imaginary part of this equation taken in the ordinary units is proportional to $\hbar$.

Omitting terms proportional to $\hbar$ and $\hbar^2$ in Eqs. (38) and (41) and taking into account Eq. (40), we obtain the Einstein–Friedmann equations

$$\left(\frac{1}{a}\,\frac{da}{d\tau}\right)^2 = \rho_{\text{tot}} - \frac{1}{a^2}\,, \qquad \frac{1}{a}\,\frac{d^2 a}{d\tau^2} = -\frac{1}{2}\,(\rho_{\text{tot}} + 3p_{\text{tot}})\,, \tag{42}$$

where the total energy density ρ_{tot} and the pressure p_{tot} are determined in Eqs. (24) and (35) with the equation of state of a condensate (36). It means that the conclusion about an anti-gravitating effect of a condensate made above remains valid after a passage to the classical limit as well. Let us note that the second equation is obtained from Eq. (34) and it contains the contribution from the operator $\mathcal{P}_{kk'}$ (33). If the quantum effects determined by this operator had been omitted, the equation of state of a condensate would have the form $p_k = 0$ as for a dust.

From the equations (42) it follows the equation for the evolution of ρ_{tot},

$$\frac{d\rho_{\text{tot}}}{d\tau} = -\,4\left(\frac{1}{a}\,\frac{da}{d\tau}\right)\rho\,. \tag{43}$$

In classical theory the value $E = a^4\rho$ is a positive constant which is determined by the values of a and ρ at a certain instant of time. In quantum theory it is quantized in accordance with Eq. (16). We shall study the effect of quantization of E on the properties of the universe.

6. Universe in the state of true vacuum of scalar field

Let us consider the exactly solvable model with $\rho_\sigma = 0$ and large masses of a condensate, $M_k \gg 1$. Eq. (23) is strictly equivalent to the stationary equation (16). The state vector $|f_k\rangle$ is characterized by the additional index n which numbers the states of the universe in the effective potential well

$$U(a) = a^2 - 2aM_k\,. \tag{44}$$

Solving Eq. (16) for the case under consideration with the boundary conditions $\langle 0|f_{n,k}\rangle \neq 0$[2] and $\langle a|f_{n,k}\rangle \to 0$ at $a \to \infty$ we find the state vectors in a-representation [28]

$$\langle a|f_{n,k}\rangle = N_{n,k}\, e^{-\frac{1}{2}(a-M_k)^2}\, H_n(a - M_k)\,, \tag{45}$$

where $H_n(\xi)$ are Hermitian polynomials, $n = 0, 1, 2, \ldots$, and

$$N_{n,k} = \Bigg\{ 2^{n-1} n! \sqrt{\pi}\,(\mathrm{erf} M_k + 1) - e^{-M_k^2} \sum_{l=0}^{n-1} \frac{2^l n!}{(n-l)!}\, H_{n-l}(M_k)\, H_{n-l-1}(M_k) \Bigg\}^{-\frac{1}{2}} \tag{46}$$

is the normalization factor, $\mathrm{erf} M_k$ is the probability integral. These state vectors correspond to the eigenvalues

$$E = 2n + 1 - M_k^2\,. \tag{47}$$

The quantum-mechanical mean value

$$\langle a\rangle_{n,k} = \langle f_{n,k}|a|f_{n,k}\rangle \tag{48}$$

in the state (45) is equal to

$$\langle a\rangle_{n,k} = M_k + \langle \xi\rangle_{n,k}\,, \tag{49}$$

[2] The choice of boundary conditions at the point $a = 0$ should be stipulated by the physical properties of the system under study. In classical cosmology the universe near the initial cosmological singularity point $a = 0$ is characterized by nontrivial values of energy density. Since $|\langle 0|f_{n,k}\rangle|^2$ is the particle number density at $a = 0$, then the choice of such a boundary condition is justified from the cosmological point of view.

where

$$\langle\xi\rangle_{n,k} = \int_{-M_k}^{\infty} d\xi\, \xi\, \langle \xi + M_k | f_{n,k}\rangle^2$$

$$= N_{n,k}^2\, 2^{n-1} n!\, e^{-M_k^2} \left\{ 1 + \sum_{l=0}^{n-1} \frac{2^{l-n}}{(n-l)!}\, H_{n-l}(M_k)\, H_{n-l-1}(M_k) \right\} \quad (50)$$

is the correction which can be neglected in the case $M_k \gg 1$.

In Ref. [28] it was shown that the quantum universe can nucleate from the initial cosmological singularity point $a = 0$ with the non-zero probability. The universe being nucleated in the ground ($n = 0$) state has the Planck parameters, $M_k = 1$ and $\langle a\rangle_{0,k} = 1.11$. In this state $E = 0$ and according to Eq. (43) $\rho_{\mathrm{tot}} = \mathrm{const}$. After integrating the second equation in (42) we find that the universe in such a state will expand exponentially

$$a(\tau) = a(0) \exp\left\{\sqrt{\rho_{\mathrm{tot}}}\,\tau\right\} . \quad (51)$$

This exponential expansion lasts as long as the condition $E = 0$ is satisfied. In the models of inflation this time is about $10^{-32} - 10^{-35}$ sec [5].

Let the condition $E = 0$ be realized at some instant $\tau = \tau_2$ of the evolution of the early universe. The mass of a condensate and the scale factor here take the values M_{k_2} and a_2, respectively, and the total energy density is equal to

$$\rho_{\mathrm{tot}} = \rho_{k_2} = \frac{2M_{k_2}}{a_2^3} .$$

Introducing the Hubble expansion rate H, the deceleration parameter q and the density parameters Ω_{tot} and Ω_k in chosen system of units at some instant $\tau = \tau_i$ we have

$$H_i = \left(\frac{1}{a}\frac{da}{d\tau}\right)_i , \quad q_i = -\frac{1}{H_i^2}\left(\frac{1}{a}\frac{d^2 a}{d\tau^2}\right)_i , \quad \Omega_{\mathrm{tot}} = \frac{\rho_{\mathrm{tot}}}{H_i^2} , \quad \Omega_{k_i} = \frac{\rho_{k_i}}{H_i^2} . \quad (52)$$

Then from (42) we obtain

$$\Omega_{\mathrm{tot}} = 1 , \qquad q_2 = -1 , \quad (53)$$

where we take into account that under consideration of the early universe the curvature term a_2^{-2} can be dropped. Hence the early universe is accelerating.

In the quantum model under study the evolution of the universe is described as transitions with the non-zero probabilities between the states of the universe with different masses of a condensate [28]. An increase in this mass leads to an expansion of the universe. Breaking the condition $E = 0$ results in change from exponential expansion to power law.

Let us consider the case $E \neq 0$ and $M_k \gg 1$. We choose the instant of time τ_1 for which the scale factor $a(\tau_1) \equiv a_1$ satisfies the condition $a_1 = M_{k_1}$[3]. Then taking into account (47) from (42) we obtain

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2_1 = \rho_{n_1}\,, \qquad \left(\frac{1}{a}\frac{d^2a}{d\tau^2}\right)_1 = \tfrac{3}{2}\rho_{k_1} - \rho_{n_1}\,, \tag{54}$$

where

$$\rho_{n_1} = \frac{2n+1}{a_1^4}\,, \qquad \rho_{k_1} = \frac{2}{a_1^2}\,. \tag{55}$$

These relations can be interpreted as the Einstein–Friedmann equations (at fixed instant τ_1) for the spatially flat universe with the total energy density $\rho_{\rm tot} = \rho_{n_1}$ and the pressure $p_{\rm tot} = p_{k_1} + p_{n_1}$, where $p_{k_1} = -\rho_{k_1}$, $p_{n_1} = \frac{1}{3}\rho_{n_1}$[4].

If we set $\mathcal{P}_{kk'} = 0$, the second equation reduces to

$$\left(\frac{1}{a}\,\frac{d^2a}{d\tau^2}\right)_1 = -\rho_{n_1}\,. \tag{56}$$

It implies that in the classical limit with $\rho_\sigma = 0$, which does not take into account the quantum effects caused by $\mathcal{P}_{kk'}$, an expansion of the universe will be decelerating at any instant of time for which $a_1 = M_{k_1}$, while quantum model predicts a possibility of an accelerating expansion at $(3/2)\,\rho_{k_1} > \rho_{n_1}$ even when $\rho_\sigma = 0$.

The relations (54) can be rewritten as

$$\Omega_{\rm tot} = 1\,, \qquad q_1 = 1 - \tfrac{3}{2}\Omega_{k_1}\,. \tag{57}$$

In the epoch when $\Omega_{k_1} > 2/3$ the expansion of the universe will be accelerating due to anti-gravitating effect of a condensate.

According to Eqs. (54) and (55) the instant $\tau = \tau_1$ can be associated with the radiation-dominated epoch. In order to consider the matter-dominated era the additional conjectures about the production of ordinary matter from a condensate are required.

[3] The equations (42) are equivalent to the Hamilton equations of classical cosmology. They do not take into account the dispersions of a and the conjugate momentum π_a. According to the Ehrenfest theorem one can replace in (42) the classical value of the scale factor a with the mean value (49) at every instant of time.

[4] In this case the term with curvature $-a^{-2}$ in (42) cannot be distinguished from the term ρ_k. Effectively the curvature makes a kind of renormalization of density of a condensate whose action is totally compensated in the first relation (54). But the second relation contains a trail from a condensate with vacuum-type equation of state.

7. Concluding remarks

This quantum model of the universe can be reduced to the standard model of classical cosmology in the limit of large quantum numbers. We suppose [32] that ordinary matter is produced in the decay of ϕ quanta to dark matter particles, baryons, leptons or to their antiparticles. Particles and antiparticles can annihilate between themselves and contribute to the observed cosmic microwave background radiation with excess of γ quanta over matter (at a ratio $\eta = n_{\mathrm{B}}/n_\gamma \sim 10^{-10}$ in our Universe). The part of a condensate which does not decay to the instant of observation τ_0 forms dark energy. Using the energy conservation law, one can write

$$M_{k_0} = M_m + M_\Lambda + M_\gamma + Q\,, \tag{58}$$

where M_m is the total mass of all baryons, leptons and dark matter, $M_\Lambda = \frac{1}{2}\,a_0^3 \rho_\Lambda$ is the mass of a condensate which does not decay in a comoving volume $\frac{1}{2}\,a_0^3$ with the energy density ρ_Λ and the equation of state $p_\Lambda = -\rho_\Lambda$, $M_\gamma = \frac{1}{2} E_\gamma a_0^{-1}$ is the mass of relativistic matter produced by decaying condensate, $E_\gamma = \mathrm{const}$, Q is the total kinetic energy of relative motion of decay products of ϕ quanta. From Eq. (58) it follows the representation for the density ρ_{k_0}

$$\rho_{k_0} = \frac{2M_{k_0}}{a_0^3} = 2\frac{M_m + Q}{a_0^3} + \rho_\Lambda + \frac{E_\gamma}{a_0^4}\,. \tag{59}$$

In the approximation $Q \sim 0$ (when this term is small in comparison with other summands in Eq. (58)) we obtain the standard cosmological model [5–7]. The numerical estimations of Ω_m and Ω_Λ made in the model (58) are in agreement with observational data [32].

In order to estimate the value of the decelerating parameter q_0 in the matter-dominated epoch we choose the instant of time $\tau = \tau_0$ for which the universe is in the state with $E = M_{k_0}^2$. This state is characterized by the quantum number $n_0 \simeq M_{k_0}^2 \gg 1$ and the condition $a_0 = M_{k_0}$. Then from (42) we obtain

$$\Omega_{\mathrm{tot}} = 1\,, \qquad q_0 = -\tfrac{1}{2}\,, \tag{60}$$

where $\rho_{\mathrm{tot}} = \rho_{k_0}$. These equations demonstrate that in the state under consideration the curvature term a^{-2} compensates for the density ρ and the universe looks like spatially flat.

The WMAP3 data [3] give $\Omega_{\mathrm{tot}} = 1.003^{+0.013}_{-0.017}$ and $q_0 = -0.63^{+0.07}_{-0.07}$ for the present-day Universe. These values correspond to the matter energy density $\Omega_m = 0.24^{+0.03}_{-0.04}$ and the dark energy density $\Omega_\Lambda = 0.76^{+0.04}_{-0.06}$.

If we suppose that the properties of our Universe are described by the quantum model considered in this paper (see also Ref. [28]), then a condensate of massive excitation quanta of oscillations of primordial matter which behaves as an anti-gravitating medium can play the role of the dark energy. Since this condensate is not a vacuum, the well-known contradiction with quantum field theory does not arise.

At the end of the paper we note that the voids discovered recently in distant regions of space which do not contain ordinary matter [33] may be filled with a condensate of primordial matter quanta described above that has an anti-gravitating effect on matter produced by this condensate.

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