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BASICS OF BFKL APPROACH*

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I present an introductory discussion of the BFKL approach to the theoretical description of QCD processes at high energy and fixed (not growing with energy) momentum transfers. The role of the gluon reggeization which determines the multi-Regge form of QCD amplitudes with gluon exchanges is emphasized. The region of applicability of the BFKL approach is discussed and the BFKL representation of elastic scattering amplitudes with quantum numbers different from the gluon ones is derived.

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1. Introduction

The BFKL equation [1–4] became famous owing to the prediction of the rapid growth of the γ^*p cross section at increasing energy afterwards discovered experimentally. Therefore BFKL is usually associated with the evolution equation for the unintegrated gluon distribution. The parton distributions serve now as the inherent part in the theoretical description of hard QCD processes. In hadron collisions cross sections of processes with a hard scale Q^2 are given by the convolution

$$d\sigma_{AB}(s) = \sum_{a,b} \int_0^1 dx_a \int_0^1 dx_b f_A^a\left(x_a, Q^2\right) f_B^b\left(x_b, Q^2\right) \hat{\sigma}_{ab}\left(x_a x_b s, Q^2\right)\,, \quad (1)$$

where s is the squared total energy in the c.m. system, $f_A^a(x, Q^2)$ is the density of the probability to find the parton a in the hadron A carrying a fraction x of its momentum and $\sigma_{ab}(x_a x_b s, Q^2)$ is the partonic cross section.

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Evolution of the parton distributions with $\tau = \ln\left(Q^2/\Lambda^2_{\rm QCD}\right)$ is determined by the DGLAP [5–9] equations:

$$\frac{d}{d\tau} f_A^a\left(x,Q^2\right) = \frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\sum_b \int_x^1 \frac{dz}{z} P_b^a\left(\frac{x}{z}\right) f_A^b\left(z,Q^2\right)\,, \tag{2}$$

where $P_a^b(z)$ are the parton splitting functions. The DGLAP equations are discussed in detail in the lectures by Martin [10]. They permit to sum the terms strengthened in each order of perturbative series by powers of $\ln Q^2$. These logarithms are called collinear since they are picked up from the region of small angles between parton momenta. There are logarithms of another kind, which are called soft ones, arising at integration over ratios of parton energies. These logarithms are present both in parton distributions and in partonic cross sections. At small values of the ratio $x = Q^2/s$ the soft logarithms appear to be even more important than the collinear ones.

At small-x dominant partons are gluons. The equation describing evolution of the unintegrated gluon distribution $\mathcal{F}(x,\vec{k})$ with change of $\ln(1/x)$ has the structure

$$\frac{\partial \mathcal{F}}{\partial \ln(1/x)} = \mathcal{K} \bigotimes \mathcal{F}\,, \tag{3}$$

where $\mathcal{K}$ is the BFKL kernel and $\bigotimes$ means convolution with respect to transverse momenta. Originally the kernel was found in the leading order (LO). The equation with this kernel permits to sum the leading terms $(\alpha_{\rm S}\ln(1/x))^n$. The summation leads to rising cross sections

$$\sigma \sim s^{\omega_P}\,, \qquad \omega_P = 4N_c\frac{\alpha_{\rm s}}{\pi}\ln 2\,. \tag{4}$$

Just this result brought fame to the BFKL equation since the sharp rise of the proton structure function with decreasing x was discovered in the experiments on deep inelastic e–p scattering at HERA [11].

But the region of applicability of the BFKL approach is much wider. The approach gives the description of QCD scattering amplitudes in the region of large s and fixed momentum transfer t, $s \gg |t|$ (Regge region), with various colour states in the t-channel. The evolution equation for the unintegrated gluon distribution appears in this approach as a particular result for the imaginary part of the forward scattering amplitude ($t = 0$ and vacuum quantum numbers in the t-channel). It is worthwhile to add that the approach was developed, and is more suitable, for the description of processes with only one hard scale, such as $\gamma^*\gamma^*$ scattering with both photon virtualities of the same order, where the DGLAP evolution is absent.

The leading logarithmic approximation (LLA) can provide only qualitative predictions, because it does not fix neither scale of energy nor scale of transverse momenta entering in strong coupling $\alpha_s(k_\perp)$. They can be determined in the next-to-leading approximation (NLA), when the terms $\alpha_S(\alpha_S \ln(1/x))^n$ are resummed. Therefore the normalization of cross sections and the exponent ω_P in (4) called Pomeron intercept can be fixed only in the NLA.

Evidently the power growth (4) of cross sections violate the Froissart bound [12]

$$\sigma_{\rm tot} < \mathrm{const}\,(\ln s)^2 \tag{5}$$

following from the unitarity. The violation of the Froissart bound can not be removed by calculation of radiative corrections at any fixed $NNN\ldots NL$ order and requires other methods. The most popular now are non-linear generalisations of the BFKL equation, related to the idea of saturation of parton densities [13]. The saturation is discussed in details in the lectures by Mueller [14] and Iancu [15]. A general approach to the unitarization problem is reformulating of QCD in terms of a gauge-invariant effective field theory for the reggeized gluon interactions [16].

2. The gluon reggeization

The basis of the BFKL approach is the gluon reggeization. The notion reggeization of elementary particles in perturbation theory was introduced in [17]. In terms of the relativistic partial wave amplitude $A_j(t)$, analytically continued to complex j values, it means that the non analytic (proportional to Kroneker delta-symbols) terms, arising in the Born approximation on account of one-particle exchanges in the t-channel, disappear as a consequence of radiative corrections. In other words, reggeization of an elementary particle with spin j_0 and mass m means that at large s and fixed t Born amplitudes with exchange of this particle in the t-channel acquire the factor $s^{j(t)-j_0}$, with $j(m^2) = j_0$, as a result of radiative corrections. This phenomenon was discovered for the first time in QED in the backward Compton scattering [17]. It was called reggeization because just such form of amplitudes is given by the Regge poles — moving poles in the complex angular momentum plane (j-plane) introduced by Regge [18]; therefore these poles are called Regge poles (Reggeons), and the functions $j(t)$ are called the Regge trajectories. The value $j(0)$ is called the intercept, and the derivative $j'(0)$ is called the slope of the trajectory.

For relativistic particles the theory of complex angular momenta was developed by Gribov. This theory had outstanding significance in the elementary particle physics. In the sixties of the last century it was the main and almost unique tool of the theoretical analysis of strong interactions. The Regge–Gribov theory is expanded in the lectures by Landshoff [19] and Predazzi [20]. Remind that as compared with ordinary particles Reggeons possess an additional quantum number, the signature. The fundamental role in the Gribov theory belongs to the Reggeon with vacuum quantum numbers and positive signature (parity under $s \leftrightarrow u$ exchange), which is called Pomeron in honour of outstanding Soviet physicist I.Ya. Pomeranchuk. The Pomeron determines behaviour of total cross sections at high energy. Originally it was introduced (with the intercept equal unity) [21, 22] to provide constant cross sections at asymptotically large energy. Very important and intriguing is also another Reggeon, differing from the Pomeron by C and P-parity and called Odderon [23, 24]. The Odderon is responsible for the difference of particle–particle and particle–antiparticle cross sections.

QCD is an unique theory where all elementary particles Reggeize. In contrast to QED, where the electron does Reggeize in perturbation theory [17], but the photon remains elementary [25], in QCD the gluon does Reggeize [26–28], [1, 2] as well as the quark [29, 30]. The reggeization is very important for the theoretical description of high energy processes with fixed momentum transfers. Especially important is the gluon reggeization, because non-decreasing with energy cross sections are provided by gluon exchanges. In each order of perturbation theory amplitudes with negative signature do dominate, owing to cancellation of the leading logarithmic terms in amplitudes with positive signature (which are pure imaginary in the LLA due to this cancellation). Therefore the primary Reggeon in QCD turns out to be the reggeized gluon, which has negative signature. The Pomeron and the Odderon emerge as compound states of two and three reggeized gluons respectively.

The gluon reggeization determines the form of QCD amplitudes at large energies and limited transverse momenta. At that the process $A + B \rightarrow A' + B'$ with the gluon quantum numbers in the t-channel can be depictured by the diagram of Fig. 1 and its amplitude has the Regge form

$$\mathcal{A}_{AB}^{A'B'} = \frac{s}{t}\Gamma^c_{A'A}\left[\left(\frac{-s}{-t}\right)^{\omega(t)} + \left(\frac{+s}{-t}\right)^{\omega(t)}\right]\Gamma^c_{B'B}\,, \tag{6}$$

which is valid in the NLA as well as in the LLA. In (6) $s = (p_A + p_B)^2$; $\omega(t)$ is called gluon trajectory (in fact the trajectory is $j(t) = 1 + \omega(t)$), c is a colour index and $\Gamma^c_{P'P}$ are the particle–particle–Reggeon (PPR) vertices which do not depend on s. Notice that the form (6) represents correctly the analytical structure of the scattering amplitude, which is quite simple in the elastic case.

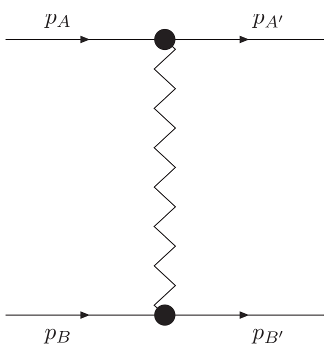

Fig. 1. The process $A+B \to A'+B'$ with the gluon quantum numbers and negative signature. The zig-zag line represents reggeized gluon exchange.

The gluon reggeization determines the form not only elastic, but also inelastic amplitudes in the multi-Regge kinematics (MRK), which is the most important at high energy. We call MRK the kinematics where all particles have limited transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with s) invariant mass of any pair of the jets. This kinematics gives dominant contributions to cross sections of QCD processes. At that in the LLA each jet is actually one particle. In the NLA one of jets can contain a couple of particle. Such kinematics is called also quasi multi-Regge kinematics (QMRK). We use the notion of jets and extend the notion of MRK, so that it includes the QMRK, in order to unify consideration.

In perturbation theory the MRK amplitudes are determined by gluon exchanges in channels with fixed transferred momenta. Despite of a great number of contributing Feynman diagrams it turns out that in the MRK these amplitudes have a simple factorised form. Quite uncommonly that radiative corrections to these amplitudes don't destroy this form, but give only simple Regge factors. At that the form (6) remains valid for the case when A' or B' represent jets. In this case the PPR vertices $\Gamma^c_{A'A}$ and $\Gamma^c_{B'B}$ are the effective vertices for $A \to A'$ and $B \to B'$ transitions owing to interactions with the reggeized gluons.

For consideration of the MRK kinematics it is suitable to use the Sudakov decomposition of momenta. For any momentum p the decomposition looks as

$$p = \beta p_1 + \alpha p_2 + p_\perp \,, \tag{7}$$

where p_1 and p_2 are the light-like vectors,

$$(p_1 + p_2)^2 = 2p_1 p_2 = s \,, \qquad p^2 = s\alpha\beta + p_\perp^2 = s\alpha\beta - \vec{p}^{\,2} \,. \tag{8}$$

Here and below the vector sign is used for components of momenta transverse to the p_1, p_2 plane. For high energy collision of particles A and B with momenta p_A and p_B we can choose p_1 and p_2 in the p_A, p_B plane, so that

$$p_A = p_1 + \left(m_A^2/s\right) p_2 \,, \qquad p_B = p_2 + \left(m_B^2/s\right) p_1 \,,$$
$$(p_A + p_B)^2 \simeq s = 2p_1p_2 \,. \tag{9}$$

Let us consider production of $n+1$ jets: $A + B \to A' + J_1 + \ldots + J_n + B'$ (see Fig. 2).

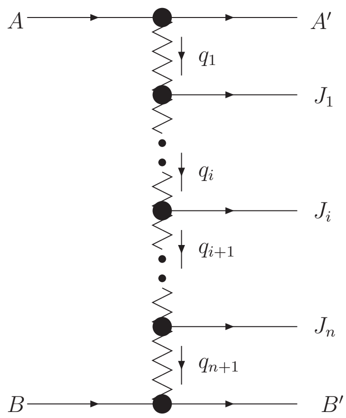

Fig. 2. Schematic representation of the process $A + B \to A' + J_1 + \ldots + J_n + B'$ in the MRK. The zig-zag lines represent reggeized gluon exchange; the black circles denote the Reggeon vertices; q_i are the Reggeon momenta; c_i are the colour indices.

If we denote momenta of the jets k_i, $i = 0 \div n+1$,

$$k_i = \beta_i p_1 + \alpha_i p_2 + k_{i\perp} \,, \qquad s\alpha_i\beta_i = k_i^2 - k_{i\perp}^2 = k_i^2 + \vec{k}_i^{\,2} \,, \tag{10}$$

then we have

$$\frac{1}{s} \sim \alpha_0 \ll \alpha_1 \ldots \ll \alpha_n \ll \alpha n+1 \simeq 1 \,,$$
$$\frac{1}{s} \sim \beta_{n+1} \ll \beta_n \ldots \ll \beta_1 \ll \beta_0 \simeq 1 \,. \tag{11}$$

Eqs. (10) and (11) secure that the squared invariant masses of neighbouring jets

$$s_i = (k_{i-1} + k_i)^2 \approx s\beta_{i-1}\alpha_i = \frac{\beta_{i-1}}{\beta_i}\left(k_i^2 + \vec{k}_i^{\,2}\right) \tag{12}$$

are large in comparison with the squared transverse momenta

$$s_i \gg \vec{k}_i^2 \sim \mid t_i \mid = \mid q_i^2 \mid \,, \tag{13}$$

where

$$t_i = q_i^2 \approx q_{i\perp}^2 = -\vec{q}_i^{\,2}\,, \tag{14}$$

and their product is proportional to s:

$$\prod_{i=1}^{n+1} s_i = s \prod_{i=1}^{n} ({k_i}^2 + \vec{k}_i^2)\,. \tag{15}$$

Dominating in each order of perturbation theory amplitudes can be presented by Fig. 2.

Multi-particle amplitudes have a complicated analytical structure. It is not simple even in the MRK (see, for instance, [31, 32]). Fortunately, only real parts of these amplitudes are used in the BFKL approach in the NLA as well as in the LLA. Restricting ourselves to the real parts (although it is not explicitly indicated below) we can write (see [33] and references therein)

$$\begin{aligned}\mathcal{A}_{AB}^{A'B'+n} &= 2s\Gamma_{A'A}^{c_1}\left[\prod_{i=1}^{n}\frac{1}{t_i}\gamma_{c_ic_{i+1}}^{J_i}(q_i,q_{i+1})\left(\frac{s_i}{\sqrt{\vec{k}_{i-1}^2\vec{k}_i^2}}\right)^{\omega(t_i)}\right]\\ &\quad\times\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{\sqrt{\vec{k}_n^2\vec{k}_{n+1}^2}}\right)^{\omega(t_{n+1})}\Gamma_{B'B}^{c_{n+1}}\,,\end{aligned} \tag{16}$$

where the vertices $\Gamma_{A'A}^{a}$ and $\Gamma_{B'B}^{b}$ are the same as in (6), and $\gamma_{c_ic_{i+1}}^{J_i}(q_i,q_{i+1})$ are the Reggeon–Reggeon–particle (RRP) vertices, *i.e.* the effective vertices for production of jets J_i with momenta $k_i{=}q_i - q_{i+1}$ in collisions of Reggeons with momenta q_i and $-q_{i+1}$ and colour indices c_i and c_{i+1}. Actually in the LLA only one gluon can be produced in the RRP vertex; in the NLA a jet can contain two gluons or $q\bar{q}$ pair. Pay attention that we have taken definite energy scales in the Regge factors in Eq. (16) as well as in Eq. (6). In the LLA the energy scales are not important at all. In the NLA we could take, in principle, an arbitrary scale s_R; in this case the PPR and RRP vertices would become dependent on s_R. Of course, physical results must be scale-independent.

For brevity in the following we call the forms (6) and (16) reggeized forms, and talking about the gluon reggeization we mean these forms. The gluon reggeization hypothesis is extremely powerful since an infinite number of amplitudes is expressed in terms of the gluon Regge trajectory and several Reggeon vertices. In the LLA the gluon reggeization was proved in [34] (see also [35]). Now it is proved in the NLA as well (see [36] and references therein). The proof is based on the "bootstrap" relations, required by compatibility of the gluon reggeization with the s-channel unitarity. It turns out that fulfilment of all these relations ensure the reggeized form of energy dependent radiative corrections order by order in perturbation theory. It is quite nontrivial, that an infinite number of the bootstrap relations for the multi- particle production amplitudes can be fulfilled if the Reggeon vertices and trajectory satisfy several bootstrap conditions. All these conditions are derived and are proved to be satisfied.

3. Fixed order calculations

The idea of the gluon reggeization arose as the result of the fixed order calculations in non-Abelian gauge theories with spontaneously broken gauge invariance [2, 28]. The dispersion method of calculations based on unitarity and analyticity suggested in [28] and developed in [2] appeared to be very effective. It greatly simplifies calculations of even Born amplitudes, which can be easily found using the t-channel unitarity. For elastic scattering amplitudes $\mathcal{A}_{AB}^{A'B'}(s,t)$ the t-channel discontinuities are presented by Fig. 3, where the dash line means the substitution

$$\frac{1}{t+i0} \to -2\pi i\delta(t) \tag{17}$$

in the gluon propagator. The straight lines can represent both quarks and gluons. Note that in order to preserve usual analytical properties of the

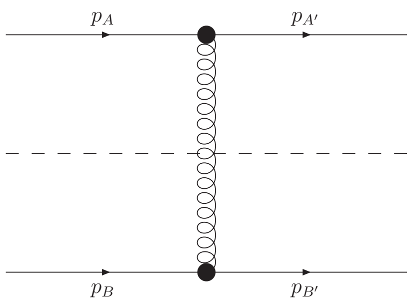

Fig. 3. The t-channel discontinuity of the amplitude of the process $A+B \to A'+B'$.

amplitude we have to use covariant gauges for intermediate gluons. We will use the Feynman one. In the Regge region it is convenient to exploit the substitution

$$g^{\mu\nu} \to \frac{2}{s} p_2^{\mu} p_1^{\nu} \tag{18}$$

for the tensor $g^{\mu\nu}$ in the numerator of the gluon propagator between vertices with indices μ and ν and momenta close to p_1 and p_2 correspondingly. It is possible because in the decomposition

$$g^{\mu\nu} = \frac{2}{s}(p_2^{\mu} p_1^{\nu} + p_1^{\mu} p_2^{\nu}) + g_{\perp}^{\mu\nu} \tag{19}$$

the last two terms give negligible contributions. Therefore we obtain

$$2i\mathrm{Im}_t\mathcal{A}_{AB}^{A'B'}(s,t)=-4\pi is\delta(t)\mathit{\Gamma}^c_{A'A}\mathit{\Gamma}^c_{B'B}\,, \tag{20}$$

where Im_t denotes the t-channel imaginary part and the vertices $\mathit{\Gamma}^c_{A'A}$ and $\mathit{\Gamma}^c_{B'B}$ are the gluon interaction vertices with the gluon colour index c and polarization vectors ip_2/s and ip_1/s correspondingly. Renormalizability of the theory requires decreasing with t terms of order s, so that (20) unambiguously determines the amplitude:

$$\mathcal{A}_{AB}^{A'B'}(s,t)=\frac{2s}{t}\mathit{\Gamma}^c_{A'A}\mathit{\Gamma}^c_{B'B}\,. \tag{21}$$

From comparison with (6) one can see that in fact the vertices $\mathit{\Gamma}^c_{A'A}$ and $\mathit{\Gamma}^c_{B'B}$ are the RRP vertices in the LO, *i.e.* the RRP vertices can be easily found assuming the form (6). In the helicity basis all these vertices have identical form:

$$\mathit{\Gamma}^c_{P'P}=gT^c_{P'P}\delta_{\lambda_{P'}\lambda_P}\,, \tag{22}$$

where $T^c_{P'P}$ represent now the matrix elements of the colour group generators in corresponding representations and λs are helicities of the partons. Except for a common coefficient the vertices (22) can be written without calculations, because they are given by forward matrix elements of the conserved current. In (22) the s-channel helicity conservation is explicitly exhibited. Note that for gluons and massive quarks it is valid only in the LO.

It is easy to rewrite (22) in terms of Dirac spinors for quarks and physical polarization vectors for gluons. The vertex for $q(p)\to q(p')$ transition with momenta p and p' having predominant components along p_1 can be presented as

$$\mathit{\Gamma}^c_{Q'Q}=g\bar{u}(p')t^c\frac{\not{p}_2}{2pp_2}u(p)\,, \tag{23}$$

where t^c are the colour group generators in the fundamental representation; for antiquarks correspondingly

$$\mathit{\Gamma}^c_{\bar{Q}'\bar{Q}}=-g\bar{v}(p)t^c\frac{\not{p}_2}{2pp_2}v(p')\,. \tag{24}$$

The vertices for gluon transitions acquire a simple form in physical gauges. For gluons with predominant components of momenta along p_1 we will use physical polarization vectors $e(p)p_2=0$ in the light-cone gauge $e(p')p_2=0$, so that

$$e(p)=e(p)_\perp-\frac{(e(p)_\perp p_\perp)}{p_2p}p_2\,,\qquad e(p')=e(p')_\perp-\frac{(e(p')_\perp p'_\perp)}{p_2p'}p_2\,, \tag{25}$$

and

$$\Gamma^c_{G'G} = -g(e^*(p')_\perp e(p)_\perp)T^c_{G'G}\,, \tag{26}$$

with the colour generators in the adjoint representation. For momenta with predominant components along p_2 we have to replace in these formulas $p_2 \to p_1$ (evidently, this replacement in (25) means change of the gauge).

Dispersion approach requires knowledge of production amplitudes in the MRK. Again in the Born approximation they can be calculated in the LO without large efforts using the t-channel unitarity. In the LO only gluons can be produced. Amplitudes $\mathcal{A}^{A'GB'}_{AB}$ are calculated using the t_1 and t_2-channel discontinuities. Schematically they are presented in Fig. 4, where the reggeized form (6) of the $2 \to 2$ amplitudes must be taken in the lowest order, so that it is given by (21), evidently with the substitutions $s \to s_2 = (p'_B + k)^2 = (p_B + q_1)^2$, $t \to t_2$ in the case of Fig. 4(a) and $s \to s_1 = (p'_A + k)^2 = (p_A - q_2)^2$, $t \to t_1$ in the case of Fig. 4(b).

Here we meet a complication. Till now the gluon–gluon–Reggeon vertices were defined only for physical gluon polarizations. But in order to use the Feynman gauges in the amplitudes of Fig. 4 it is necessary to have the vertices in the gauge invariant form. It is easy to see that the required form can be obtained from (26) by the substitution ($e^{'*} \equiv e^*(p')$, $e \equiv e(p)$)

$$\left(e^{'*}_\perp e_\perp\right) \to \left(e^{'*}e\right) - \frac{1}{(p_i p)}\left(\left(p_i e^{'*}\right)(p'e) + \left(pe^{'*}\right)(p_i e)\right) + \frac{(pp')}{(p_i p)^2}\left(p_i e^{'*}\right)(p_i e)\,, \tag{27}$$

where $p_i = p_2$ ($p_i = p_1$) for the gluons with predominant components of momenta along p_1 (p_2), so that $(p_i p) \simeq (p_i p') \gg (pp')$. Using the vertices in the covariant form, one can easily find the contributions $\mathcal{A}_a$ and $\mathcal{A}_b$ with the discontinuities corresponding to the diagrams Fig. 4(a) and Fig. 4(b):

$$\begin{aligned}\mathcal{A}_a &= 2s\Gamma^{c_1}_{A'A}\frac{1}{t_1}gT^c_{c_1c_2}e^*_\mu(k) \\ &\times\left(-q_1^\mu - q_2^\mu + 2p_1^\mu\frac{kp_2}{p_1p_2} - p_2^\mu\left(\frac{q_2^2}{kp_2} + 2\frac{kp_1}{p_1p_2}\right)\right)\frac{1}{t_2}\Gamma^{c_2}_{B'B}\,, \quad (28)\\ \mathcal{A}_b &= 2s\Gamma^{c_1}_{A'A}\frac{1}{t_1}gT^c_{c_1c_2}e^*_\mu(k) \\ &\times\left(-q_1^\mu - q_2^\mu + p_1^\mu(\frac{q_1^2}{kp_2} + 2\frac{kp_2}{p_1p_2} - 2p_2^\mu\frac{kp_1}{p_1p_2}\right)\frac{1}{t_2}\Gamma^{c_2}_{B'B}\,. \quad (29)\end{aligned}$$

Here $k = q_1 - q_2$, $e(k)$ and c are the gluon momentum, polarization vector and colour index. It is easy to see that the amplitude $\mathcal{A}^{A'GB'}_{AB}$

$$\mathcal{A}^{A'GB'}_{AB} = 2s\Gamma^{c_1}_{A'A}\frac{1}{t_1}\gamma^c_{c_1c_2}(q_1,q_2)\frac{1}{t_2}\Gamma^{c_2}_{B'B}\,, \tag{30}$$

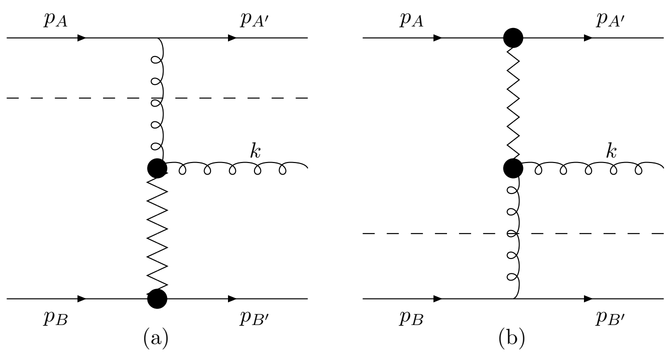

Fig. 4. Schematic representation of the discontinuities of the $A+B \to A'+G+B'$ amplitude in the t_1 (a) and t_2 (b) channels.

with

$$\gamma^c_{c_1c_2}(q_1,q_2) = gT^c_{c_1c_2}e^*_\mu(k)C^\mu(q_2,q_1)\,, \tag{31}$$

$$\begin{aligned} C^\mu(q_2,q_1) &= -q_1^\mu - q_2^\mu + p_1^\mu\left(\frac{q_1^2}{kp_1} + 2\frac{kp_2}{p_1p_2}\right) - p_2^\mu\left(\frac{q_2^2}{kp_2} + 2\frac{kp_1}{p_1p_2}\right) \\ &= -q_{1\perp}^\mu - q_{2\perp}^\mu - \frac{p_1^\mu}{2(kp_1)}\left(k_\perp^2 - 2q_{1\perp}^2\right) + \frac{p_2^\mu}{2(kp_2)}\left(k_\perp^2 - 2q_{2\perp}^2\right)\,, \end{aligned} \tag{32}$$

has correct discontinuities both in the t_1 and t_2 channels. It means that it is correct amplitude, because contributions $\sim s$ without singularities in the t_1 and t_2 channels are forbidden by the renormalizability. Therefore the vertex (31) is in fact the Reggeon–Reggeon–Gluon vertex. In the LO it is the only RRP vertex.

Note that the vertex is gauge invariant: $C^\mu(q_2,q_1)k_\mu = 0$. In the physical light cone gauges the vertex simplifies. In the gauge $e(k)k = e(k)p_2 = 0$

$$e^*_\mu(k)C^\mu(q_2,q_1) = -2e^*_\perp(k)\left(q_{1\perp} - k_\perp\frac{q_{1\perp}^2}{k_\perp^2}\right)\,, \tag{33}$$

and in the gauge $e(k)k = e(k)p_1 = 0$

$$e^*_\mu(k)C^\mu(q_2,q_1) = -2e^*_\perp(k)\left(q_{2\perp} + k_\perp\frac{q_{2\perp}^2}{k_\perp^2}\right)\,. \tag{34}$$

Thus *assuming* the gluon reggeization the LO Reggeon vertices are found without noticeable efforts. It is quite easy also to find in the LO the gluon trajectory. To do this it is sufficient to find the lowest order contribution to the s-channel discontinuity of some elastic amplitude with negative signature and to compare it with (6). Of course, the trajectory must be process-independent. This requirement serves as a check of self-consistency of the reggeization hypothesis.

In the lowest order only two-particle intermediate states do contribute in the unitarity relation

$$\mathrm{Im}_s \mathcal{A}_{AB}^{A'B'} = \frac{1}{2} \sum_{\tilde{A},\tilde{B}} \int \mathcal{A}_{AB}^{\tilde{A}\tilde{B}} \left(\mathcal{A}_{A'B'}^{\tilde{A}\tilde{B}}\right)^* d\Phi_{\tilde{A}\tilde{B}} \,, \tag{35}$$

where Im_s stands for the s-channel imaginary part, $\sum_{\tilde{A},\tilde{B}}$ means sum over discrete quantum numbers of the intermediate particles, $d\Phi_{\tilde{A},\tilde{B}}$ is the element of their phase space,

$$d\Phi_{\tilde{A},\tilde{B}} = (2\pi)^D \delta^D \left(p_A + p_B - p_{\tilde{A}} - p_{\tilde{B}}\right) \frac{d^{D-1}p_{\tilde{A}}}{(2\pi)^{D-1} 2\epsilon_{\tilde{A}}} \frac{d^{D-1}p_{\tilde{B}}}{(2\pi)^{D-1} 2\epsilon_{\tilde{B}}} \,. \tag{36}$$

Here and in the following $D = 4 + 2\epsilon$ is the space-time dimension taken different from 4 for regularization of infrared, collinear and ultraviolet divergences. Using (21) and

$$\begin{aligned} \delta^D(p) &= \frac{2}{s}\delta(\alpha)\delta(\beta)\delta^{D-2}(p_\perp)\,, \qquad \frac{d^{D-1}p}{(2\pi)^{D-1}2\epsilon} = \delta(p^2 - m^2)d^D p\,, \\ d^D p &= \frac{s}{2} d\alpha d\beta \frac{d^{D-2}p_\perp}{(2\pi)^{D-1}}\,, \qquad p_{\tilde{A}}^2 \simeq s\alpha_{\tilde{A}}\,, \qquad p_{\tilde{B}}^2 \simeq s\beta_{\tilde{B}}\,, \end{aligned} \tag{37}$$

we obtain

$$\mathrm{Im}_s \mathcal{A}_{AB}^{A'B'} = s \sum_{\tilde{A}} \Gamma_{\tilde{A}A}^{c} \Gamma_{A'\tilde{A}}^{c'} \sum_{\tilde{B}} \Gamma_{\tilde{B}B}^{c} \Gamma_{A'\tilde{A}}^{c'} \frac{1}{(2\pi)^{(D-2)}} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2} \left(\vec{q} - \vec{q}_1\right)^2} \,. \tag{38}$$

Account of the signature means anti-symmetrization in the colour indices in the first (or second) sum. The commutation relations of the colour group generators give

$$\begin{aligned} \frac{1}{2}\left(\Gamma_{\tilde{A}A}^{c} \Gamma_{A'\tilde{A}}^{c'} - \Gamma_{\tilde{A}A}^{c'} \Gamma_{A'\tilde{A}}^{c}\right) &= -i\frac{g}{2} f_{cc'a} \Gamma_{A'A}^{a}\,, \\ -i\frac{g}{2} f_{cc'a} \Gamma_{\tilde{B}B}^{c} \Gamma_{B'\tilde{B}}^{c'} &= -g^2 \frac{N_c}{4} \Gamma_{B'B}^{a}\,, \end{aligned} \tag{39}$$

where $f_{cc'a}$ are the group structure constants, $f_{cc'a} f_{cc'b} = N_c$, N_c in the number of colours. Comparing (38) with account of (39) with (6) we obtain

$$\omega(t) = \frac{g^2 N_c t}{2(2\pi)^{D-1}} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2}(\vec{q} - \vec{q}_1)^2} = -g^2 \frac{N_c \Gamma(1-\epsilon)}{(4\pi)^{D/2}} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} (\vec{q}^{\,2})^\epsilon \,. \tag{40}$$

Thus, we see that assuming the gluon reggeization all the Reggeon vertices and the Regge trajectory can be easily obtained in the LO. To find the vertices it is sufficient to calculate elastic and one-gluon production amplitudes in the Born approximation; to find the trajectory it is enough to calculate in the lowest order the s-channel discontinuity of some elastic scattering amplitude.

Originally the reggeized form of elastic amplitudes was established in the LO in two loops [28]. The three-loop calculations [2] confirmed this form and permitted to formulate the reggeization hypothesis for inelastic amplitudes.

Now the vertices and the trajectory are known in the NLO. Of course, in this order neither the calculation, nor the results are not so simple as in the LO. To find the PPR vertices one has to calculate non-logarithmic terms in one-loop elastic amplitudes; to obtain the RRG vertex it is necessary to compute with such accuracy a gluon production amplitude in the MRK.

4. BFKL equation

The gluon reggeization gives the amplitudes with the gluon quantum numbers and negative signature. Other amplitudes are found in the BFKL approach from the s-channel unitarity. In the unitarity relations the contribution of order s, which we are interested in, is given by the MRK. Large logarithms come from integration over longitudinal momenta of the produced jets. In the LLA, where production of each additional particle must give the large logarithm ($\ln s$), each jet is in fact a gluon. For elastic amplitudes we have (see Fig. 5)

$$\mathrm{Im}_s \mathcal{A}_{AB}^{A'B'} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{\{f\}}\int \mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n}\left(\mathcal{A}_{A'B'}^{\tilde{A}\tilde{B}+n}\right)^* d\Phi_{\tilde{A}\tilde{B}+n}\,, \tag{41}$$

where $\sum_{\{f\}}$ means sum over discrete quantum numbers of the intermediate particles, the amplitudes $\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n}$ and $\left(\mathcal{A}_{A'B'}^{\tilde{A}\tilde{B}+n}\right)$ are defined by (16) and $d\Phi_{\tilde{A},\tilde{B}+n}$ is the element of the phase space volume,

$$\begin{aligned} d\Phi_{\tilde{A}\tilde{B}+n} \;=\; & \frac{2}{s}\,(2\pi)^D\,\delta\left(1+\frac{m_A^2}{s}-\sum_{i=0}^{n+1}\alpha_i\right)\delta\left(1+\frac{m_B^2}{s}-\sum_{i=0}^{n+1}\beta_i\right) \\ & \times\delta^{(D-2)}\left(\sum_{i=0}^{n+1}k_{i\perp}\right)\frac{d\beta_{n+1}}{2\beta_{n+1}}\frac{d\alpha_0}{2\alpha_0}\prod_{i=1}^{n}\frac{d\beta_i}{2\beta_i}\prod_{i=0}^{n+1}\frac{d^{D-2}k_{i\perp}}{(2\pi)^{D-1}}\,. \end{aligned} \tag{42}$$

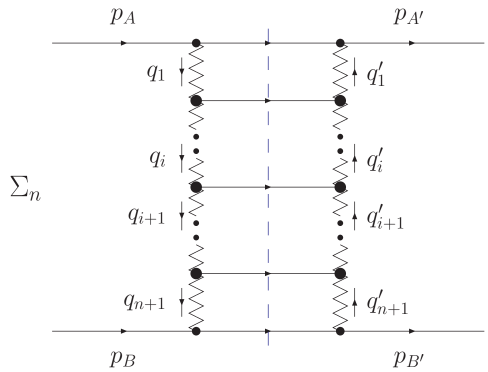

Fig. 5. The s-channel discontinuity of the amplitude of the process $A+B \to A'+B'$.

In the integrand of (41) dependence on the longitudinal components enters only from the Regge factors, so that the integration over α_i and β_i can be explicitly performed. In order to present the discontinuities in a compact way it is convenient to use the operator notations in the transverse momentum and colour space. We will use also notations which accumulate all quantum numbers. Thus, $\langle \mathcal{G}_i \mathcal{G}_j|$ and $|\mathcal{G}_i \mathcal{G}_j\rangle$ are *bra*- and *ket*-vectors for the t-channel states of two reggeized gluons with transverse momenta $q_{i\perp}$ and $q_{j\perp}$ and colour indices c_i and c_j correspondingly. It is convenient to distinguish the states $|\mathcal{G}_i \mathcal{G}_j\rangle$ and $|\mathcal{G}_j \mathcal{G}_i\rangle$. We will associate the first of them with the case when the Reggeon $\mathcal{G}_i$ is contained in the amplitude with initial particles (in the left part of Fig. 5), and the second with the case when it is contained in the amplitude with final particles (in the right part of Fig. 5). It is convenient to introduce the scalar product

$$\left\langle \mathcal{G}_i \mathcal{G}_j | \mathcal{G}'_i \mathcal{G}'_j \right\rangle = q_{i\perp}^2 q_{j\perp}^2 \delta\left(q_{i\perp} - r'_{i\perp}\right) \delta\left(q_{j\perp} - q'_{j\perp}\right) \delta_{c_i c'_i} \delta_{c_j c'_j} \, . \tag{43}$$

These states are complete, and with the scalar product (43) the completeness means

$$\langle \Psi | \Phi \rangle = \int \frac{d^{D-2} q_{1\perp} d^{D-2} q_{2\perp}}{q_{1\perp}^2 q_{2\perp}^2} \langle \Psi | \mathcal{G}_1 \mathcal{G}_2 \rangle \langle \mathcal{G}_1 \mathcal{G}_2 | \Phi \rangle . \tag{44}$$

In these notations the discontinuity is presented in the form (see Fig. 6):

$$\delta\left(\vec{q}_A - \vec{q}_B\right) \mathrm{Im}_s \mathcal{A}_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \left\langle A' \bar{A} \left| e^{Y \widehat{\mathcal{K}}} \right| \bar{B}' B \right\rangle , \tag{45}$$

where $q_A = p_{A'} - p_A$, $q_B = p_B - p_{B'}$, $\langle A' \bar{A}|$ and $|\bar{B}' B\rangle$ are the t-channel states representing impact factors of scattering particles, $Y = \ln(s/s_0)$, s_0 is an energy scale, and $\hat{\mathcal{K}}$ is the BFKL kernel. At that

$$\left\langle \mathcal{G}_1 \mathcal{G}_2 | \bar{B}' B \right\rangle = \delta(\vec{q}_B - \vec{q}_1 - \vec{q}_2) \Phi_{B'B}^{c_1 c_2}(\vec{q}_1, \vec{q}_2) \, , \tag{46}$$

$$\langle A'\bar{A}|\mathcal{G}_1\mathcal{G}_2\rangle = \delta(\vec{q}_A - \vec{q}_1 - \vec{q}_2)\Phi^{c_1c_2}_{A'A}(\vec{q}_1,\vec{q}_2)\,, \tag{47}$$

where the impact factors Φ are expressed in terms of the Reggeon vertices. In the LO the expressions are quite simple (*cf.* (38)):

$$\Phi^{c_1c_2}_{A'A}(\vec{q}_1,\vec{q}_2) = \sum_{\tilde{A}} \Gamma^{c_1}_{\tilde{A}A}\Gamma^{c_2}_{A'\tilde{A}}\,, \qquad \Phi^{c_1c_2}_{B'B}(\vec{q}_1,\vec{q}_2) = \sum_{\tilde{B}} \Gamma^{c_1}_{\tilde{B}B}\Gamma^{c_2}_{B'\tilde{B}}\,. \tag{48}$$

In the NLO the impact factors are defined according to [33].

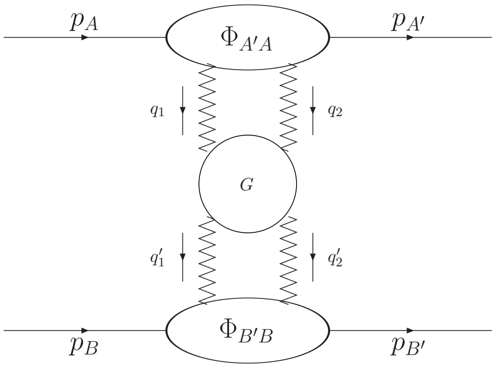

Fig. 6. Schematic representation of the process $A + B \to A' + A'$.

The kernel $\hat{\mathcal{K}}$ is given by the sum of "virtual" and "real" parts:

$$\widehat{\mathcal{K}} = \widehat{\Omega} + \widehat{\mathcal{K}}_r\,. \tag{49}$$

The "virtual" part comes from the Regge factors and is expressed in terms of the trajectories of two interacting reggeized gluons:

$$\widehat{\Omega} = \hat{\omega}_1 + \hat{\omega}_2\,. \tag{50}$$

In the momentum space

$$\langle \mathcal{G}_i|\hat{\omega}_i|\mathcal{G}'_i\rangle = \hat{\vec{q}}_i^{\,2}\,\delta\left(q_{i\perp} - q'_{i\perp}\right)\delta_{c_ic'_i}\omega\left(q_\perp^2\right)\,, \tag{51}$$

where $\omega(q_\perp^2)$ is the gluon Regge trajectory. The "real" part $\widehat{\mathcal{K}}_r$ comes from convolutions of the RRP vertices. In the LO only gluons are produced, so that

$$\left\langle \mathcal{G}_1,\mathcal{G}_2|\hat{\mathcal{K}}_r|\mathcal{G}'_1,\mathcal{G}'_2\right\rangle = \delta\left(\vec{q}_1 + \vec{q}_2 - \vec{q}'_1 - \vec{q}'_2\right)\sum_a \gamma^a_{c_1c'_1}(q_1,q'_1)\left(\gamma^a_{c_2c'_2}\left(q_2,q'_2\right)\right)^*\,, \tag{52}$$

where the RRP vertices $\gamma^a_{cc'}$ for gluon production are given in (31) and the sum goes over gluon colours and polarizations.

Let us introduce the operators $\hat{\mathcal{P}}_R$ for projection of the two-Reggeon colour states on the irreducible representations R of the colour group and use the decomposition

$$\left\langle \mathcal{G}_1, \mathcal{G}_2|\hat{\mathcal{K}}_r|\mathcal{G}'_1, \mathcal{G}'_2\right\rangle = \delta\left(\vec{q}_1 + \vec{q}_2 - \vec{q}_1^{\,\prime} - \vec{q}_2^{\,\prime}\right) \sum_R \left\langle c_1 c_2|\hat{\mathcal{P}}_R|c'_1 c'_2\right\rangle \times 2(2\pi)^{D-1}\mathcal{K}_r^{(R)}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}\,)\,. \tag{53}$$

The most interesting representations are the singlet (Pomeron) and anti-symmetrical octet (reggeized gluon) representations. For the first of them

$$\left\langle c_1 c_2|\hat{\mathcal{P}}_0|c'_1 c'_2\right\rangle = \frac{\delta_{c_1 c_2}\delta_{c'_1 c'_2}}{N_c^2 - 1}\,, \tag{54}$$

and for the second

$$\left\langle c_1 c_2|\hat{\mathcal{P}}_8|c'_1 c'_2\right\rangle = \frac{f_{ac_1c_2} f_{ac'_1c'_2}}{N_c}\,. \tag{55}$$

Using the decomposition

$$T^a_{c_1c'_1}\left(T^a_{c_2c'_2}\right)^* = \sum_R c_R \left\langle c_1 c_2|\hat{\mathcal{P}}_R|c'_1 c'_2\right\rangle\,, \tag{56}$$

one obtains from (52), (53)

$$\mathcal{K}_r^{(\mathcal{R})}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}\,) = \frac{g^2 c_R}{(2\pi)^{D-1}}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime 2} + \vec{q}_2^{\,2}\vec{q}_1^{\,\prime 2}}{(\vec{q}_1 - \vec{q}_1^{\,\prime}\,)^2} - \vec{q}^{\,2}\right). \tag{57}$$

For the singlet and octet representations

$$c_0 = N_c\,, \qquad c_8 = \frac{N_c}{2}\,. \tag{58}$$

In the NLO the kernels $\mathcal{K}_r^{(\mathcal{R})}$ are defined according to [33]. In the particular case of the forward scattering in the LO

$$\mathcal{K}_r(\vec{q}_1, \vec{q}_1^{\,\prime}) = \frac{\mathcal{K}_r^{(0)}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{0})}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime 2}} = \frac{g^2 N_c}{(2\pi)^{D-1}}\frac{2}{(\vec{q}_1 - \vec{q}_1^{\,\prime})^2}\,. \tag{59}$$

Taken separately, the virtual and real contributions to the kernel lead to infrared singularities. But in the singlet (Pomeron) channel the singularities cancel each other. For the forward LO kernel

$$\mathcal{K}(\vec{q}_1, \vec{q}_1^{\,\prime}) = 2\omega(-\vec{q}_1^{\,2})\delta(\vec{q}_1 - \vec{q}_1^{\,\prime}) + \mathcal{K}_r(\vec{q}_1, \vec{q}_1^{\,\prime}) \tag{60}$$

we have

$$\int d^{D-2}q_2\ \mathcal{K}(\vec{q}_1,\vec{q}_2)(\vec{q}_2^{\,2})^{\gamma-1} = \frac{N_c\alpha_s}{\pi}\chi(\gamma)\left(\vec{q}_1^{\,2}\right)^{\gamma-1}\ , \qquad (61)$$

where

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)\,, \qquad \psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)\,. \qquad (62)$$

The set of functions $(\vec{q}_2^{\,2})^{\gamma-1}$ with $\gamma = 1/2 + i\nu,\ -\infty < \nu < \infty$ is complete. The maximal value of $\chi^B(\gamma)$ is $\chi(1/2) = 4\ln 2$, that corresponds to the Pomeron intercept (4).

5. Conclusion

In this paper I could give only the foundation of the BFKL theory. The bulk of the paper was devoted to the gluon reggeization, which is the corner stone of the BFKL approach. The reggeization determines QCD amplitudes with the gluon quantum numbers in the t-channels in the multi-Regge kinematics. Unfortunately, the present status of the theory for amplitudes with other quantum numbers remained almost untouched.

Now the BFKL approach is well developed in the next-to-leading approximation. For the forward case ($t = 0$ and the vacuum quantum numbers in the t-channel) BFKL kernel was found at NLO about ten years ago [37,38]. Applications of this kernel are discussed in the lectures by Papa [39] and Sabio Vera [40].

However, the BFKL approach is not limited to the forward case. From the beginning it was developed for arbitrary t and for all possible t-channel colour states. The forward kernel can carry only restrictive information about the BFKL dynamics. Moreover, the non-forward case has an advantage of smaller sensitivity to large-distance contributions, since the diffusion in the infrared region is limited by $\sqrt{|t|}$. Now the BFKL kernel is known also for the non-forward case [41].

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