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BFKL POMERON IN QCD AND IN $N = 4$ SUSY* **

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We review existing theoretical approaches for the deep-inelastic lepton–hadron interactions and the high energy hadron–hadron scattering in the Regge kinematics. It is demonstrated, that the gluon in QCD is reggeized and the Pomeron is a composite state of the reggeized gluons. A gauge invariant effective action for the reggeized gluon interactions is constructed. Remarkable properties of the BFKL equation for the Pomeron wave function in QCD and supersymmetric gauge theories in leading and next-to-leading approximations are outlined. It is shown, that due to the AdS/CFT correspondence the BFKL Pomeron is equivalent to the reggeized graviton in the extended $N = 4$ supersymmetric model. Its intercept at large coupling constant is calculated in this model.

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1. Introduction

The inclusive electron–proton scattering in the Bjorken kinematics (see Fig. 1)

$$2pq \sim Q^2 = -q^2 \to \infty \,, \qquad x = \frac{Q^2}{2pq} \,, \qquad 0 \leq x \leq 1 \tag{1}$$

is very important, because it gives a direct information about the distributions $n^q(x)$ of quarks inside the proton as a function of their energy ratio $x = |\vec{k}|/|\vec{p}|$ $(|\vec{p} \to \infty|)$. Indeed, in the framework of the Feynman–Bjorken quark–parton model [1, 2] we can obtain the following simple expression for the structure functions $F_{1,2}(x)$ of this process

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$$\frac{1}{x}\,F_2(x) = 2F_1(x) = \sum_{i=q,\bar q} Q_i^2\, n^i(x)\,, \tag{2}$$

where the quark charges are $Q_u = \frac{2}{3}\,,\; Q_d = -\frac{1}{3}$.

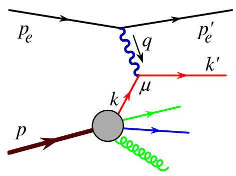

Fig. 1.

It turns out, that the partonic picture is valid also in renormalizable field theories, if one would restrict the parton transverse momenta $\vec{k}_\perp$ by an ultraviolet cut-off $k_\perp^2 < \Lambda^2 \sim Q^2$ [3]. In these theories the running coupling constant $\alpha = g^2/(4\pi)$ in the leading logarithmic approximation (LLA) is

$$\alpha(Q^2) = \frac{\alpha_\mu}{1+\beta\frac{\alpha_\mu}{4\pi}\,\ln\frac{Q^2}{\mu^2}}\,, \tag{3}$$

where α_μ is its value at the renormalization point μ. In quantum electrodynamics (QED) and quantum chromodynamics (QCD) the coefficients β have an opposite sign

$$\beta_{\rm QED} = -n_e\frac{4}{3}\,, \qquad \beta_{\rm QCD} = \frac{11}{3}N_c - n_f\frac{2}{3}\,, \tag{4}$$

where N_c is the rank of the gauge group ($N_c = 3$ for QCD), n_e and n_f are numbers of the leptons and the quarks, which can be considered as massless for given Q^2.

Landau and Pomeranchuck argued, that due to the negative sign of $\beta_{\rm QED}$ the Landau pole in the photon propagator is generated, which leads to vanishing the physical electric charge in the local limit. Respectively, in QCD the non-Abelian interaction disappears at large Q^2 and, as a result of the asymptotic freedom, we have an approximate Bjorken scaling: the structure functions depend on Q^2 only logarithmically [4]. Thus, the experiments on the deep-inelastic *ep* scattering performed at SLAC in the end of sixties discovered, that at strong interactions the Landau "zero charge" problem is absent.

In the infinite momentum frame $|\vec{p}| \to \infty$ it is helpful to introduce the Sudakov variables for parton momenta

$$\vec{k_i} = \beta_i\,\vec{p} + \vec{k}_i^{\perp}\,, \qquad \left(\vec{k}_i^{\perp},\vec{p}\right) = 0\,, \qquad \sum_i \vec{k_i} = \vec{p}\,. \tag{5}$$

The parton distributions are defined in terms of the proton wave function Ψ_m as follows

$$n^i(x) = \sum_m \int \prod_{r=1}^{m-1} \frac{d\beta_r\, d^2 k_r^\perp}{(2\pi)^2}\, |\Psi_m|^2 \sum_{r\in i} \delta(\beta_r - x)\,. \tag{6}$$

They are functions of $\Lambda \sim Q$ because the factor $|\Psi_m|^2 \sim \prod_{r=1}^m Z_r$ depends on Λ through the wave function renormalization constants $\sqrt{Z_r}$ and Λ is an upper limit in integrals over the transverse momenta $k_r^\perp$. Taking into account the cascade-type dynamics of the parton number growth with Λ, we can obtain in LLA the evolution equations of Dokshizer, Gribov, Lipatov, Altarelli and Parisi (DGLAP) [5]

$$\frac{d}{d\xi(Q^2)} n_i(x) = -w_i\, n_i(x) + \sum_r \int_x^1 \frac{dy}{y} w_{r\to i}\left(\frac{x}{y}\right) n_r(y)\,, \tag{7}$$

$$w_i = \sum_k \int_0^1 dx\, x\, w_{i\to k}(x)\,, \tag{8}$$

where

$$\xi\left(Q^2\right) = \frac{N_c}{2\pi} \int_{\mu^2}^{Q^2} \frac{d\vec{k}_\perp^2}{\vec{k}_\perp^2} \alpha\left(\vec{k}_\perp^2\right)\,. \tag{9}$$

This equation has a clear probabilistic interpretation: the number of partons n_i decreases due to their decay to other partons in the opening phase space $d\xi(Q^2)$ and increases due to the fact that the products of the decay of other partons r can contain the partons of the type i [3].

Momenta of parton distributions

$$n_i^j = \int_0^1 dx\, x^{j-1}\, n_i(x) \tag{10}$$

satisfy the renormalization group equations

$$\frac{d}{d\xi(Q^2)} n_i^j = \sum_r w_{r\to i}^j n_r^j \tag{11}$$

and are related to the matrix elements of twist-2 operators

$$n_i(j) = \left\langle p|O_i^j|p\right\rangle\,. \tag{12}$$

The twist t is defined as a difference between their canonical dimension d measured in units of mass and the Lorentz spin j of the corresponding tensor. The quantities $w^j_{r\to i}$ are elements of the anomalous dimension matrix for the operators O^j_i.

Hadron–hadron scattering in the Regge kinematics (see Fig. 2)

$$s = (p_A + p_B)^2 = (2E)^2 \gg \vec{q}^{\,2} = -(p_{A'} - p_A)^2 \sim m^2 \tag{13}$$

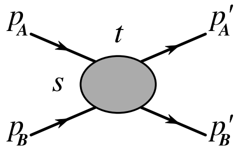

Fig. 2.

is usually described in terms of an t-channel exchange of the Reggeon (see Fig. 3)

$$A_p(s,t) = \xi_p(t)\, g(t)\, s^{j_p(t)}\, g(t)\,, \;\; j_p(t) = j_0 + \alpha' t\,, \tag{14}$$

$$\xi_p = \frac{e^{-i\pi j_p(t)} + p}{\sin(\pi j_p)}\,, \tag{15}$$

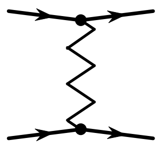

Fig. 3.

where $j_p(t)$ is the Regge trajectory which is assumed to be linear, j_0 and α' are its intercept and slope, respectively. The signature factor ξ_p is a complex quantity depending on the Reggeon signature $p = \pm 1$. A special Reggeon–Pomeron is introduced to explain an approximately constant behavior of total cross-sections at high energies and a fulfillment of the Pomeranchuck theorem $\sigma_{h\bar{h}}/\sigma_{hh} \to 1$. Its signature p is positive and its intercept is close to unity $j_0^p = 1 + \Delta$, $\Delta \ll 1$. The field theory of the Pomeron interactions was constructed by Gribov around 40 years ago.

The particle production at high energies can be investigated in the multi-Regge kinematics (see Fig. 4)

$$s \gg s_1\,, \qquad s_2\,, \ldots\,, \qquad s_{n+1} \gg t_1\,, \qquad t_2\,, \ldots\,, \qquad t_{n+1}\,, \tag{16}$$

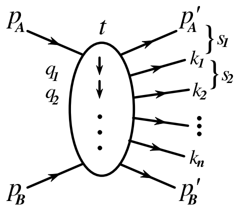

Fig. 4.

where s_r are squares of the sum of neighboring particle momenta k_{r-1}, k_r and $-t_r$ are squares of the momentum transfers $\vec{q}_r$. This amplitude can be also expressed in terms of the Reggeon exchanges in each of t_r-channels

$$A_{2\to 2+n} \sim \prod_{r=1}^{n+1} s_r^{j_p(t_r)} \,. \tag{17}$$

2. High energy scattering in QCD

2.1. Leading logarithmic approximation

In the Born approximation of QCD the scattering amplitude for two colored particle scattering in the Regge kinematics is factorized (see Fig. 2)

$$M_{AB}^{A'B'}(s,t)\Big|_{\text{Born}} = \Gamma_{A'A}^{c}\,\frac{2s}{t}\,\Gamma_{B'B}^{c}\,, \qquad \Gamma_{A'A}^{c} = g\,T_{A'A}^{c}\,\delta_{\lambda_{A'}\lambda_A}\,, \tag{18}$$

where T^c are the generators of the color group SU(N_c) in the corresponding representation and λ_r are helicities of the colliding and final state particles. The helicity conservation is related to the fact, that the t-channel gluon for small $t = -\vec{q}^{\,2}$ interacts with the conserved color charge. As a result of that the matrix elements of this charge between different states are zero and do not depend on their spin projections.

In LLA the scattering amplitude in QCD turns out to be of the Regge form [6]

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)\Big|_{\text{Born}}\, s^{\omega(t)}\,, \qquad \alpha_{\text{s}} \ln s \sim 1\,, \tag{19}$$

where the gluon Regge trajectory is

$$\omega(-|q|^2) = -\int \frac{d^2k}{4\pi^2}\,\frac{\alpha_{\text{s}} N_c\,|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_{\text{s}} N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}\,. \tag{20}$$

Here the fictious gluon mass λ was introduced to regularize the infrared divergence. This trajectory was calculated also in two-loop approximation in QCD [7] and in supersymmetric gauge theories [8].

Further, the gluon production amplitude in the multi-Regge kinematics can be written in the factorized form [6]

$$M_{2\to 1+n} = 2s\Gamma^{c_1}_{A'A}\frac{s_1^{\omega_1}}{|q_1|^2}gT^{d_1}_{c_2c_1}C(q_2,q_1)\frac{s_2^{\omega_2}}{|q_2|^2}\dots C(q_n,q_{n-1})\frac{s_n^{\omega_n}}{|q_n|^2}\Gamma^{c_n}_{B'B}\,. \quad (21)$$

The Reggeon–Reggeon–gluon vertex for the produced gluon with a definite helicity is

$$C(q_2,q_1) = \frac{q_2\,q_1^*}{q_2^*-q_1^*}\,, \quad (22)$$

where we used the complex notations for the transverse components of particle momenta.

2.2. Effective action for reggeized gluons

Initially calculations of scattering amplitudes in Regge kinematics were performed by an iterative method based on analyticity, unitarity and renormalizability of the theory [6]. The s-channel unitarity was incorporated partly in the form of bootstrap equations for the amplitudes generated by reggeized gluons exchange. But later it turned out that for this purpose one can also use an effective field theory for reggeized gluons [9, 10].

We shall write below the effective action valid at high energies for interactions of particles inside each cluster having their rapidities y in a certain interval

$$y = \frac{1}{2}\ln\frac{\epsilon_k+|k|}{\epsilon_k-|k|}\,, \qquad |y-y_0|<\eta\,, \qquad \eta \ll \ln s\,. \quad (23)$$

The corresponding gluon and quark fields are

$$v_\mu(x) = -iT^a v_\mu^a(x)\,, \qquad \psi(x)\,, \qquad \bar{\psi}(x)\,, \qquad \left[T^a,T^b\right] = if_{abc}T^c\,. \quad (24)$$

In the case of the supersymmetric models one can take into account also the fermion and scalar fields with known Yang–Mills and Yukawa interactions. Let us introduce now the fields describing the production and annihilation of reggeized gluons [9]

$$A_\pm(x) = -iT^aA^a_\pm(x)\,. \quad (25)$$

Under the global color group rotations the fields are transformed in the standard way

$$\delta v_\mu(x) = [v_\mu(x),\chi]\,, \qquad \delta\psi(x) = -\chi\,\psi(x)\,, \qquad \delta A(x) = [A(x),\chi]\,, \quad (26)$$

but under the local gauge transformations with $\chi(x)\to 0$ at $x\to\infty$ we have

$$\delta v_\mu(x) = \frac{1}{g}[D_\mu,\chi(x)]\,, \qquad \delta\psi(x) = -\chi(x)\,\psi(x)\,, \qquad \delta A_\pm(x) = 0\,. \quad (27)$$

In quasi-multi-Regge kinematics particles are produced in groups (clusters) with fixed masses. These groups have significantly different rapidities corresponding to the multi-Regge asymptotics. In this case one obtains the following kinematical constraint on the Reggeon fields

$$\partial_\mp A_\pm(x) = 0\,, \qquad \partial_\pm = n^\mu_\pm \partial_\mu\,, \tag{28}$$

$n^\mu_\pm = \delta^\mu_0 \pm \delta^\mu_3$. For QCD the corresponding effective action local in the rapidity y has the form [9]

$$S = \int d^4x\,(L_0 + L_{\rm ind})\;, \tag{29}$$

where L_0 is the usual Yang–Mills Lagrangian

$$L_0 = i\bar\psi\hat{D}\psi + \frac{1}{2}{\rm Tr}\,G^2_{\mu\nu}, \qquad D_\mu = \partial_\mu + gv_\mu,\; G_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu] \tag{30}$$

and the induced contribution is given below

$$L_{\rm ind} = {\rm Tr}\,(L^k_{\rm ind} + L^{GR}_{\rm ind})\,, \qquad L^k_{\rm ind} = -\partial_\mu A^a_+ \partial_\mu A^a_-\,. \tag{31}$$

Here the Reggeon–gluon interaction can be presented in terms of Wilson P-exponents

$$\begin{aligned} L^{GR}_{\rm ind} &= -\frac{1}{g}\partial_+\,P\exp\left(-g\frac{1}{2}\int\limits_{-\infty}^{x^+} v_+\left(x'\right) d\left(x'\right)^+\right)\partial^2_\sigma A_- \\ &\quad -\frac{1}{g}\partial_-\,P\exp\left(-g\frac{1}{2}\int\limits_{-\infty}^{x^-} v_-\left(x'\right) d\left(x'\right)^+\right)\partial^2_\sigma A_+ \\ &= \left(v_+ - gv_+\frac{1}{\partial_+}v_+ + g^2 v_+\frac{1}{\partial_+}v_+\frac{1}{\partial_+}v_+ - \ldots\right)\partial^2_\sigma A_- \\ &\quad + \left(v_- - gv_-\frac{1}{\partial_-}v_- + g^2 v_-\frac{1}{\partial_-}v_-\frac{1}{\partial_-}v_- - \ldots\right)\partial^2_+ A_\sigma\,. \end{aligned} \tag{32}$$

One can formulate the Feynman rules directly in momentum space [10]. For this purpose it is needed to take into account the gluon momentum conservation for induced vertices

$$k^\pm_0 + k^\pm_1 + \ldots + k^\pm_r = 0\,. \tag{33}$$

Some simple examples of induced Reggeon–gluon vertices are

$$\Delta^{\nu_0+}_{a_0c} = \vec{q}^{\,2}_{\perp}\,\delta_{a_0c}\,\left(n^+\right)^{\nu_0}\,, \qquad \Delta^{\nu_0\nu_1+}_{a_0a_1c} = \vec{q}^{\,2}_{\perp}\,T^c_{a_1a_0}\,\left(n^+\right)^{\nu_1}\frac{1}{k_1^+}(n^+)^{\nu_0}\,, \tag{34}$$

$$\Delta^{\nu_0\nu_1\nu_2+}_{a_0a_1a_2c} = \vec{q}^{\,2}_{\perp}\,\left(n^+\right)^{\nu_0}\left(n^+\right)^{\nu_1}\left(n^+\right)^{\nu_2}\left(\frac{T^a_{a_2a_0}\,T^c_{a_1a}}{k_1^+k_2^+}+\frac{T^a_{a_2a_1}\,T^c_{a_0a}}{k_0^+k_2^+}\right)\,. \tag{35}$$

They can be used for the construction of the effective vertices for the gluon–gluon–Reggeon and gluon–Reggeon–Reggeon interactions [9]. In the general case the induced vertices factorize in the form

$$\Delta^{\nu_0\nu_1\ldots\nu_r+}_{a_0a_1\ldots a_rc} = (-1)^r\vec{q}^{\,2}_{\perp}\prod_{s=0}^{r}\left(n^+\right)^{\nu_s}\,2\,\mathrm{Tr}\left(T^cG_{a_0a_1\ldots a_r}\right)\,, \tag{36}$$

where T^c are the color generators in the fundamental representation. In more detail, $G_{a_0a_1\ldots a_r}$ can be written as [10]

$$G_{a_0a_1\ldots a_r} = \sum_{\{i_0,i_1,\ldots,i_r\}}\frac{T^{a_{i_0}}T^{a_{i_1}}T^{a_{i_2}}\ldots T^{a_{i_r}}}{k^+_{i_0}(k^+_{i_0}+k^+_{i_1})\ldots(k^+_{i_0}+k^+_{i_1}+\ldots+k^+_{i_{r-1}})}\,. \tag{37}$$

These vertices satisfy the following recurrent relations (Ward identities) [9]

$$\begin{aligned} &k_r^+\,\Delta^{\nu_0\nu_1\ldots\nu_r+}_{a_0a_1\ldots a_rc}\left(k_0^+,\ldots,k_r^+\right) \\ &= -\left(n^+\right)^{\nu_r}\sum_{i=0}^{r-1} i f_{aa_ra_i}\,\Delta^{\nu_0\ldots\nu_{r-1}+}_{a_0\ldots a_{i-1}aa_{i+1}\ldots a_{r-1}c}\left(k_0^+,\ldots,k^+_{i-1},k_i^++k_r^+,k^+_{i+1},\ldots\right)\,. \end{aligned} \tag{38}$$

With the use of this effective theory one can calculate the tree amplitude for the production of a cluster of gluons, or a gluon and a pair of fermions or scalar particles (in the case of an extended supersymmetric model) in the collision of two reggeized gluons [10]. The square of the amplitude for three particle production integrated over the momenta of these particles is the main ingredient to construct the corresponding contribution to the BFKL kernel in the next-to-next-to-leading approximation. In principle it is also possible to calculate the loop corrections to the Reggeon-particle vertices with the use of the effective action, however for $N=4$ SUSY one can use also for this purpose amplitudes presented by Bern, Dixon and Smirnov in [11].

The production amplitudes give a possibility to calculate the total cross-section [6]

$$\sigma_{\rm t} = \sum_n\int d\Gamma_n\left|M_{2\to 1+n}\right|^2\,, \tag{39}$$

where Γ_n is the phase space for the produced particle momenta in the multi-Regge kinematics.

3. BFKL dynamics and integrability

3.1. Möbius invariance

Using the fact, that the production amplitudes in QCD are factorized, one can write a Bethe–Salpeter-type equation for the total cross-section $\sigma_{\rm t}$. Using also the optical theorem it can be presented as the equation of Balitsky, Fadin, Kuraev and Lipatov (BFKL) for the Pomeron wave function [6]

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,, \qquad \Delta = -\frac{\alpha_{\rm s}N_c}{2\pi}\,E\,, \tag{40}$$

where $\sigma_{\rm t} \sim s^{\Delta}$ and the BFKL Hamiltonian in the coordinate representation is [12]

$$\begin{aligned} H_{12} \;=\; & \ln|p_1p_2|^2 + \frac{1}{p_1p_2^*}\left(\ln|\rho_{12}|^2\right)p_1p_2^* \\ & + \frac{1}{p_1^*p_2}(\ln|\rho_{12}|^2)p_1^*p_2 - 4\psi(1)\,, \qquad \rho_{12} = \rho_1 - \rho_2\,. \end{aligned} \tag{41}$$

Here we used the complex notations for transverse coordinates and their canonically conjugated momenta

$$\rho_r = x_r + iy_r\,, \qquad \rho_r^* = x_r - iy_r\,, \qquad p_r = i\partial_r\,, \qquad p_r^* = i\partial_r^*\,. \tag{42}$$

The Hamiltonian is invariant under the Möbius transformations [13]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,. \tag{43}$$

The corresponding generators are

$$\vec{M} = \sum_r \vec{M}_r\,, \qquad M_r^3 = \rho_r\partial_r\,, \qquad M_r^+ = \partial_r\,, \qquad M_r^- = -\rho_r^2\partial_r\,. \tag{44}$$

The eigenvalues of the Casimir operators of the Möbius group

$$M^2 = \left(\sum_r \vec{M}_r\right)^2 = -\sum_{r<r'}\rho_{rr'}^2\partial_r\partial_{r'}\,, \qquad M^{*2} = \left(M^2\right)^*\,, \tag{45}$$

are $m(m-1)$ and $\widetilde{m}(\widetilde{m}-1)$, where m and $\widetilde{m}$ are the conformal weights defined by the relations [13]

$$m = \gamma + \frac{n}{2}\,, \qquad \widetilde{m} = \gamma - \frac{n}{2}\,, \qquad \gamma = \frac{1}{2} + i\nu\,, \tag{46}$$

for the principal series of unitary representations of the Möbius group. Here γ is the anomalous dimension of the twist-2 operators and n is the conformal spin. The eigenfunctions of the Casimir operators and the Hamiltonian for the BFKL Pomeron are known [13]

$$\Psi_{m,\widetilde{m}}(\vec{\rho}_1,\vec{\rho}_2) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}} . \tag{47}$$

The corresponding eigenvalues E of H_{12} are

$$E_{m,\widetilde{m}} = \epsilon_m + \epsilon_{\widetilde{m}} \,, \qquad \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1) \,, \tag{48}$$

where $\psi(m) = (\ln \Gamma(m))'$ has the integral representation

$$\psi(m) - \psi(1) = \int_0^1 dx\, \frac{1-x^{m-1}}{1-x} \,. \tag{49}$$

The ground state energy is negative

$$E_0 = E_{\frac{1}{2},\frac{1}{2}} = -8\ln 2 \,, \tag{50}$$

and therefore the intercept Δ of the BFKL Pomeron in LLA is positive and the Froissart theorem in this approximation is not fulfilled. We should take into account also the diagrams with many reggeized gluons in t-channel.

3.2. Large-N_c limit and holomorphic factorization

The Bartels–Kwiecinski–Praszalowicz equation for colorless composite states of several reggeized gluons has the following form [14]

$$E\Psi\left(\vec{\rho}_1,\ldots,\vec{\rho}_n\right) = H\Psi\left(\vec{\rho}_1,\ldots,\vec{\rho}_n\right) \,, \qquad H = \sum_{k<l} \frac{\vec{T}_k\vec{T}_l}{-N_c} H_{kl} \,, \tag{51}$$

where H_{kl} is the BFKL Hamiltonian. To simplify the structure of the equation for colorless composite states in a general case of n reggeized gluons we consider the multi-color limit$N_c \to \infty$ [15]. According to 't Hooft only planar diagrams are essential in the multi-color QCD. In our case it means, that providing, that we describe the color structure of the gluon r by an hermitian matrix $A_a T_r^a$ of the rank N_c with its Green function presented by a pair of quark and anti-quark lines, only cylinder-type diagrams in the t-channel survive. It is enough to consider the irreducible contribution for which the wave function has the following color structure

$$\Psi_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \dots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) = \sum_{\{i_1,\dots,i_n\}} f_{m,\widetilde{m}}(\overrightarrow{\rho_{i_1}}, \dots, \overrightarrow{\rho_{i_n}}; \overrightarrow{\rho_0})\, tr\,(T^{a_{i_1}}, \dots, T^{a_{i_n}})\ , \quad (52)$$

where the summation is performed over all permutations $\{i_1 \dots i_n\}$ of gluons $1, 2, \dots, n$. We consider the solutions with the definite values of the conformal weights m, $\widetilde{m}$. At large N_c each term in the sum satisfies the Schrödinger equation and therefore for the function $f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, \dots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0})$ symmetric under the cyclic transmutations

$$f_{m,\widetilde{m}}\,(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, \dots \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) = f_{m,\widetilde{m}}(\overrightarrow{\rho_n}, \overrightarrow{\rho_1}, \dots \overrightarrow{\rho_{n-1}}; \overrightarrow{\rho_0}) \quad (53)$$

we obtain the simplified BKP equation

$$E_{m,\widetilde{m}}\, f_{m,\widetilde{m}} = H\, f_{m,\widetilde{m}}\,, \qquad H = \frac{1}{2}\sum_{r=1}^{n} H_{r,r+1}\,. \quad (54)$$

For $N_c \to \infty$ only neighboring gluons interact each with another and the factor $1/2$ is related to the fact, that each pair of these gluons is in an adjoint representation. Here it was implied, that $H_{n,n+1} = H_{1,n}$.

It is remarkable, that the Hamiltonian H in the multicolor QCD has the property of the holomorphic separability [15]:

$$H = \tfrac{1}{2}(h + h^*)\,, \qquad [h, h^*] = 0\,, \quad (55)$$

where the holomorphic and anti-holomorphic Hamiltonians

$$h = \sum_{k=1}^{n} h_{k,k+1}\ , \qquad h^* = \sum_{k=1}^{n} h^*_{k,k+1} \quad (56)$$

are expressed in terms of the corresponding BFKL Hamiltonians [12]:

$$h_{k,k+1} = \ln(p_k\, p_{k+1}) + p_k^{-1}\ln(\rho_{k,k+1})\, p_k + p_{k+1}^{-1}\ln(\rho_{k,k+1})\, p_{k+1} + 2\,\gamma\,. \quad (57)$$

Owing to the holomorphic separability of H, the wave function $f_{m,\widetilde{m}}$ has the property of the holomorphic factorization [15]:

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \dots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) = \sum_{r,l} c_{r,l}\, f^r_m(\rho_1, \dots, \rho_n; \rho_0)\ f^l_{\widetilde{m}}\,(\rho_1^*, \dots, \rho_n^*; \rho_0^*)\ , \quad (58)$$

where r and l enumerate degenerate solutions of the Schrödinger equations in the holomorphic and anti-holomorphic sub-spaces:

$$\epsilon_m\, f_m = h\, f_m\,, \qquad \epsilon_{\widetilde{m}}\, f_{\widetilde{m}} = h^*\, f_{\widetilde{m}}\,, \qquad E_{m,\widetilde{m}} = \tfrac{1}{2}(\epsilon_m + \epsilon_{\widetilde{m}})\,. \quad (59)$$

Similarly to the case of two-dimensional conformal field theories, the coefficients $c_{r,l}$ are fixed by the single-valuedness condition for the wave function $f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2},\ldots,\overrightarrow{\rho_n};\overrightarrow{\rho_0})$ in the two-dimensional $\overrightarrow{\rho}$-space. Note, that in these conformal models the holomorphic factorization of the Green functions is a consequence of the invariance of the operator algebra under the infinitely dimensional Virasoro group.

3.3. Integrability of the BKP equation

It is easily verified, that for the pair Hamiltonian one can write another representation [12]

$$h_{k,k+1}=\rho_{k,k+1}\,\ln(p_k\,p_{k+1})\,\rho_{k,k+1}^{-1}+2\ln(\rho_{k,k+1})+2\,\gamma\,. \tag{60}$$

As a result, there are two different normalization conditions for the wave function

$$\|f\|_1^2=\int\prod_{r=1}^{n}d^2\rho_r\left|\prod_{r=1}^{n}\rho_{r,r+1}^{-1}\,f\right|^2,\qquad \|f\|_2^2=\int\prod_{r=1}^{n}d^2\rho_r\left|\prod_{r=1}^{n}p_r\,f\right|^2 \tag{61}$$

compatible with the hermiticity properties of H. It is related to the fact, that the transposed Hamiltonian h^{t} is related to h by two different similarity transformations [12]

$$h^{\mathrm{t}}=\prod_{r=1}^{n}p_r\,h\,\prod_{r=1}^{n}p_r^{-1}=\prod_{r=1}^{n}\rho_{r,r+1}^{-1}\,h\,\prod_{r=1}^{n}\rho_{r,r+1}\,. \tag{62}$$

Therefore h commutes

$$[h,A]=0 \tag{63}$$

with the differential operator [12]

$$A=\rho_{12}\rho_{23}\ldots\rho_{n1}\,p_1p_2\ldots p_n\,. \tag{64}$$

Furthermore [17], there is a family $\{q_r\}$ of mutually commuting integrals of motion

$$[q_r,q_s]=0\,,\;\;[q_r,h]=0\,. \tag{65}$$

They are given below

$$q_r=\sum_{i_1<i_2<\ldots<i_r}\rho_{i_1i_2}\,\rho_{i_2i_3}\ldots\rho_{i_ri_1}\,p_{i_1}\,p_{i_2}\ldots p_{i_r}\,. \tag{66}$$

In particular, q_n is equal to A and q_2 is proportional to the Casimir operator of the Möbius group M^2.

The generating function for these integrals of motion coincides with the transfer matrix $T(u)$ for the XXX model [17]

$$T(u) = \mathrm{tr}\,(L_1(u)L_2(u)\dots L_n(u)) = \sum_{r=0}^{n} u^{n-r}\, q_r\,, \tag{67}$$

where the L-operators are

$$L_k(u) = \begin{pmatrix} u+\rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u-\rho_k\, p_k \end{pmatrix}. \tag{68}$$

The transfer matrix is the trace of the monodromy matrix $t(u)$:

$$T(u) = \mathrm{tr}\,(t(u))\,, \qquad t(u) = L_1(u)L_2(u)\dots L_n(u)\,. \tag{69}$$

It can be verified that $t(u)$ satisfies the Yang–Baxter (YB) equation [17]

$$t^{s_1}_{r'_1}(u)\, t^{s_2}_{r'_2}(v)\, l^{r'_1 r'_2}_{r_1 r_2}(v-u) = l^{s_1 s_2}_{s'_1 s'_2}(v-u)\, t^{s'_2}_{r_2}(v)\, t^{s'_1}_{r_1}(u)\,, \tag{70}$$

where $l(w)$ is the L-operator for the well-known Heisenberg spin model

$$l^{s_1 s_2}_{s'_1 s'_2}(w) = w\, \delta^{s_1}_{s'_1}\, \delta^{s_2}_{s'_2} + i\, \delta^{s_1}_{s'_2}\, \delta^{s_2}_{s'_1}\,. \tag{71}$$

The commutativity of $T(u)$ and $T(v)$

$$[T(u), T(v)] = 0 \tag{72}$$

is a consequence of the YB equation.

3.4. Hidden Lorentz symmetry

If one will parametrize $t(u)$ in the form

$$t(u) = \begin{pmatrix} j_0(u)+j_3(u) & j_-(u) \\ j_+(u) & j_0(u)-j_3(u) \end{pmatrix}, \tag{73}$$

the YB equation is reduced to the following Lorentz-covariant relations for the currents $j_\mu(u)$:

$$[j_\mu(u), j_\nu(v)] = [j_\mu(v), j_\nu(u)] = \frac{i\,\epsilon_{\mu\nu\rho\sigma}}{2(u-v)}\left(j^\rho(u) j^\sigma(v) - j^\rho(v) j^\sigma(u)\right). \tag{74}$$

Here $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetric tensor ($\epsilon_{1230} = 1$) in the four-dimensional Minkowski space and the metric tensor $\eta^{\mu\nu}$ has the signature $(1,-1,-1,-1)$. This form is compatible with the invariance of the YB equations under the Lorentz transformations.

The generators for the spacial rotations coincide with that of the Möbius transformations $\overrightarrow{M}$. The commutation relations for the Lorentz algebra are given below:

$$\left[M^s, M^t\right] = i\epsilon_{stu}\, M^u\,, \qquad \left[M^s, N^t\right] = i\epsilon_{stu}\, N^u\,, \qquad \left[N^s, N^t\right] = i\epsilon_{stu}\, M^u\,, \tag{75}$$

where $\overrightarrow{N}$ are the Lorentz boost generators.

The commutativity of the transfer matrix $T(u)$ with the local Hamiltonian h

$$[T(u), h] = 0 \tag{76}$$

is a consequence of the relation:

$$[L_k(u)\, L_{k+1}(u), h_{k,k+1}] = -i\,(L_k(u) - L_{k+1}(u)) \tag{77}$$

for the pair Hamiltonian $h_{k,k+1}$, which is easily to verify by direct calculations.

In turn, this relation follows from the Möbius invariance of $h_{k,k+1}$ and from the identity:

$$\left[h_{k,k+1}, \left[\left(\overrightarrow{M_{k,k+1}}\right)^2, \overrightarrow{N_{k,k+1}}\right]\right] = 4\,\overrightarrow{N_{k,k+1}}\,, \tag{78}$$

where

$$\overrightarrow{M_{k,k+1}} = \overrightarrow{M_k} + \overrightarrow{M_{k+1}}\,, \qquad \overrightarrow{N_{k,k+1}} = \overrightarrow{M_k} - \overrightarrow{M_{k+1}} \tag{79}$$

are the Lorentz group generators for the two gluon state. To check this identity one should take into account, that the pair Hamiltonian $h_{k,k+1}$ depends only on the Casimir operator $\left(\overrightarrow{M_{k,k+1}}\right)^2$ and therefore it is diagonal

$$h_{k,k+1}\, |m_{k,k+1}\rangle = \left(\psi(m_{k,k+1}) + \psi(1 - m_{k,k+1}) - 2\,\psi(1)\right) |m_{k,k+1}\rangle \tag{80}$$

in the conformal weight representation:

$$\left(\overrightarrow{M_{k,k+1}}\right)^2 |m_{k,k+1}\rangle = m_{k,k+1}(m_{k,k+1} - 1)\, |m_{k,k+1}\rangle\,. \tag{81}$$

Further, using the commutation relations between $\overrightarrow{M_{k,k+1}}$ and $\overrightarrow{N_{k,k+1}}$ and taking into account that $\left(\overrightarrow{M_k}\right)^2 = 0$, one can verify that the operator $\overrightarrow{N_{k,k+1}}$ has non-vanishing matrix elements only between the states $|m_{k,k+1}\rangle$ and $|m_{k,k+1} \pm 1\rangle$. As a result, the above identity for $\overrightarrow{N_{k,k+1}}$ turns out to be a consequence of the known recurrence relations for the ψ-functions:

$$\psi(m) = \Psi(m-1) + 1/(m-1)\,, \qquad \psi(1-m) = \psi(2-m) + 1/(m-1)\,. \tag{82}$$

3.5. Algebraic Bethe ansatz

The pair Hamiltonian $h_{k,k+1}$ can be expressed in terms of a small-u asymptotics for the fundamental $\hat{L}$-operator of the integrable Heisenberg model with spins being the generators of the Möbius group [18]

$$\widehat{L}_{k,k+1}(u) = P_{k,k+1}(1 + i\,u\,h_{k,k+1} + \ldots)\,. \tag{83}$$

The fundamental $\hat{L}$-operator has the same representation in the auxiliary space acting on functions $f(\rho_k, \rho_{k+1})$ and $P_{k,k+1}$ is defined by the relation

$$P_{k,k+1}\; f(\rho_k, \rho_{k+1}) = f(\rho_{k+1}, \rho_k)\,. \tag{84}$$

The operator $\widehat{L}_{k,k+1}$ satisfies the linear YB equation

$$L_k(u)\,L_{k+1}(v)\;\widehat{L}_{k,k+1}(u-v) = \widehat{L}_{k,k+1}(u-v)\;L_{k+1}(v)\,L_k(u)\,. \tag{85}$$

This equation can be solved in terms of Γ-functions in a way similar to what was done above for $h_{k,k+1}$ and the proportionality constant is fixed from the triangle YB equation

$$\widehat{L}_{13}(u)\,\widehat{L}_{23}(v)\;\widehat{L}_{12}(u-v) = \widehat{L}_{12}(u-v)\;\widehat{L}_{23}(v)\,\widehat{L}_{13}(u)\,. \tag{86}$$

To find a representation of the operators obeying the Yang–Baxter commutation relations, the algebraic Bethe ansatz can be used [19]. To begin with, in the above parametrization of the monodromy matrix $t(u)$ in terms of the currents $j_\mu(u)$, one should construct the pseudovacuum state $|0\rangle$ satisfying the equations

$$j_+(u)\,|0\rangle = 0\,. \tag{87}$$

However, these equations have a non-trivial solution only if the above L-operators are regularized

$$L_k^\delta(u) = \begin{pmatrix} u + \rho_k\,p_k - i\,\delta & p_k \\ -\rho_k^2\,p_k + 2\,i\,\rho_k\delta & u - \rho_k\,p_k + i\delta \end{pmatrix} \tag{88}$$

by introducing a small conformal weight $\delta \to 0$ for reggeized gluons Another possibility is to use the dual space corresponding to $\delta = -1$ [18]. For the above regularization the pseudovacuum state is

$$|\delta\rangle = \prod_{k=1}^{n} \rho_k^{2\delta}\,. \tag{89}$$

It is also an eigenstate of the transfer matrix:

$$T(u)\,|\delta\rangle = 2\,j_0(u)\,|\delta\rangle = \left((u - i\,\delta)^n + (u + i\,\delta)^n\right)\,|\delta\rangle\,. \tag{90}$$

Furthermore, all excited states are obtained by applying to $|\delta\rangle$ the product of the operators $j_-(v)$

$$|v_1 v_2 \dots v_k\rangle = j_-(v_1)\, j_-(v_2) \dots j_-(v_k)\ |\delta\rangle\,. \tag{91}$$

They are eigenfunctions of the transfer matrix $T(u)$ with the eigenvalues:

$$\widetilde{T}(u) = (u + i\delta)^n \prod_{r=1}^{k} \frac{u - v_r - i}{u - v_r} + (u - i\delta)^n \prod_{r=1}^{k} \frac{u - v_r + i}{u - v_r}\,, \tag{92}$$

providing that the spectral parameters $v_1, v_2, \dots, v_k$ are chosen to be solutions of the set of the Bethe equations [19]

$$\left(\frac{v_s - i\delta}{v_s + i\delta}\right)^n = \prod_{r \neq s} \frac{v_s - v_r - i}{v_s - v_r + i} \tag{93}$$

for $s = 1, 2 \dots k$.

Due to the above relations the Baxter function defined as follows

$$Q(u) = \prod_{r=1}^{k} (u - v_r) \tag{94}$$

satisfies the Baxter equation [18, 19]

$$\widetilde{T}(u)\, Q(u) = (u - i\delta)^n\, Q(u + i) + (u + i\delta)^n\, Q(u - i)\,. \tag{95}$$

Here $\widetilde{T}(u)$ is an eigenvalue of the transfer matrix $T(u)$. One can slightly simplify the Baxter equation by choosing $\delta = -1$ [18].

The eigenfunctions of h and q_k can be expressed in terms of a solution $Q^{(k)}(u)$ of this equation using the Sklyanin ansatz [20]

$$|v_1 v_2 \dots v_k\rangle = Q^{(k)}(\widehat{u}_1)\ Q^{(k)}(\widehat{u}_2)\ \dots\ Q^{(k)}(\widehat{u}_{n-1})\, |\delta\rangle\,, \tag{96}$$

where the integral operators $\widehat{u}_r$ are zeros of the current $j_-(u)$

$$j_-(u) = c \prod_{r=1}^{n-1} (u - \widehat{u}_r)\,. \tag{97}$$

Thus, the problem of finding the wave functions and intercepts of composite states of reggeized gluons is reduced to the solution of the Baxter equation [18, 19]. We shall consider the Baxter–Sklyanin approach later.

3.6. Duality symmetry

The integrals of motion q_r and the Hamiltonian h are invariant under the cyclic permutation of gluon indices $i \to i+1$ ($i = 1, 2 \ldots n$), corresponding to the Bose symmetry of the Reggeon wave function at $N_c \to \infty$. It is remarkable that above operators are invariant also under the more general canonical transformation [16]:

$$\rho_{i-1,i} \to p_i \to \rho_{i,i+1}\,, \tag{98}$$

combined with reversing the order of the operator multiplication.

This duality symmetry is realized as an unitary transformation only for a vanishing total momentum:

$$\overrightarrow{p} = \sum_{r=1}^{n} \overrightarrow{p_r} = 0\,. \tag{99}$$

The wave function $\psi_{m,\widetilde{m}}$ of the composite state with $\overrightarrow{p} = 0$ can be written in terms of the eigenfunction $f_{m\widetilde{m}}$ of the integrals of motion q_k and q_k^* for $k = 1, 2 \ldots n$ as follows

$$\psi_{m,\widetilde{m}}(\overrightarrow{\rho_{12}}, \overrightarrow{\rho_{23}}, \ldots, \overrightarrow{\rho_{n1}}) = \int \frac{d^2\rho_0}{2\,\pi}\, f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, \ldots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0})\,. \tag{100}$$

Taking into account the hermiticity of the total Hamiltonian [17]:

$$H^+ = \prod_{k=1}^{n} |\rho_{k,k+1}|^{-2}\, H \prod_{k=1}^{n} |\rho_{k,k+1}|^2 = \prod_{k=1}^{n} |p_k|^2\, H \prod_{k=1}^{n} |p_k|^{-2}\,, \tag{101}$$

the solution $\psi^+_{\widetilde{m},m}$ of the complex-conjugated Schrödinger equation for $\overrightarrow{p} = 0$ can be expressed in terms of $\psi_{\widetilde{m},m}$ as follows :

$$\psi^+_{\widetilde{m},m}(\overrightarrow{\rho_{12}}, \overrightarrow{\rho_{23}}, \ldots) = \prod_{k=1}^{n} |\rho_{k,k+1}|^{-2} \left(\psi_{\widetilde{m},m}(\overrightarrow{\rho_{12}}, \overrightarrow{\rho_{23}}, \ldots)\right)^*\,. \tag{102}$$

Because $\psi_{m,\widetilde{m}}$ is also an eigenfunction of the integrals of motion $A = q_n$ and A^* with their eigenvalues λ_m and $\lambda_m^* = \lambda_{\widetilde{m}}$ [12]:

$$A\psi_{m,\widetilde{m}} = \lambda_m \psi_{m,\widetilde{m}}\,, \quad A^*\psi_{m,\widetilde{m}} = \lambda_{\widetilde{m}}\psi_{m,\widetilde{m}}\,, \quad A = \rho_{12}\ldots\rho_{n1}p_1\ldots p_n\,, \tag{103}$$

one can verify that the duality symmetry takes a form of the following integral equation for $\psi_{m,\widetilde{m}}$ [16]:

$$\frac{\psi_{m,\widetilde{m}}(\overrightarrow{\rho_{12}}, \ldots, \overrightarrow{\rho_{n1}})}{|\lambda_m|\,2^n} = \int \prod_{k=1}^{n-1} \frac{d^2\rho'_{k-1,k}}{2\pi} \prod_{k=1}^{n} \frac{e^{i\overrightarrow{\rho_{k,k+1}}\overrightarrow{\rho'_k}}}{\left|\rho'_{k,k+1}\right|^2}\, \psi^*_{\widetilde{m},m}(\overrightarrow{\rho'_{12}}, \ldots, \overrightarrow{\rho'_{n1}})\,. \tag{104}$$

In the next section we consider the application of this general approach to the three-gluon composite state in QCD.

4. Odderon states in QCD

4.1. Three-Reggeon state

In a particular case of the Odderon, being a composite state of three reggeized gluons with the charge parity $C = -1$ and signature $P_j = -1$, the color factor coincides with the known completely symmetric tensor d_{abc}. Therefore taking into account, that in this state each pair of gluons is projected into the adjoint representation, the equation for the coefficient $f_{m,\widetilde{m}}$ in front of d_{abc} is simplified as follows [14]

$$E_{m,\widetilde{m}}\, f_{m,\widetilde{m}} = \tfrac{1}{2}\,\left(H_{12} + H_{13} + H_{23}\right)\, f_{m,\widetilde{m}}\,. \tag{105}$$

The eigenvalue of this equation is related to the high-energy behavior of the difference of the total cross-sections σ_{pp} and $\sigma_{p\overline{p}}$ for interactions of particles p and anti-particles $\overline{p}$ with a target

$$\sigma_{pp} - \sigma_{p\overline{p}} \sim s^{\Delta_{m,\widetilde{m}}}\,. \tag{106}$$

Due to the Bose symmetry the wave function is completely symmetric

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2},\overrightarrow{\rho_3};\overrightarrow{\rho_0}) = f_{m,\widetilde{m}}(\overrightarrow{\rho_2},\overrightarrow{\rho_1},\overrightarrow{\rho_3};\overrightarrow{\rho_0}) = f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_3},\overrightarrow{\rho_2};\overrightarrow{\rho_0})\,. \tag{107}$$

Note, that the other solution proportional to a completely anti-symmetric tensor f_{abc} has the anti-symmetric wave function and describes the state with the Pomeron quantum numbers $C = P_j = 1$.

In the case of the Odderon the conformal invariance fixes the solution of the Schrödinger equation [13]

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2},\overrightarrow{\rho_3};\overrightarrow{\rho_0}) = \left(\frac{\rho_{12}\,\rho_{23}\,\rho_{31}}{\rho_{10}^2\,\rho_{20}^2\,\rho_{30}^2}\right)^{m/3}\left(\frac{\rho_{12}^*\,\rho_{23}^*\,\rho_{31}^*}{\rho_{10}^{*2}\,\rho_{20}^{*2}\,\rho_{30}^{*2}}\right)^{\widetilde{m}/3} f_{m,\widetilde{m}}(\overrightarrow{x}) \tag{108}$$

up to an arbitrary function $f_{m,\widetilde{m}}(\overrightarrow{x})$ of one complex variable x being the anharmonic ratio of four coordinates

$$x = \frac{\rho_{12}\,\rho_{30}}{\rho_{10}\,\rho_{32}}\,. \tag{109}$$

Owing to the Bose symmetry of the Odderon wave function $f_{m,\widetilde{m}}(\overrightarrow{x})$ has simple transformation properties under the substitutions $x \to 1 - x$, $x \to 1/x$ [16].

The wave function $\psi_{m,\widetilde{m}}(\overrightarrow{\rho_{ij}})$ at $\overrightarrow{q} = 0$ can be written as

$$\psi_{m,\widetilde{m}}(\overrightarrow{\rho_{ij}}) = \left(\frac{\rho_{23}}{\rho_{12}\rho_{31}}\right)^{m-1}\left(\frac{\rho_{23}^*}{\rho_{12}^*\rho_{31}^*}\right)^{\widetilde{m}-1}\chi_{m,\widetilde{m}}(\overrightarrow{z})\,, \qquad z = \frac{\rho_{12}}{\rho_{32}}\,, \tag{110}$$

where

$$\chi_{m,\widetilde{m}}(\overrightarrow{z}) = \int \frac{d^2x\, f_{m,\widetilde{m}}(\overrightarrow{x})}{2\,\pi\, |x-z|^4} \left(\frac{(x-z)^3}{x(1-x)}\right)^{2m/3} \left(\frac{(x^*-z^*)^3}{x^*(1-x^*)}\right)^{2\widetilde{m}/3} . \quad (111)$$

In fact this function is proportional to $f_{1-m,1-\widetilde{m}}(\overrightarrow{z})$:

$$\chi_{m,\widetilde{m}}(\overrightarrow{z}) \sim (x(1-x))^{2(m-1)/3}\, (x^*(1-x^*))^{2(\widetilde{m}-1)/3}\, f_{1-m,1-\widetilde{m}}(\overrightarrow{z})\,, \quad (112)$$

which is a certain realization of the linear dependence between two representations $(m,\widetilde{m})$ and $(1-m,1-\widetilde{m})$. The corresponding reality property for the Möbius group representations can be presented in a form of the integral relation

$$\chi_{m,\widetilde{m}}(\overrightarrow{z}) = \int \frac{d^2x}{2\,\pi}\, (x-z)^{2m-2}\, (x^*-z^*)^{2\widetilde{m}-2}\, \chi_{1-m,1-\widetilde{m}}(\overrightarrow{x}) \quad (113)$$

for an appropriate choice of phases for the functions $\chi_{m,\widetilde{m}}$ and $\chi_{1-m,1-\widetilde{m}}$.

4.2. Duality equation for Odderon

The duality equation for $\chi_{m,\widetilde{m}}(\overrightarrow{z})$ can be written in the pseudo-differential form [16]:

$$|z(1-z)|^2 (i\partial)^{2-m} (i\partial^*)^{2-\widetilde{m}} \varphi_{1-m,1-\widetilde{m}}(\overrightarrow{z}) = \left|\lambda_{m,\widetilde{m}}\right| \left(\varphi_{1-m,1-\widetilde{m}}(\overrightarrow{z})\right)^* , \quad (114)$$

where

$$\varphi_{1-m,1-\widetilde{m}}(\overrightarrow{z}) = \, (z(1-z))^{1-m}\, (z^*(1-z^*))^{1-\widetilde{m}}\, \chi_{m,\widetilde{m}}(\overrightarrow{z})\,. \quad (115)$$

The normalization condition for the wave function

$$\left\|\varphi_{m,\widetilde{m}}\right\|^2 = \int \frac{d^2x}{|x(1-x)|^2}\, \left|\varphi_{m,\widetilde{m}}(\overrightarrow{x})\right|^2 \quad (116)$$

is compatible with the duality symmetry.

For the holomorphic factors $\varphi^{(m)}(x)$ the duality equations have the simple form [16]:

$$a_m \varphi^{(m)}(x) = \lambda^m \varphi^{(1-m)}(x)\,, \qquad a_{1-m} \varphi^{(1-m)}(x) = \lambda^{1-m} \varphi^{(m)}(x)\,, \quad (117)$$

where

$$a_m = x(1-x)\, p^{1+m}\,, \qquad p = i\frac{\partial}{\partial x}\,. \quad (118)$$

If we consider p as a coordinate and $x-1/2$ as a momentum, the duality equation for the most important case $m=1/2$ can be reduced to the Schrödinger equation with the potential $V(p)=\sqrt{\lambda}\,p^{-3/2}$ [16].

For each eigenvalue λ there are three independent solutions $\varphi_i^{(m)}(x,\lambda)$ of the third-order ordinary differential equation corresponding to the diagonalization of the operator A [12]

$$-ix(1-x)\left(x(1-x)\partial^2+(2-m)\left((1-2x)\partial-1+m\right)\right)\partial\,\varphi=\lambda\,\varphi\,. \quad (119)$$

In the region $x\to 0$ they can be chosen as follows [21]

$$\varphi_r^{(m)}(x,\lambda) \;=\; \sum_{k=1}^{\infty} d_k^{(m)}(\lambda)\;x^k\,, \qquad d_1^{(m)}(\lambda)=1\,. \quad (120)$$

$$\varphi_s^{(m)}(x,\lambda) \;=\; \sum_{k=0}^{\infty} a_k^{(m)}(\lambda),\;x^k+\,\varphi_r^{(m)}(x,\lambda)\;\ln x\,, \qquad a_1^{(m)}=0\,, \quad (121)$$

$$\varphi_f^{(m)}(x,\lambda) \;=\; \sum_{k=0}^{\infty} c_{k+m}^{(m)}(\lambda)\;x^{k+m}\,, \qquad c_m^{(m)}(\lambda)=1\,. \quad (122)$$

Due to the above differential equation the coefficients a_k, c_k and d_k satisfy certain recurrence relations. From the single-valuedness condition near $\overrightarrow{x}=0$, we obtain the following representation for the wave function in x,x^*-space:

$$\varphi_{m,\widetilde{m}}(x,x^*)=\varphi_f^{(m)}(x,\lambda)\varphi_f^{(\widetilde{m})}(x^*,\lambda^*)+c_2\varphi_r^{(m)}(x,\lambda)\varphi_r^{(\widetilde{m})}(x^*,\lambda^*)$$
$$+c_1\Big(\varphi_s^{(m)}(x,\lambda)\,\varphi_r^{(\widetilde{m})}(x^*,\lambda^*)+\varphi_r^{(m)}(x,\lambda)\varphi_s^{(\widetilde{m})}(x^*,\lambda^*)\Big)+(\lambda\to-\lambda)\,. \quad (123)$$

The complex coefficients c_1,c_2 and the eigenvalues λ are fixed from the single-valuedness condition for $f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2},\overrightarrow{\rho_3};\overrightarrow{\rho_0})$ at $\overrightarrow{\rho_3}=\overrightarrow{\rho_i}$ $(i=1,2)$ and the Bose symmetry [21].

With the use of the duality equation we have [16]

$$|c_1|=|\lambda|\,. \quad (124)$$

Another relation

$$\mathrm{Im}\frac{c_2}{c_1}=\mathrm{Im}\,(m^{-1}+\widetilde{m}^{-1})\,. \quad (125)$$

can be derived [16] if we shall take into account, that the complex conjugated representations $\varphi_{m,\widetilde{m}}$ and $\varphi_{1-m,1-\widetilde{m}}$ of the Möbius group are related by the above discussed linear transformation. One can verify from the numerical results of Ref. [21] that both relations for c_1 and c_2 are fulfilled.

If we introduce for general n the time-dependent pair Hamiltonian $h_{k,k+1}(t)$ in the form

$$h_{k,k+1}(t) = \exp(i\,T(u)\,t)\ h_{k,k+1}\ \exp(-i\,T(u)\,t)\,, \tag{126}$$

in the total holomorphic Hamiltonian h we can substitute

$$h_{k,k+1} \to h_{k,k+1}(t) \tag{127}$$

due to the commutativity of h and $T(u)$. On the other hand, as a result of the rapid oscillations at $t \to \infty$ each pair Hamiltonian $h_{k,k+1}(t)$ is diagonalized in the representation, where the transfer matrix $T(u)$ is diagonal:

$$h_{k,k+1}(\infty) = f_{k,k+1}(\widehat{q_2}, \widehat{q_3}, \ldots \widehat{q_n})\,. \tag{128}$$

4.3. Odderon Hamiltonian

Generally one can present the holomorphic Hamiltonian for n reggeized gluons in the form explicitly invariant under the Möbius transformations [16]

$$h = \sum_{k=1}^{n} \left(\log\left(\frac{\rho_{k+2,0}\,\rho^2_{k,k+1}}{\rho_{k+1,0}\,\rho_{k+1,k+2}}\partial_k \right) + \log\left(\frac{\rho_{k-2,0}\,\rho^2_{k,k-1}}{\rho_{k-1,0}\,\rho_{k-1,k-2}}\partial_k \right) - 2\psi(1) \right) \tag{129}$$

by introducing the coordinate ρ_0 of the composite state.

In the case of the Odderon $h_{k,k+1}(\infty)$ is a function of the total conformal momentum M^2 and of the integral of motion $q_3 = A$ which can be written as follows:

$$A = \frac{i^3}{2}\left[M^2_{12}\,,\, M^2_{13}\right] = \frac{i^3}{2}\left[M^2_{23}\,,\, M^2_{12}\right] = \frac{i^3}{2}\left[M^2_{13}\,,\, M^2_{23}\right]\,. \tag{130}$$

Let us attempt to simplify the equation for the Odderon using for its wave function the conformal ansatz

$$f_m(\rho_1, \rho_2, \rho_3; \rho_0) = \left(\frac{\rho_{23}}{\rho_{20}\rho_{30}}\right)^m \varphi_m(x)\,, \qquad x = \frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}\,, \tag{131}$$

one can obtain the following Hamiltonian for the function $\varphi_m(x)$ in the space of the anharmonic ratio x [12]

$$\begin{aligned} h \;=\; & 6\gamma + \log\left(x^2\partial\right) + \log\left((1-x)^2\,\partial\right) + \log\left(x^2\left(\partial + \frac{m}{1-x}\right)\right) \\ & + \log\left(\partial + \frac{m}{1-x}\right) + \log\left((1-x)^2\left(\partial - \frac{m}{x}\right)\right) + \log\left(\partial - \frac{m}{x}\right)\,. \end{aligned} \tag{132}$$

It is convenient to introduce the logarithmic derivative $P \equiv x\partial$ as a new momentum. With the use of the relations of the type

$$\log(\partial) = -\log(x) + \psi(-x\partial)\,, \qquad \log(x^2\partial) = \log(\partial) + 2\log(x) - \frac{1}{P}\,,$$

and expanding the functions in a series over x, one can obtain the Odderon Hamiltonian in the normal order [16]:

$$\frac{h}{2} = -\log(x)+\psi(1-P)+\psi(-P)+\psi(m-P)-3\psi(1)+\sum_{k=1}^{\infty} x^k\, f_k(P)\,, \tag{133}$$

where

$$f_k(P) = -\frac{2}{k} + \frac{1}{2}\left(\frac{1}{P+k-m} + \frac{1}{P+k}\right) + \sum_{t=0}^{k} \frac{c_t(k)}{P+t}\,. \tag{134}$$

Here

$$c_t(k) = \frac{(-1)^{k-t}\;\Gamma(m+t)\;((t-k)\,(m+t)+m\,k/2)}{k\,\Gamma(m-k+t+1)\,\Gamma(t+1)\,\Gamma(k-t+1)}\,. \tag{135}$$

4.4. Expansion in the inversed integral of motion

Because h and $B = iA$ commute each with another, h is a function of B. In particular, for large B this function should have the form:

$$\frac{h}{2} = \log(B) + 3\gamma + \sum_{r=1}^{\infty} \frac{c_r}{B^{2r}}\,. \tag{136}$$

The first two terms of this asymptotic expansion were calculated in [12]. The series is constructed in inverse powers of B^2, because h should be invariant under all modular transformations, including the inversion $x \to 1/x$ under which B changes its sign. The same functional relation should be valid for the eigenvalues $\varepsilon/2$ and $\mu = i\,\lambda$ of these operators.

For large μ it is convenient to consider the corresponding eigenvalue equations in the P representation, where x is the shift operator

$$x = \exp\left(-\frac{d}{dP}\right)\,, \tag{137}$$

after extracting from eigenfunctions of B and h the common factor

$$\varphi_m(P) = \Gamma(-P)\,\Gamma(1-P)\,\Gamma(m-P)\,\exp(i\pi P)\,\Phi_m(P)\,. \tag{138}$$

The function $\varPhi_m(P)$ can be expanded in series over $1/\mu$

$$\varPhi_m(P) = \sum_{n=0}^{\infty} \mu^{-n} \varPhi_m^n(P)\,,\ \varPhi_m^0(P) = 1\,, \tag{139}$$

where the coefficients $\varPhi_m^n(P)$ turn out to be the polynomials of order $4n$ satisfying the recurrence relation:

$$\begin{aligned}
&\varPhi_m^n(P) = \\
&\sum_{k=1}^{P}(k-1)(k-1-m)\left((k-m)\varPhi_m^{n-1}(k-1)+(k-2)\varPhi_{1-m}^{n-1}(k-1-m)\right) \\
&-\frac{1}{2}\sum_{k=1}^{m}(k-1)(k-1-m)\left((k-m)\varPhi_m^{n-1}(k-1)+(k-2)\varPhi_{1-m}^{n-1}(k-1-m)\right)\,,
\end{aligned} \tag{140}$$

valid due to the duality equations written below after the substitution $x\mu \to x$ for a definite choice of the phase of $\varPhi_m(P)$

$$\begin{aligned}
\varPhi_{1-m}(P+1-m)-\frac{1}{\mu}P(P-1)(P-m)\varPhi_{1-m}(P-m) &= \varPhi_m(P)\,, \\
\varPhi_m(P+m)-\frac{1}{\mu}P(P-1)(P+m-1)\varPhi_m(P+m-1) &= \varPhi_{1-m}(P)\,.
\end{aligned} \tag{141}$$

Indeed, we obtain after changing the argument $P \to P-m$ in the second equation and adding it with the first one

$$\begin{aligned}
&\varPhi_m(P) - \varPhi_m(P-1) = \\
&\frac{1}{\mu}(P-1)(P-1-m)\left((P-m)\varPhi_m(P-1)+(P-2)\varPhi_{1-m}(P-m-1)\right)\,,
\end{aligned}$$

which leads to the above recurrence relation.

Note that the summation constants $\varPhi_m^n(0)$ in this recurrence relation have the anti-symmetry property

$$\varPhi_m^n(0) = -\varPhi_{1-m}^n(0)\,, \tag{142}$$

which guarantees the fulfilment of the relation

$$\varPhi_m^n(m) = \varPhi_{1-m}^n(0) \tag{143}$$

being a consequence of the duality relation. On the other hand, taking into account the last relation and the fact, that $r_m = \varPhi_{1-m}^n(0) - \varPhi_m^n(0)$ is an anti-symmetric function, we can chose $\varPhi_m^n(0) = -r_m/2$ because adding a symmetric contribution will redefine the initial condition $\varPhi_m^0(P) = 1$. The most general solution of the duality equation is our function $\varPhi_m(P)$ multiplied by an arbitrary symmetric constant.

4.5. Expression for the Odderon energy

With the use of the recurrence relation in particular we have

$$\Phi_m(1) = \Phi_m(0)\,.$$

The Odderon energy can be expressed in terms of $\Phi_m(P)$ as follows

$$\begin{aligned}\frac{\varepsilon}{2} &= \log(\mu) + 3\gamma + \frac{\partial}{\partial P}\log\Phi_m(P) \\ &\quad + \sum_{k=1}^{\infty}\mu^{-k} f_k(P-k)\frac{\Phi_m(P-k)}{\Phi_m(P)}\prod_{r=1}^{k}(P-r)(P-r+1)(P-r-m+1) \qquad (144)\end{aligned}$$

and this expression does not depend on P due to the commutativity of h and B.

Because for $P \to 1$ we have

$$f_1(P-1) \to \frac{c_0(1)}{P-1} = \frac{m}{2}\,\frac{1}{P-1}\,, \qquad f_k(P-k) \neq \infty$$

and $\Phi_m(1) = \Phi_m(0)$, one can obtain the more simple expression for energy

$$\frac{\varepsilon}{2} = \log(\mu) + 3\gamma + \frac{\Phi'_m(1)}{\Phi_m(1)} + \frac{m(1-m)}{2\mu}\,. \qquad (145)$$

It is possible to express ε through the values of the function $\Phi_m(P)$ in other integer points s

$$\begin{aligned}\frac{\varepsilon}{2} &= \log(\mu) + 3\gamma + \frac{\Phi'_m(s)}{\Phi_m(s)} \\ &\quad + \frac{(1-m)c_0(k)}{\mu^s}\frac{\Phi_m(0)}{\Phi_m(s)}\prod_{r=1}^{s-1}(s-r)(s-r+1)(s-r-m+1) \\ &\quad + \sum_{k=1}^{s-1}\mu^{-k} f_k(s-k)\frac{\Phi_m(s-k)}{\Phi_m(s)}\prod_{r=1}^{k}(s-r)(s-r+1)(s-r-m+1)\,.\end{aligned}$$

This representation is equivalent to previous one due to the recurrence relations for $\Phi_{m,1-m}(P)$ following from the duality equation.

One can fix $\Phi(P)$ at some P in an accordance with the duality relation without a loss of generality

$$\Phi_m(1+m) = \Phi_m(m) = \Phi_m(1) = \Phi_m(0) = 1\,. \qquad (146)$$

For other integer arguments of Φ_m we have the recurrence relations following from the eigenfunction equation for the integral of motion

$$
\begin{aligned}
\Phi_m(2) &= \left(1+\frac{(1-m)\,(2-m)}{\mu}\right)\,, \\
\Phi_m(s+1) &= \left(1+\frac{s(s-m)}{\mu}\,(2s-m)\right) \\
&\quad\times\Phi_m(s)-\frac{s(s-1)^2(s-m)^2(s-m-1)}{\mu^2}\Phi_m(s-1)\,, \\
\Phi_m(2+m) &= \left(1+\frac{(1+m)(2+m)}{\mu}\right)\,, \\
\Phi_m(s+1+m) &= \left(1+\frac{(s+m)s}{\mu}\,(2s+m)\right)\Phi_m(s+m) \\
&\quad-\frac{(s+m)(s+m-1)^2s^2(s-1)}{\mu^2}\,\Phi_m(s+m-1)\,.
\end{aligned}
$$

The solution of these equations is a polynomial in μ^{-1} and m

$$
\Phi_m(s)=\sum_{k=0}^{2(s-1)-1}\ \sum_{l=0}^{2k} c_{kl}\,\mu^{-k}\,m^l\,.
$$

We can calculate also subsequently the derivatives in integer points s defining

$$
\Phi'_m(1)=e(m)\,. \tag{147}
$$

Namely, one obtains

$$
\Phi'_m(2)=\left(1+\frac{(1-m)(2-m)}{\mu}\right)\,e(m)+\frac{m^2-6m+6}{\mu} \tag{148}
$$

and

$$
\begin{aligned}
\Phi'_m(s+1) &= \left(1+\frac{s(s-m)}{\mu}\,(2s-m)\right)\Phi'_m(s) \\
&\quad-\frac{s(s-1)^2(s-m)^2(s-m-1)}{\mu^2}\Phi'_m(s-1)+\frac{1}{\mu}\,\left(s(s-m)\,(2s-m)\right)' \\
&\quad\times\Phi_m(s)-\frac{1}{\mu^2}\,\left(s(s-1)^2(s-m)^2(s-m-1)\right)'\,\Phi_m(s-1)\,.
\end{aligned} \tag{149}
$$

The parameter $e(m)$ is fixed from the condition, that the duality equation for $\Phi_m(s)$ is valid at $P\to\infty$. This requirement can be formulated in a more simple way if one returns to the initial definition of the eigenfunction

$$
\Phi_m(P)=\frac{\varphi_m(P)\exp(i\pi P)}{\Gamma(-P)\,\Gamma(1-P)\,\Gamma(m-P)\mu^P}\,.
$$

and presents $\phi_m(P)$ as a sum over poles of the first and second order with the residues satisfying the recurrence relations obtained from above relations for $\Phi_m(s)$ and $\Phi'_m(s)$.

It is plausible, that the holomorphic energies for the different meromorphic solutions $\phi_m(P)$ will be generally different. Because the Odderon wave function constructed as a bilinear combination of these solutions in the holomorphic and anti-holomorphic subspaces should have a definite total energy, the quantization of μ should arise as a result of the coincidence of the holomorphic energies for different solutions similar to the case of the Baxter–Sklyanin approach [19, 20].

4.6. Odderon intercept

By solving the recurrence relation for $\Phi_m^n(P)$ and putting the result in the above expression for the energy, we obtain the following asymptotic expansion for $\varepsilon/2$ [16]:

$$\begin{aligned}\frac{\varepsilon}{2} &= \log(\mu) + 3\gamma + \left(\frac{3}{448} + \frac{13}{120}(m-1/2)^2 - \frac{1}{12}(m-1/2)^4\right)\frac{1}{\mu^2} \\ &\quad + \left(-\frac{4185}{2050048} - \frac{2151}{49280}(m-1/2)^2 + \ldots\right) \\ &\quad \times \frac{1}{\mu^4} + \left(\frac{965925}{37044224} + \ldots\right)\frac{1}{\mu^6} + \ldots . \end{aligned} \tag{150}$$

This expansion can be used with a certain accuracy even for the smallest eigenvalue $\mu = 0.20526$, corresponding to the ground-state energy $\varepsilon = 0.49434$ [21]. For the first excited state with the same conformal weight $m = 1/2$, where $\varepsilon = 5.16930$ and $\mu = 2.34392$ [9], the energy can be calculated from the above asymptotic series with a good precision. The analytic approach, developed in this section, should be compared with the method based on the Baxter equation [21].

One can derive from above formulas also the representation for the Odderon Hamiltonian in the two-dimensional space $\overrightarrow{x}$:

$$\begin{aligned}2H &= h + h^* = 12\gamma + \ln(|x|^4\,|\partial|^2) + \ln(|1-x|^4\,|\partial|^2) \\ &\quad + (x-1)^m (x^*-1)^{\widetilde{m}} \left(\ln(|\partial|^2) + \ln(|x|^4\,|\partial|^2)\right)(x-1)^{-m}(x^*-1)^{-\widetilde{m}} \\ &\quad + (-x)^m(-x^*)^{\widetilde{m}} \left(\ln(|1-x|^4\,|\partial|^2) + \ln(|\partial|^2)\right)(-x)^{-m}(-x^*)^{-\widetilde{m}} . \end{aligned} \tag{151}$$

The logarithms in this expression can be presented as integral operators with the use of the relation

$$\int\frac{d^2p}{2\pi}\exp(i\overrightarrow{p}\,\overrightarrow{y})\left(2\gamma+\ln\frac{(\overrightarrow{p})^2}{4}\right)=-2\left(\frac{\theta(|y|-\varepsilon)}{|y|^2}-2\pi\ln\frac{1}{\varepsilon}\delta^2(\overrightarrow{y})\right). \quad (152)$$

This representation can be used to find the eigenvalue of the Hamiltonian for the eigenfunction of the integrals of motion B and B^* with their vanishing eigenvalues $\mu = \mu^* = 0$:

$$\varphi_{m,\widetilde{m}}(\overrightarrow{x}) = 1 + (-x)^m(-x^*)^{\widetilde{m}} + (x-1)^m(x^*-1)^{\widetilde{m}}\,. \quad (153)$$

The corresponding wave function $f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2},\overrightarrow{\rho_3};\overrightarrow{\rho_0})$ is invariant under the cyclic permutation of coordinates $\overrightarrow{\rho_1}\to\overrightarrow{\rho_2}\to\overrightarrow{\rho_3}\to\overrightarrow{\rho_1}$ but it is symmetric under the permutations $\overrightarrow{\rho_1}\leftrightarrow\overrightarrow{\rho_2}$ only for even value of the conformal spin $n = \widetilde{m}-m$, where the norm $\left\|\varphi_{m,\widetilde{m}}\right\|_1$ (without derivatives) is divergent due to the singularities at $x = 0,\ 1\infty$. It is the reason, why the solution exists only for the case

$$\widetilde{m} - m = 2k+1\,, \qquad k = 0,\pm1,\pm2,\dots\,. \quad (154)$$

Owing to the Bose symmetry of the wave function, this state corresponds to the f-coupling and has the positive charge-parity C. It could be responsible for the small-x behavior of the structure function $g_2(x)$. Using the above representation for H, we obtain

$$2H\varphi_{m,\widetilde{m}}(\overrightarrow{x}) = E^p_{m,\widetilde{m}}\,\varphi_{m,\widetilde{m}}(\overrightarrow{x})\,, \quad (155)$$

where $E^p_{m,\widetilde{m}}$ is the corresponding eigenvalue for the Pomeron Hamiltonian

$$E^p_{m,\widetilde{m}} = \epsilon^p_m + \epsilon^p_{\widetilde{m}} \quad (156)$$

and

$$\epsilon^p_m = \psi(1-m) + \psi(m) - 2\psi(1)\,. \quad (157)$$

The minimal value of $E^p_{m,\widetilde{m}}$ is obtained at $\widetilde{m} - m = \pm1$ and corresponds to $\omega = 0$.

For the case of odd $n = \widetilde{m} - m$, the norm $|\varphi|_1$ of $\varphi_{m,\widetilde{m}}(\overrightarrow{x})$ is finite

$$\int\frac{d^2x\,\left|\varphi_{m,\widetilde{m}}(\overrightarrow{x})\right|^2}{3\pi\,|x(1-x)|^2} = \mathrm{Re}\left(\psi(m)+\psi(1-m)+\psi(\widetilde{m})+\psi(1-\widetilde{m})-4\psi(1)\right)\,. \quad (158)$$

Note, however, that the norm $|\varphi|_2$ (with derivatives) is divergent. It is possible, that the divergence disappears for a more general solution with a non-vanishing value of λ. Using the duality transformation it is possible to obtain from above function $\varphi_m(\vec{x})$ a new solution for the Odderon symmetric in the coordinates $\vec{\rho_k}$ [22]. The solution is normalized according to the norm $|\varphi|_2$ compatible with the hermiticity properties of the BFKL Hamiltonian. Note, that the intercept j_0 for this solution exceeds the intercepts for the solutions vanishing at $\rho_{ij} \to 0$ and is exactly one. Solutions of the BKP equation for Pomeron and Odderon with $n = 1$ have the same energies, which is related with the interpretation of the duality symmetry at large N_c as a symmetry between the Reggeon states with the gluon quantum numbers and an opposite signature [22].

5. BFKL Pomeron in supersymmetric models

5.1. One-loop corrections to the BFKL equation

One can calculate the integral kernel for the BFKL equation also in two loops [23]. Its eigenvalue can be written as follows

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,, \qquad \hat{a} = g^2 N_c/(16\pi^2)\,, \tag{159}$$

where

$$\chi(n,\gamma) = 2\psi(1) - \psi(\gamma + |n|/2) - \psi(1-\gamma+|n|/2) \tag{160}$$

and $\psi(x) = \Gamma'(x)/\Gamma(x)$. The one-loop correction $\Delta(n,\gamma)$ in QCD contains the non-analytic terms — the Kroniker symbols $\delta_{|n|,0}$ and $\delta_{|n|,2}$ [8]. But in $N = 4$ SUSY they are canceled and the result for $\Delta(n,\gamma)$ has the hermitian separability [8, 24]

$$\Delta(n,\gamma) \;=\; \phi(M)+\phi(M^*) - \frac{\rho(M)+\rho(M^*)}{2\hat{a}/\omega}\,, \qquad M = \gamma + \frac{|n|}{2}\,, \tag{161}$$

$$\rho(M) \;=\; \beta'(M) + \frac{1}{2}\zeta(2)\,, \qquad \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]. \tag{162}$$

It is interesting, that all functions entering in these expressions have the property of the maximal transcendentality [24]. In particular, $\phi(M)$ can be written in the form

$$\phi(M) \;=\; 3\zeta(3) + \psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\psi(1) - \psi(M)\Big), \tag{163}$$

$$\Phi(M) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\psi'(k+1) - \frac{\psi(k+1)-\psi(1)}{k+M}\right). \tag{164}$$

Here $\psi(M)$ has the transcendentality equal to 1, its derivatives $\psi^{(n)}$ have transcedentalities $n+1$, the additional poles in the sum over k increase the transcendentality of the function $\Phi(M)$ up to 3 which is also the transcendentality of $\zeta(3)$. The maximal transcendentality hypothesis is valid also for the anomalous dimensions of twist-2 -operators in $N = 4$ SUSY [25, 26] contrary to the case of QCD [27]. This result will be discussed later.

5.2. BFKL Pomeron in $N = 4$ SUSY and graviton

Generally the BFKL equation in the diffusion approximation can be written in the simple form [6]

$$j = 2 - \Delta - D\,\nu^2\,, \tag{165}$$

where ν is related to the anomalous dimension of the twist-2 operators as follows [23]

$$\gamma = 1 + \frac{j-2}{2} + i\nu\,. \tag{166}$$

The parameters Δ and D are functions of the coupling constant $\hat{a}$ and are known up to two loops. Higher order perturbative corrections can be obtained with the use of the effective action [9, 10]. For large coupling constants one can expect, that the leading Pomeron singularity in $N = 4$ SUSY is moved to the point $j = 2$ and asymptotically the Pomeron coincides with the graviton Regge pole. This assumption is related to the AdS/CFT correspondence, formulated in the framework of the Maldacena hypothesis, that $N = 4$ SUSY is equivalent to the superstring model living on the 10-dimensional anti-de-Sitter space [28–30]. Therefore it is natural to impose on the BFKL equation in the diffusion approximation the physical condition, that for the conserved energy-momentum tensor $\vartheta_{\mu\nu}(x)$ having $j = 2$ the anomalous dimension γ is zero. As a result, we obtain, that the parameters Δ and D coincide [26]. In this case one can solve the above BFKL equation for γ

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)\,. \tag{167}$$

Using the dictionary developed in the framework of the AdS/CFT correspondence [29], one can rewrite the BFKL equation in the form of the graviton Regge trajectory [26]

$$j = 2 + \frac{\alpha'}{2}\,t\,, \qquad t = E^2/R^2\,, \qquad \alpha' = \frac{R^2}{2}\,\Delta\,. \tag{168}$$

On the other hand, Gubser, Klebanov and Polyakov predicted the following asymptotics of the anomalous dimension at large $\hat{a}$ and j [31]

$$\gamma_{|\hat{a},j\to\infty} = -\sqrt{j-2}\,\Delta^{-1/2}_{|j\to\infty} = \sqrt{2\pi j}\,\hat{a}^{1/4}\,. \tag{169}$$

As a result, one can obtain the explicit expression for the Pomeron intercept at large coupling constants [26, 32]

$$j = 2 - \varDelta\,, \qquad \varDelta = \frac{1}{2\pi}\,\hat{a}^{-1/2}\,. \tag{170}$$

6. Discussion of obtained results

It was demonstrated, that Pomeron in QCD is a composite state of reggeized gluons. To calculate the Regge trajectory and various couplings one can use the effective action approach. In LLA BFKL equation has the Möbius invariance, which allows to find its exact solution. Moreover, in the generalized LLA the BFKL dynamics for interactions of several gluons in the multi-color QCD is integrable and equivalent to the Heisenberg spin model. In particular, the Reggeon interaction has the duality symmetry. For the Odderon case the duality symmetry allows to construct solutions of the Schrödinger equation and calculate their intercepts in terms of expansion in the inverse eigenvalue λ of the integral of motion. In the next-to-leading approximation in $N = 4$ SUSY the equation for the Pomeron wave function has remarkable properties including the analyticity in the conformal spin n and the maximal transcendentality. In this model the BFKL Pomeron coincides with the reggeized graviton, which gives a possibility to calculate its intercept at large coupling constants.

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