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MAXIMAL TRANSCENDENTALITY AND INTEGRABILITY* **

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The Hamiltonian describing possible interactions of the Reggeized gluons in the leading logarithmic approximation (LLA) of the multicolor QCD has the properties of conformal invariance, holomorphic separability and duality. It coincides with the Hamiltonian of the integrable Heisenberg model with the spins being the Möbius group generators. With the use of the Baxter–Sklyanin representation we calculate intercepts of the colorless states constructed from three and four Reggeized gluons and anomalous dimensions of the corresponding high twist operators. The integrability properties of the BFKL equation at a finite temperature are reviewed. Maximal transcendentality is used to construct anomalous dimensions of twist-2 operators up to 4 loops. It is shown that the asymptotic Bethe Ansatz in the 4-loop approximation is not in an agreement with predictions of the BFKL equation in $N=4$ SUSY.

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1. Introduction

1.1. BFKL equation

The scattering amplitude $A(s,t)$ at high energies $2E=\sqrt{s}$ and fixed momentum transfers $q=\sqrt{-t}$ in QCD and other gauge theories in the leading logarithmic approximation (LLA) $\alpha_{\rm s}\ln s\sim 1$ can be obtained from

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the solution of the BFKL equation [1]. The next-to-leading corrections to its kernel were also calculated in QCD and in supersymmetric models [2]. The BFKL equation is used for the description of structure functions for the deep-inelastic lepton–hadron scattering together with the DGLAP equation [3] (see the reviews [4]). In the impact parameter space $\overrightarrow{\rho}$ the BFKL equation has the Schrödinger-like form [4]

$$E\, f(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}) = H_{12}\; f(\overrightarrow{\rho_1}, \overrightarrow{\rho_2})\,, \tag{1}$$

where in LLA the eigenvalue E of the ground state is related to the intercept Δ of the Pomeron as follows

$$\Delta = -\frac{g^2 N_c}{8\pi^2}\, E \tag{2}$$

and the Hamiltonian is given below in the holomorphically separable form:

$$\begin{aligned} H_{12} \;&=\; h_{12} + h_{12}^*\,, \\ h_{12} \;&=\; \ln(p_1 p_2) + \frac{1}{p_1}(\ln \rho_{12}) p_1 + \frac{1}{p_2}(\ln \rho_{12}) p_2 + 2\gamma\,. \end{aligned} \tag{3}$$

Here $\gamma = -\psi(1)$ is the Euler constant. We introduced the complex components of the gluon coordinates $\rho_k = x_k + i y_k$, ρ_k^* ($\rho_{12} = \rho_1 - \rho_2$) and their canonically conjugated momenta p_k, p_k^*.

In LLA the BFKL equation is invariant under the Möbius group transformations [4]

$$\rho_k \to \frac{a\,\rho_k + b}{c\,\rho_k + d}\,, \tag{4}$$

where a, b, c, d are arbitrary complex parameters. Its solutions belong to the principal series of unitary representations of the Möbius group. For this series the conformal weights

$$m = 1/2 + i\nu + n/2\,, \qquad \widetilde{m} = 1/2 + i\nu - n/2\,, \tag{5}$$

are expressed in terms of the anomalous dimension $\gamma = 1 + 2i\nu$ of the twist-2 operators $O_{m,\widetilde{m}}(\overrightarrow{\rho_0})$ (with real ν) and their integer conformal spin n. The conformal weights are related to the eigenvalues

$$M^2 f_{m,\widetilde{m}} = m(m-1) f_{m,\widetilde{m}}\,, \qquad M^{*2} f_{m,\widetilde{m}} = \widetilde{m}(\widetilde{m}-1) f_{m,\widetilde{m}}\,, \tag{6}$$

of the Casimir operators M^2 and M^{*2} of the Möbius group (for the Pomeron the number of Reggeons is $n = 2$)

$$M^2 = \left(\sum_{k=1}^{n} M_k^a\right)^2 = \sum_{r<s} 2\, M_r^a\, M_s^a = -\sum_{r<s} \rho_{rs}^2 \partial_r \partial_s\,,\; M^{*2} = (M^2)^*\,. \tag{7}$$

Here M_k^a are the group generators

$$M_k^3 = \rho_k \partial_k \,, \qquad M_k^+ = \partial_k \,, \qquad M_k^- = -\rho_k^2 \partial_k \tag{8}$$

and $\partial_k = \partial/(\partial \rho_k)$.

The eigenfunctions and eigenvalues of the BFKL Hamiltonian are given below [5]:

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}; \overrightarrow{\rho_0}) = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\,\rho_{20}^*}\right)^{\widetilde{m}} , \tag{9}$$

$$E_{m,\widetilde{m}} = 4\,\mathrm{Re}\left(\psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2}\right)\right) - 4\psi(1) \,. \tag{10}$$

The minimum of $E_{m,\widetilde{m}}$ is obtained for $\nu = n = 0$ and equals

$$\min E_{m,\widetilde{m}} = -8\,\ln 2 \,. \tag{11}$$

Therefore, the total cross-section in LLA grows rather rapidly

$$\sigma_t \sim g^4 s^{\Delta} \,, \qquad \Delta = \frac{g^2}{\pi^2} N_c \ln 2 \tag{12}$$

and exceeds the Froissart limit $\sigma_t \leq \ln^2 s$.

In the next-to-leading approximation [2] the value of the intercept, obtained with the use of the BML procedure, is significantly smaller $\Delta \approx 0.2$ [6]. A self-consistent approach to the construction of the unitary S-matrix at high energies in the perturbative QCD should be based on the use of the effective action for Reggeized gluon interactions [7]. But we consider below a more simple method related to the solution of the Bartels, Kwiecinski and Praszalowicz (BKP) equations for the composite states of n Reggeized gluons [8].

1.2. Multi-Reggeon states in the multi-color QCD

The Bartels–Kwiecinski–Praszalowicz equation [8] for the n-gluon composite states is given below

$$E_{m,\widetilde{m}}\,\psi_{m,\widetilde{m}} = H\,\psi_{m,\widetilde{m}} \,, \qquad H = \sum_{1\leq r<l\leq n} H_{rl}\,\frac{T_r^a T_l^a}{(-N_c)} \,. \tag{13}$$

Here T_r^a implies the gauge group generator acting on the color index of the gluon r. The wave functions and energies are enumerated by the conformal weights m and $\widetilde{m}$ for the corresponding representations of the Möbius group. The intercepts $\Delta_{m,\widetilde{m}}$ entering in the asymptotic contribution to the total cross-section $\sigma_t \sim s^{\Delta}$ from the corresponding Feynman diagrams are proportional to $E_{m,\widetilde{m}}$

$$\Delta_{m,\widetilde{m}} = -\frac{g^2 N_c}{8\pi^2}\, E_{m,\widetilde{m}} \,. \tag{14}$$

In a particular case of the Odderon [9], being a composite state of three Reggeized gluons with the charge parity $C = -1$ and signature $P_j = -1$, the color factor coincides with the known completely symmetric tensor d_{abc}.

To simplify the structure of the equation for colorless composite states in a general case of n Reggeized gluons we consider the multi-color limit $N_c \to \infty$ [4]. It is remarkable that the Hamiltonian H in the multicolor QCD apart from the Möbius invariance has the property of the holomorphic separability [10]:

$$H = \frac{1}{2}(h + h^*), \;\; [h, h^*] = 0\,, \tag{15}$$

where the holomorphic and anti-holomorphic contributions

$$h = \sum_{k=1}^{n} h_{k,k+1}\,,\; h^* = \sum_{k=1}^{n} h^*_{k,k+1} \tag{16}$$

are expressed in terms of the BFKL Hamiltonians h_{12} [10].

Owing to the holomorphic separability of H, the wave function $f_{m,\widetilde{m}}$ has the property of the holomorphic factorization [10]:

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1}\,, \dots\,, \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) = \sum_{r,l} c_{r,l}\, f^r_m(\rho_1\,, \dots\,, \rho_n; \rho_0)\, f^l_{\widetilde{m}}(\rho^*_1\,, \dots\,, \rho^*_n; \rho^*_0)\,, \tag{17}$$

where r and l enumerate the degenerate solutions of the Schrödinger equations in the holomorphic and anti-holomorphic sub-spaces:

$$\epsilon_m\, f_m = h\, f_m\,, \qquad \epsilon_{\widetilde{m}}\, f_{\widetilde{m}} = h^*\, f_{\widetilde{m}}\,, \qquad E_{m,\widetilde{m}} = \frac{1}{2}(\epsilon_m + \epsilon_{\widetilde{m}})\,. \tag{18}$$

Similarly to the case of two-dimensional conformal field theories, the coefficients $c_{r,l}$ are fixed by the single-valuedness condition for the wave function $f_{m,\widetilde{m}}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, \dots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0})$ in the two-dimensional $\overrightarrow{\rho}$-space. Note that in these conformal models the holomorphic factorization of the Green functions is a consequence of the invariance of the operator algebra under the infinitely dimensional Virasoro group [11].

The holomorphic Hamiltonian commutes with the differential operator [12]

$$A = \rho_{12}\rho_{23} \dots \rho_{n1}\, p_1 p_2 \dots p_n \,. \tag{19}$$

Furthermore, [13], one can construct many mutually commuting integrals of motion

$$q_r = \sum_{i_1<i_2<\ldots<i_r} \rho_{i_1 i_2}\,\rho_{i_2 i_3}\ldots\rho_{i_r i_1}\,p_{i_1}\,p_{i_2}\ldots p_{i_r}\,,\ [q_r,h]=0\,. \tag{20}$$

In particular, q_n is equal to A and q_2 is proportional to M^2.

The generating function for these integrals of motion coincides with the transfer matrix $T(u)$ for the XXX model [13, 14]:

$$T(u) = \mathrm{Tr}\,(L_1(u)L_2(u)\ldots L_n(u)) = \sum_{r=0}^{n} u^{n-r}\,q_r\,, \tag{21}$$

where the L-operators are

$$L_k(u) = \begin{pmatrix} u+\rho_k\,p_k & p_k \\ -\rho_k^2\,p_k & u-\rho_k\,p_k \end{pmatrix} = \begin{pmatrix} u\,0 \\ 0\,u \end{pmatrix} + \begin{pmatrix} 1 \\ -\rho_k \end{pmatrix} \begin{pmatrix} \rho_k\,1 \end{pmatrix}\,p_k\,.$$

The transfer matrix is the trace of the monodromy matrix $t(u)$:

$$T(u) = \mathrm{Tr}\,(t(u))\,, \qquad t(u) = L_1(u)L_2(u)\ldots L_n(u)\,. \tag{22}$$

It can be verified that $t(u)$ satisfies the Yang–Baxter (YB) equation [13,14]

$$t^{s_1}_{r_1'}(u)\,t^{s_2}_{r_2'}(v)\,l^{r_1'r_2'}_{r_1 r_2}(v-u) = l^{s_1 s_2}_{s_1' s_2'}(v-u)\,t^{s_2'}_{r_2}(v)\,t^{s_1'}_{r_1}(u)\,, \tag{23}$$

where $l(w)$ is the L-operator for the well-known Heisenberg spin model

$$l^{s_1 s_2}_{s_1' s_2'}(w) = w\,\delta^{s_1}_{s_1'}\,\delta^{s_2}_{s_2'} + i\,\delta^{s_1}_{s_2'}\,\delta^{s_2}_{s_1'}\,. \tag{24}$$

Really the BKP Hamiltonian coincides with the local Hamiltonian of the Heisenberg spin model, in which spins are generators of the Möbius group [15]. The general method of solving such models was suggested by Sklyanin [16].

The integrability is closely related to the duality symmetry of the Reggeized gluon interactions [17]. Using the results of Ref. [12] the equation for Odderon was solved approximately [18]. A new Odderon solution with a larger intercept was constructed in Ref. [19]. The integrability appears also in the problem of finding the anomalous dimensions for local operators for $N=4$ SUSY [20]. This model is assumed to be equivalent to the superstring theory on the 10-dimensional anti-de-Sitter space [21].

2. Baxter–Sklyanin representation

2.1. Sklyanin Ansatz

Thus, the problem of finding solutions of the Schrödinger equation for the Reggeized gluon interaction is reduced to the search of a representation for the monodromy matrix satisfying the Yang–Baxter bilinear relations [13]. It is convenient to work in the conjugated space [15], where the monodromy matrix is parametrized as follows,

$$\widetilde{t}(u) = \widetilde{L}_n(u) \dots \widetilde{L}_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{25}$$

where $\widetilde{L}_k(u)$ is given below

$$\widetilde{L}_k(u) = \begin{pmatrix} u + p_k\rho_{k0} & -p_k\rho_{k0}^2 \\ p_k\rho_{k0}^2 & u - p_k\rho_{k0} \end{pmatrix}. \tag{26}$$

The pseudo-vacuum state annihilated by the operators $C(u)$ and $C^*(u)$ has the form [15]

$$\Psi^{(0)}(\overrightarrow{\rho_1}, \overrightarrow{\rho_2}, \dots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) = \prod_{k=1}^{n} \frac{1}{|\rho_{k0}|^4}. \tag{27}$$

To construct the n-Reggeon states with physical values of conformal weights $m, \widetilde{m}$ in the framework of the Bethe Ansatz one can use the Baxter–Sklyanin approach [14, 16]. To begin with, we should introduce the Baxter function satisfying the equation (see [15, 22, 23])

$$\Lambda^{(n)}(\lambda;\vec{\mu})Q(\lambda;m,\vec{\mu}) = (\lambda+i)^n Q(\lambda+i;m,\vec{\mu}) + (\lambda-i)^n Q(\lambda-i;m,\vec{\mu}), \tag{28}$$

where $\Lambda^{(n)}(\lambda)$ is the eigenvalue of the monodromy matrix

$$\begin{aligned} \Lambda^{(n)}(\lambda;\vec{\mu}) &= \sum_{k=0}^{n} (-i)^k \mu_k \lambda^{n-k}, \\ \mu_0 &= 2, \qquad \mu_1 = 0, \qquad \mu_2 = m(m-1). \end{aligned} \tag{29}$$

Here we assume [22], that the eigenvalues $\mu_k = i^k q_k$ of integrals of motion are real.

The eigenfunctions of the holomorphic Schrödinger equation can be expressed through the Baxter function $Q(\lambda)$ using the Sklyanin Ansatz [16]:

$$\begin{aligned} f(\rho_1, \rho_2, \dots, \rho_n; \rho_0) &= Q(\widehat{\lambda}_1;\, m, \vec{\mu}) \\ &\times Q(\widehat{\lambda}_2;\, m, \vec{\mu}) \dots Q(\widehat{\lambda}_{n-1}; m, \vec{\mu})\, \Psi^{(0)}, \end{aligned} \tag{30}$$

where $\widehat{\lambda}_r$ are the operator zeroes of the matrix element $B(u)$ of the monodromy matrix:

$$B(u) = -P \prod_{r=1}^{n-1} (u - \widehat{\lambda}_r) \,, \qquad P = \sum_{k=1}^{n} p_k \,. \tag{31}$$

2.2. Holomorphic factorization and quantization

In Ref. [22] a unitary transformation of the wave function for the composite state of n Reggeized gluons was constructed for the transition from the coordinate representation to Baxter–Sklyanin one in which the operators $\widehat{\lambda_r}$ are diagonal (see also [23]). As a consequence of the single-valuedness condition for its kernel the arguments of the Baxter functions $Q(\lambda)$ and $Q(\lambda^*)$ in the holomorphic and anti-holomorphic sub-spaces are quantized (see [22, 23]):

$$\lambda = \sigma + i\frac{N}{2} \,, \qquad \lambda^* = \sigma - i\,\frac{N}{2} \,, \tag{32}$$

where σ and N are real and integer numbers, respectively.

In Ref. [22] a general method of solving the Baxter equation for the n-Reggeon composite state was proposed and the wave functions and intercepts of the composite states of three and four Reggeons were calculated. It turns out [22] that there is a set of independent Baxter functions $Q^{(t)}$ ($t = 0, 1, \ldots, n-1$) having multiple poles simultaneously in the upper and lower half-λ planes in the points $\lambda = ik$ ($k = 0, \pm 1, \pm 2, \ldots$). Using all these functions one can construct the normalizable total Baxter function $Q_{m,\widetilde{m},\vec{\mu}}\left(\overrightarrow{\lambda}\right)$ without poles at $\sigma = 0$ [22],

$$Q_{m,\widetilde{m},\vec{\mu}}\left(\overrightarrow{\lambda}\right) = \sum_{t,l} C_{t,l}\, Q^{(t)}\left(\lambda;\, m, \vec{\mu}\right)\, Q^{(l)}\left(\lambda^*;\, \widetilde{m}, \vec{\mu^s}\right) \,, \tag{33}$$

by adjusting for this purpose the coefficients $C_{t,l}$.

The total energy $E_{m,\widetilde{m}}$ can be expressed in terms of the Baxter function (see Ref. [22]):

$$E = i \lim_{\lambda,\lambda^* \to i} \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\lambda^*} \ln\left[(\lambda - i)^{n-1}(\lambda^* - i)^{n-1}\, |\lambda|^{2\,n}\; Q_{m,\widetilde{m},\vec{\mu}}\left(\overrightarrow{\lambda}\right)\right] \,. \tag{34}$$

Since the function $Q_{m,\widetilde{m},\vec{\mu}}\left(\overrightarrow{\lambda}\right)$ is a bilinear combination of the Baxter functions $Q^{(t)}(\lambda)$ and $Q^{(l)}(\lambda^*)$ ($t\,, l = 1, 2, \ldots, n$), the holomorphic energies for all solutions $Q^{(t)}$ should be the same. This leads to a quantization of the integrals of motion q_k [22].

Let us rewrite the Baxter equation for the n Reggeon composite state in a real form introducing the new variable $x \equiv -i\lambda$:

$$\Omega(x,\vec{\mu})\, Q(x,\vec{\mu}) = (x+1)^n\, Q(x+1,\vec{\mu}) + (x-1)^n\, Q(x-1,\vec{\mu})\,, \qquad (35)$$

where

$$\Omega(x,\vec{\mu}) = \sum_{k=0}^{n} (-1)^k\, \mu_k\, x^{n-k} \qquad (36)$$

and

$$\mu_0 = 2\,, \qquad \mu_1 = 0\,, \qquad \mu_2 = m(m-1)\,,$$

assuming that the eigenvalues of the integrals of motion μ_k $(k > 2)$ are real numbers.

2.3. Meromorphic solutions of the Baxter equation

To solve the Baxter equation we introduce a set of the auxiliary functions for $r = 1, 2, \ldots, n-1$ [22]

$$f_r(x,\vec{\mu}) = \sum_{l=0}^{\infty} \left[\frac{\widetilde{a}_l(\vec{\mu})}{(x-l)^r} + \frac{\widetilde{b}_l(\vec{\mu})}{(x-l)^{r-1}} + \ldots + \frac{\widetilde{g}_l(\vec{\mu})}{x-l} \right], \qquad (37)$$

where the coefficients $\widetilde{a}_l, \ldots, \widetilde{g}_l$ satisfy recurrent relations obtained by inserting f_r instead of $Q(x)$ in the Baxter equation, but with other initial conditions

$$\widetilde{a}_0 = 1, \qquad \widetilde{b}_0 = \ldots = \widetilde{g}_0 = 0\,. \qquad (38)$$

Note that all functions $f_r(x,\vec{\mu})$ are expressed in terms of a subset of pole residues $\widetilde{a}_l, \ldots, \widetilde{z}_l$ for $f_{n-1}(x,\vec{\mu})$ and therefore they can be obtained from it.

There are n "minimal" independent solutions $Q^{(t)}(x,\vec{\mu})$ $(t = 0, 1, 2, \ldots, n-1)$ of the Baxter equation having t-order poles at positive integer x and $(n-1-t)$-order poles at negative integer x [22]:

$$Q^{(t)}(x,\vec{\mu}) = \sum_{r=1}^{t} C_r^{(t)}(\vec{\mu})\, f_r(x,\vec{\mu}) + \beta^{(t)}(\vec{\mu}) \sum_{r=1}^{n-1-t} C_r^{(n-1-t)}(\vec{\mu^s}) f_r(-x,\vec{\mu^s})\,, \quad (39)$$

where the meromorphic functions $f_r(x,\vec{\mu})$ were defined above and $\mu_r^s = (-1)^r \mu_r$. Such form of the solution is related to the invariance of the Baxter equation under the substitution $x \to -x\,, \vec{\mu} \to \vec{\mu^s}$.

The coefficients $C_r^{(t)}(\vec{\mu})\,,\ C_r^{(n-1-t)}(\vec{\mu^s})$ and $\beta^t(\vec{\mu})$ are obtained imposing the validity of the Baxter equation at $x \to \infty$:

$$\lim_{x\to\infty} x^{n-2}\, Q^{(t)}(x,\vec{\mu}) = 0\,.$$

This leads to a system of $n-2$ linear equations for the coefficients $C_r^{(t)}$. We normalize $Q^{(t)}(x,\vec{\mu})$ by choosing

$$C_t^{(t)}(\vec{\mu}) = C_{n-1-t}^{(n-1-t)}(\vec{\mu}) = 1\,. \tag{40}$$

It is important to notice that three subsequent solutions $Q^{(r)}$ for $r = 1,2,\ldots,n-2$ are linearly dependent [22]:

$$\left[\delta^{(r)}(\vec{\mu}) + \pi\cot(\pi x)\right] Q^{(r)}(x,\vec{\mu}) = Q^{(r+1)}(x,\vec{\mu}) + \alpha^{(r)}(\vec{\mu})Q^{(r-1)}(x,\vec{\mu})\,. \tag{41}$$

Indeed, the left and right-hand sides satisfy the Baxter equation everywhere including $x \to \infty$ and have the same singularities. Therefore, due to the uniqueness of the 'minimal' solutions the quantity $\pi\cot(\pi x)\,Q^{(r)}(x,\vec{\mu})$ can be expressed as a linear combination of $Q^{(r-1)}(x,\vec{\mu})$, $Q^{(r)}(x,\vec{\mu})$ and $Q^{(r+1)}(x,\vec{\mu})$. Furthermore, the coefficient in front of $Q^{(r+1)}(x,\vec{\mu})$ is chosen to be 1 taking into account our normalization of $Q^{(r)}(x,\vec{\mu})$.

The Baxter function in the total $\vec{x}$-space is a bilinear combination of holomorphic and anti-holomorphic functions $Q^{(r)}$. Therefore, the holomorphic energy expressed in terms of the residues $a_0 = 1, b_0, a_1, b_1$ of the poles closest to zero,

$$\epsilon = \frac{b_1}{a_1} + n = b_0 - \frac{\mu_{n-1}}{\mu_n}\,, \tag{42}$$

should be the same for all solutions $\epsilon^{(0)} = \epsilon^{(1)} = \ldots = \epsilon^{(n)}$. It leads to a quantization of the integrals of motion μ_k and the energy E [22].

The total energy of the composite state of n Reggeons is the sum of the holomorphic and anti-holomorphic energies:

$$E_{m,\widetilde{m}} = \epsilon_m(\vec{\mu}) + \epsilon_{\widetilde{m}}(\vec{\mu}^{s*})\,. \tag{43}$$

It can be obtained from the Schrödinger equation for the wave function $\phi_{m,\widetilde{m}}$ in the Baxter–Sklyanin representation in the limit $\lambda,\lambda^* \to i$ [22]. We can obtain the analogous expression

$$E_{m,\widetilde{m}} = \epsilon_m(\vec{\mu}^{s}) + \epsilon_{\widetilde{m}}(\vec{\mu}^{*}) \tag{44}$$

by taking instead another limit $\lambda,\lambda^* \to -i$. These two expressions for the energies were derived from the Schrödinger equation with the hermitian Hamiltonian [22]. Therefore, they should coincide for the quantized values of $\vec{\mu}$:

$$\epsilon_m(\vec{\mu}) + \epsilon_{\widetilde{m}}(\vec{\mu}^{s*}) = \epsilon_m(\vec{\mu}^{s}) + \epsilon_{\widetilde{m}}(\vec{\mu}^{*})\,. \tag{45}$$

It gives an additional constraint on the spectrum of the integrals of motion. One of the possible solutions of this constraint is that $\vec{\mu}$ should be real or pure imaginary. Note, however, that provided the wave function $Q(\vec{x})$ does not contain all possible bilinear combinations of the Baxter functions $Q^{(r)}$ and $Q^{(s)*}$, the quantization conditions can be not so restrictive.

2.4. Anomalous dimensions and intercepts of Reggeons

The Q^2-dependence of the inclusive probabilities $n_i(x, \ln Q^2)$ to have a parton i with the momentum fraction x inside a hadron with the large momentum $|\vec{p}| \to \infty$ can be found from the DGLAP evolution equation [3]. The eigenvalues of its integral kernels describing the inclusive parton transitions $i \to k$ coincide with the matrix elements $\gamma_j^{ki}(\alpha)$ of the anomalous dimension matrix for the twist-2 operators O^j with the Lorentz spins $j = 2, 3, \ldots$.

For example, in the case of the pure Yang–Mills theory with the gauge group $\mathrm{SU}(N_c)$ we have only one multiplicatively renormalized operator. Similarly, in the $N = 4$ supersymmetric gauge theory [21] there is one supermultiplet of twist-2 operators [20]. Its anomalous dimension is singular at the non-physical point $\omega = j - 1 \to 0$. In this limit one can calculate the anomalous dimension in all orders of perturbation theory [5]

$$\gamma_{\omega\to 0} = \frac{\alpha N_c}{\pi\omega} - \Psi''(1)\left(\frac{\alpha N_c}{\pi\omega}\right)^4 + \ldots \tag{46}$$

from the eigenvalue of the integral kernel for the BFKL equation in LLA [1] at $n = 0$:

$$\omega_{\mathrm{BFKL}} = \frac{\alpha\, N_c}{\pi}\left[2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)\right] . \tag{47}$$

Using next-to-leading corrections [2] it is possible also to predict the residues of the poles $\sim \alpha^n/\omega^{n-1}$.

One can find from the BFKL equation the anomalous dimensions of higher twist operators by solving the eigenvalue equation near other singular points $\gamma = -k$ $(k = 1, 2, \ldots)$. But it is more important to calculate the anomalous dimensions for the so-called quasi-partonic operators (see Ref. [25]) constructed from several gluonic or quark fields and responsible for the unitarization of structure functions at high energies. The simplest operator of such type is the product of the twist-2 gluon operators. In the limit $N_c \to \infty$ this operator is multiplicatively renormalized [24].

Let us return now to the high energy asymptotics of irreducible Feynman diagrams in which each of n Reggeized gluons at $N_c \to \infty$ interacts only with two neighbors. In the Born approximation the corresponding Green function is a product of free gluon propagators $\prod_{r=1}^{n} \ln|\rho_r - \rho'_r|^2$. For small coupling constants α_{s} the full dimension for the operator related to the composite state of n Reggeized gluons is approximately equal to the position of the pole $(m + \widetilde{m})/2 \approx n/2$ in the eigenvalue ω of the Schrödinger equation $\omega(m, \widetilde{m}; \mu_3, \ldots, \mu_n)$ (see [22]),

$$\frac{m + \widetilde{m}}{2} = \frac{n}{2} - \gamma^{(n)}\,, \qquad \gamma^{(n)} = c^{(n)}\,\frac{\alpha_{\mathrm{s}}\, N_c}{\omega} + O\left(\left[\frac{\alpha_{\mathrm{s}}\, N_c}{\omega}\right]^2\right) . \tag{48}$$

Here $\gamma^{(n)}$ is the anomalous dimension. This expression can be obtained also from the equation for matrix elements of quasi-partonic operators [25] written with a double-logarithmic accuracy [24]. In particular, for the Odderon we obtain from the Baxter equation that $c^{(3)} = 0$. Note, however, that for the BLV solution [19] γ has the singularity at $\omega = 0$. For $n = 4$ a pole singularity indeed was found near $(m + \widetilde{m})/2 = 2$ [22]. Moreover, in this paper the anomalous dimensions γ_3 and γ_4 were calculated for arbitrary α/ω (see [22]), which is important for finding multi-Reggeon contributions to the deep-inelastic processes at small Bjorken's variable x.

From the above quantization conditions one can calculate for $m = \widetilde{m} = 1/2$ the first roots for the Odderon numerically (see [22]):

$$\mu_1 = 0.205257506\dots\,, \qquad \mu_2 = 2.3439211\dots\,, \qquad \mu_3 = 8.32635\dots \tag{49}$$

with the corresponding energies

$$E_1 = 0.49434\dots\,, \qquad E_2 = 5.16930\dots\,, \qquad E_3 = 7.70234\dots \tag{50}$$

in an agreement with Ref. [18]. The eigenvalues for this state were computed as functions of m for $0 < m < 1$ (see [22]). The energy decreases from $E = E_1$ at $m = 1/2$ in a monotonic way. Only $m = 0, 1$ and $\frac{1}{2}$ are physical values. For other m the curve describes the behavior of the anomalous dimension for corresponding high-twist operators. The energy vanishes at $m = 0, 1$ $(n = \pm 1)$, which follows from its explicit expression given in Ref. [22]

$$E(m, \mu \equiv 0) = \frac{\pi}{\sin(\pi m)} + \psi(m) + \psi(1 - m) - 2\psi(1)\,.$$

Note that $E(m, \mu \equiv 0)$ describes an eigenvalue for which the function $Q^{(1)}$ does not enter in the bilinear combination of the total wave function $Q_{m,\widetilde{m},\mu}$ and therefore here our general method of quantization does not work.

We obtain numerically at $m \to 0$

$$E(m) = 2.152m - 2.754m^2 + \dots\,, \quad \mu_1(m) = 0.375\sqrt{m} - 0.0228m + \dots\,. \tag{51}$$

The state with $m = 1$ and $\widetilde{m} = 0$ (or *vice versa*) is therefore the ground state of the Odderon corresponding to $|n| = 1$. It has a vanishing energy for $\nu \to 0$ and is situated below the eigenstates with $m = \widetilde{m} = 1/2$. Note, that generally this solution is different from that found in Ref. [19] because for it μ is non-zero. The first eigenstate with $n = 2$ was also investigated [22]. The energy proceeds to decrease. The eigenstate with $|n| = 2$ is absent on this trajectory because μ is pure imaginary in this interval and vanishes only at $m = 1$ and $m = 2$.

Let us consider now the Baxter equation for the Quarteton (4 Reggeons state) [22]. A new integral of motion $\mu_4 = q_4$ appears here. The eigenvalues μ and q_4 are assumed to be real, which is compatible with a single-valuedness of the wave function in the $\vec{\rho}$-space. Following the general method presented above one can search solutions of the Baxter equation for the Quarteton in the form of a series of poles. Our quantization procedure gives for $m = \widetilde{m} = 1/2$ [22]

$$\begin{aligned} \mu &= 0\,, \qquad & q_4 = 0.1535892\,, \qquad & E = -1.34832\,, \\ \mu &= 0.73833\,, \qquad & q_4 = -0.3703\,, \qquad & E = 2.34105\,. \end{aligned}$$

One finds for the first eigenvalue with $m = 0$, $\widetilde{m} = 1$, corresponding to $|n| = 1$

$$\mu = 0\,, \qquad q_4 = 0.12167\,, \qquad E = -2.0799\,.$$

The state of the Quarteton with $|n| = 1$ has $m = 0$, $\widetilde{m} = 1$. Its energy is lower than the energy of the above state with $m = \widetilde{m} = \frac{1}{2}$.

The eigenvalue with $\mu = 0$ as a function of m for $0 < m < \frac{1}{2}$ was also calculated (see [22]). Contrary to the Odderon case, the energy eigenvalue does not vanish for $m = 0$. It decreases with m for $0 < m < \frac{1}{2}$ and takes the value $E = -2.0799$ at $m = 0$.

The state with $m = 3/2$ (corresponding to $n = 2$, $\nu = 0$) can be considered as a ground state for the Quarteton because for it the eigenvalue of q_4 is real. It has a large negative energy $E = -5.863$ lower then the energy $E = -5.545$ of the BFKL Pomeron constructed from two Reggeized gluons [22]. But to prove that this state is a physical ground state one should construct a bilinear combination of the corresponding Baxter functions to verify the normalizability of the corresponding solution.

2.5. BFKL Pomeron in the thermostat

In the experiments at RHIC one of the footprints of the quark-gluon plasma is a decrease of the number of produced ψ-mesons due to breaking the confining quark-anti-quark potential at large temperature T. Therefore, it is interesting to investigate the properties of the BFKL Pomeron as a composite state of two Reggeized gluons at a non-zero t-channel temperature [26].

The Green functions at a finite temperature should satisfy the additional symmetry: they should be periodic under the shift of the Euclidean time $x_4 \to x_4 + 1/T$. It leads to the quantization of the corresponding Euclidean energies $E_l = 2\pi l\, T$ in the t-channel. After an analytic continuation of these Green functions to the s-channel with the Regge kinematics $s \gg T^2 \sim -t > 0$ one should impose on them the periodicity condition to the transformation $y \to y + 1/T$ of one of the transverse s-channel coordinates y. Respectively, the canonically conjugated momentum is quantized $k_y^{(l)} = 2\pi\, l\, T$. In this cylinder-type topology it is convenient to introduce the rescaled variables ρ and p:

$$\rho = x + iy \to \frac{1}{2\pi T}\,\rho\,, \qquad p^{(l)} = \frac{p_x^{(l)} - ip_y^{(l)}}{2} \to \pi\, T p^{(l)} \tag{52}$$

with the temperature constraints

$$0 < \mathrm{Im}\rho < 2\pi\,, \qquad \mathrm{Im}\, p^{(l)} = \frac{l}{2}\,, \qquad [p, \rho] = i\,. \tag{53}$$

In this case the BFKL equation is modified, but the holomorphic separability remains [26]

$$H_{12}\Psi = \Psi\,, \qquad H_{12} = h_{12} + h_{12}^*\,, \tag{54}$$

where the holomorphic Hamiltonian is

$$h_{12} = \sum_{r=1}^{2} \left[\Omega(q_r) + \frac{1}{p_r}\, G(\rho_{12})\, p_r \right]\,. \tag{55}$$

The kinetic energy for the Reggeized gluon is

$$\Omega(q) = \frac{\pi T}{2\lambda} + \frac{1}{2}\,[\psi(1+iq) + \psi(1-iq) - 2\psi(1)] \tag{56}$$

and the Green function for the cylinder topology is

$$G(\rho_{12}) = -\frac{\pi\, T}{2\lambda} + \ln\left(2\sinh\frac{\rho_{12}}{2}\right)\,. \tag{57}$$

It turns out that the BFKL equation at a non-zero temperature can be also solved. The reason is that one can find the conformal transformation

$$\rho_r = \ln\, \rho'_r\,, \tag{58}$$

after which the Hamiltonian and the integral of motion take the form, corresponding to the zero temperature [26],

$$h_{12} \;=\; \ln(p'_1\, p'_2) + \frac{1}{p'_1}\log(\rho'_{12})\; p'_1 + \frac{1}{p'_2}\log(\rho'_{12})\; p'_2 - 2\psi(1)\,, \tag{59}$$

$$A \;=\; -(\rho'_{12})^2 \frac{\partial}{\partial\rho'_1}\frac{\partial}{\partial\rho'_2}\,. \tag{60}$$

To verify it one should use the following operator identity

$$\frac{1}{2}\left[\psi\left(1+z\frac{\partial}{\partial z}\right)+\psi\left(-z\frac{\partial}{\partial z}\right)\right]=\ln z+\ln\frac{\partial}{\partial z}\,. \tag{61}$$

Moreover, for the case of n Reggeized gluons the Hamiltonian coincides again with the local Hamiltonian of the integrable Heisenberg spin model, but with the spins realizing another representation of the Möbius group generators [26]:

$$M_k=\partial_k\,,\qquad M_+=e^{-\rho_k}\,\partial_k\,,\qquad M_-=-e^{\rho_k}\,\partial_k\,. \tag{62}$$

It is interesting that the eigenvalue equation for the integral of motion M^2 in the Pomeron case coincides for $t=0$ with the Baxter equation for the above Heisenberg spin model [26].

3. Maximal transcendentality and anomalous dimensions

3.1. Anomalous dimensions of twist-2 operators

The anomalous dimension of twist-2 operators in $N=4$ SUSY in one-loop approximation was calculated comparatively recently [20]. It turns out that it is proportional to $\psi(j-1)-\psi(j)$. In Ref. [20] it was claimed that in this model the evolution equations for the so-called quasi-partonic operators [25] are integrable in LLA. Later the integrability for $N=4$ SUSY was generalized to other operators [27] and to higher loops [28].

The anomalous dimension for twist-2 operators was calculated in 2 loops in Ref. [30] confirming the result obtained with the use of the maximal transcendentality hypothesis [29].

The universal anomalous dimension for the twist-2 operators was found with the use of the maximal transcendentality property in $N=4$ SUSY up to three loops [29–31]:

$$\gamma(j)=\hat{\alpha}\gamma_1(j)+\hat{\alpha}^2\gamma_2(j)+\hat{\alpha}^3\gamma_3(j)+\dots\,,\quad \hat{\alpha}=\frac{\alpha_{\rm s}N_c}{4\pi}\,, \tag{63}$$

where

$$\gamma_1(j+2) = -4S_1(j)\,, \tag{64}$$

$$\frac{\gamma_2(j+2)}{8} = 2S_1\left(S_2+S_{-2}\right)-2S_{-2,1}+S_3+S_{-3} \tag{65}$$

$$\begin{aligned}\frac{\gamma_3(j+2)}{32} = {} & -12\,\left(S_{-3,1,1}+S_{-2,1,2}+S_{-2,2,1}\right)\\ & +6\left(S_{-4,1}+S_{-3,2}+S_{-2,3}\right)-3\,S_{-5}-2\,S_3\,S_{-2}-S_5\\ & -2\,S_1^2\left(3\,S_{-3}+S_3-2\,S_{-2,1}\right)-S_2\left(S_{-3}+S_3-2\,S_{-2,1}\right)\\ & +24\,S_{-2,1,1,1}-S_1\left(8S_{-4}+S_{-2}^2+4S_2S_{-2}+2S_2^2\right)\\ & -S_1\left(3S_4-12S_{-3,1}-10S_{-2,2}+16S_{-2,1,1}\right)\,.\end{aligned} \tag{66}$$

The harmonic sums are defined in a recursive way below:

$$
\begin{aligned}
S_a(j) &= \sum_{m=1}^{j} \frac{1}{m^a}\,, \qquad S_{a,b,c,\dots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\, S_{b,c,\dots}(m)\,, \\
S_{-a}(j) &= \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\,, \qquad S_{-a,b,\dots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\, S_{b,\dots}(m)\,, \\
\overline{S}_{-a,b,c\cdots}(j) &= (-1)^j S_{-a,b,\dots}(j) + S_{-a,b,\cdots}(\infty)\Big(1-(-1)^j\Big)\,.
\end{aligned} \tag{67}
$$

During the last years there was a great progress in the investigation of the $N=4$ SYM theory in a framework of the AdS/CFT correspondence [32]. This model at a strong-coupling regime, $\alpha_{\rm s} N_c \to \infty$, is equivalent to a classical supergravity in the anti-de Sitter space $\mathrm{AdS}_5 \times \mathrm{S}^5$. In particular, a very interesting prediction [33] was obtained for the large-j behavior of the anomalous dimension of twist-2 operators

$$
\gamma(j) = a(z)\ln j\,, \qquad z = \frac{\alpha_{\rm s} N_c}{\pi} \tag{68}
$$

in the strong coupling regime:

$$
\lim_{z\to\infty} a = -\left(\frac{\alpha_{\rm s} N_c}{\pi}\right)^{1/2} + \dots\,. \tag{69}
$$

Note that in our normalization $\gamma(j)$ contains the extra factor $-1/2$ in comparison with that in Ref. [33].

On the other hand, with the use of the asymptotic behavior of the two-loop anomalous dimension γ_2 one can suggest a resummation procedure based on the solution of the following algebraic equation constructed from two first terms of the small-$\alpha_{\rm s}$ expansion of the coefficient a [30]:

$$
\frac{\alpha_{\rm s} N_c}{\pi} = -\widetilde{a} + \frac{\pi^2}{12}\widetilde{a}^2\,. \tag{70}
$$

Using this equation the following large-$\alpha_{\rm s}$ behavior of $\tilde{a}$ can be obtained:

$$
\lim_{\alpha_s\to} \tilde{a} \approx -1.1632\left(\frac{\alpha_{\rm s} N_c}{\pi}\right)^{1/2} + \dots\,, \tag{71}
$$

in a rather good agreement with the above result based on the AdS/CFT correspondence. Moreover, the small-$\widetilde{a}$ expansion of the solution of this equation,

$$
\widetilde{a} = -\frac{\alpha_{\rm s} N_c}{\pi} + \frac{\pi^2}{12}\left(\frac{\alpha_{\rm s} N_c}{\pi}\right)^2 - \frac{1}{72}\,\pi^4\left(\frac{\alpha_{\rm s} N_c}{\pi}\right)^3 + \dots\,, \tag{72}
$$

also coincides with a good accuracy with exact calculations up to three loops [31]:

$$a = -\frac{\alpha_s N_c}{\pi} + \frac{\pi^2}{12}\left(\frac{\alpha_s N_c}{\pi}\right)^2 - \frac{11}{720}\pi^4\left(\frac{\alpha_s N_c}{\pi}\right)^3 + \dots \,. \tag{73}$$

The anomalous dimension should be zero for $j = 2$ due to the energy-momentum conservation. It is natural to consider the slope $b = \gamma'(2)$ of the anomalous dimension in this point. To resum the perturbation theory for this quantity one can use the same procedure as above. Namely, it is possible to write the following algebraic equation [30]:

$$\frac{\pi^2}{6}\,\frac{\alpha_s N_c}{\pi} = -\widetilde{b} + \frac{1}{2}\,\widetilde{b}^2\,. \tag{74}$$

Its perturbative solution is

$$\widetilde{b} = -\frac{\pi^2}{6}\,\frac{\alpha_s N_c}{\pi} + \frac{\pi^4}{72}\left(\frac{\alpha_s N_c}{\pi}\right)^2 - \frac{1}{432}\pi^6\left(\frac{\alpha_s N_c}{\pi}\right)^3 + \dots \,. \tag{75}$$

Again this expansion is in a rather good agreement with the exact result up to three loops [31]:

$$b = -\frac{\pi^2}{6}\,\frac{\alpha_s N_c}{\pi} + \frac{\pi^4}{72}\left(\frac{\alpha_s N_c}{\pi}\right)^2 - \frac{1}{540}\pi^6\left(\frac{\alpha_s N_c}{\pi}\right)^3 + \dots \,. \tag{76}$$

Therefore, one can attempt to estimate the strong coupling behavior of b from the above resummation:

$$\lim_{\alpha_s\to\infty} \tilde{b} = \frac{\pi}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{\pi}}\,. \tag{77}$$

It should be compared with the exact result obtained from AdS/CFT correspondence [31]:

$$\lim_{\alpha_s\to\infty} b = \frac{\pi}{2}\sqrt{\frac{\alpha_s N_c}{\pi}}\,. \tag{78}$$

It is important also that the behavior of the anomalous dimension near the singularity at $\omega = j - 1 \to 0$ is in an agreement with the prediction of the BFKL equation in the next-to-leading logarithmic approximation [29]:

$$\lim_{\omega\to 0}\gamma(j) = \frac{4}{\omega}\frac{\alpha_s N_c}{\pi} + 0\left(\frac{\alpha_s N_c}{\pi}\right)^2 + \frac{32\zeta(3)}{\omega^2}\left(\frac{\alpha_s N_c}{\pi}\right)^3 + \dots \,. \tag{79}$$

3.2. Beisert–Eden–Staudacher equation

Using the integrability and maximal transcendentality the integral equation for the anomalous dimension at large j was constructed in all orders of perturbation theory [34, 35]. Its asymptotic behavior in this region is given below:

$$\lim_{j\to\infty}\gamma(j)=-\frac{1}{2}\gamma_K\ln j\,,\quad \gamma=8g^2\sigma(0)=4g\sqrt{2}f(0)\,,\quad g=\sqrt{\frac{\alpha_s N_c}{2\pi}}\,. \tag{80}$$

Here γ_K is the so-called cusp anomalous dimension. It is expressed through the solution of the Eden–Staudacher (ES) equation,

$$\epsilon\, f(x)=\frac{t}{e^t-1}\left(\frac{J_1(x)}{x}-\int\limits_0^\infty dx'\,K(x,x')\,f(x')\right)\,, \tag{81}$$

$$K(x,y)=\frac{J_1(x)\,J_0(y)-J_1(y)\,J_0(x)}{x-y}\,,\qquad \epsilon=\frac{1}{g\sqrt{2}} \tag{82}$$

as follows

$$\gamma_K=4\,g\,\sqrt{2}\,f(0)\,. \tag{83}$$

Using the Mellin transformation

$$f(x)=\int\limits_{-i\infty}^{i\infty}\frac{d\,j}{2\pi\,i}\,e^{x\,j}\phi(j)\,,\qquad \lim_{j\to\infty}\phi(j)=\frac{\gamma_K}{4g\sqrt{2}\,j}\,, \tag{84}$$

one can present $\phi(j)$ as the sum [37]

$$\phi(j)=\sum_{n=1}^{\infty}\phi_{n,\epsilon}(j)\,(\delta_{n,1}-a_{n,\epsilon})\,,\qquad \gamma_K=4g^2(1-a_{1,\epsilon})\,, \tag{85}$$

where

$$\phi_{n,\epsilon}(j)=\sum_{s=1}^{\infty}\frac{\left(\sqrt{(j+s\,\epsilon)^2+1}+j+s\epsilon\right)^{-n}}{\sqrt{(j+s\epsilon)^2+1}}\,. \tag{86}$$

The coefficients $a_{n,\epsilon}$ satisfy the set of linear algebraic equations [37]

$$a_{n,\epsilon}=\sum_{n'=1}^{\infty}K_{n,n'}(\epsilon)\,\left(\delta_{n',1}-a_{n',\epsilon}\right)\,, \tag{87}$$

where the kernel is given by the perturbative expansion

$$K_{n,n'}(\epsilon) = \sum_{R=0}^{\infty} (-1)^R \, \frac{\zeta(2R+n+n')}{(2\epsilon)^{2R+n+n'}} \, S^R_{n,n'} \, . \qquad (88)$$

The coefficients $S^R_{n,n'}$

$$S^R_{n,n'} = 2n \, \frac{(2R+n+n'-1)! \, (2R+n+n')!}{R! \, (R+n)! \, (R+n')! \, (R+n+n')!} \, , \qquad (89)$$

are integer numbers. As a result, the anomalous dimension has the property of the maximal transcendentality in all loops:

$$\gamma_K(\epsilon) = 8 \sum_{k=1}^{\infty} \left(-\frac{1}{4\epsilon^2} \right)^k \sum_{[s_t]} c_{[s_t]} \prod_r \zeta(s_r) \, , \qquad \sum_t s_t = 2k-2 \, , \qquad (90)$$

with the integer coefficients $c_{[s_t]}$ expressed as sums of products of $S^R_{n,n'}$ [37].

It turns out that the solution of the ES equation does not have a consistent asymptotics at large coupling constants [37] in accordance with the fact that the correct equation should include effects of the so-called dressing phase. The necessity of these corrections was understood in the direct four-loop calculations [38]. Beisert, Eden and Staudacher (BES) calculated the dressing phase and constructed a new equation for γ_K [35]. Its perturbative expansion is different from the perturbative expansion of the BS equation only by the change of the sign in the terms, in which the zeta-functions with odd integer arguments appear two times modulo 4. It is possible to derive from this equation the following asymptotics of the cusp anomalous dimension at large coupling constants [37, 39]:

$$\lim_{\alpha_{\rm s} N_c \to \infty} \gamma_K = 2 \left(\frac{\alpha_{\rm s} N_c}{\pi} \right)^1 \Big/ 2 \qquad (91)$$

in agreement with the AdS/CFT prediction [33].

3.3. Anomalous dimension in 4 loops

To calculate the anomalous dimension of the twist-2 operators in 4 loops one can apply the integrability approach based on the asymptotic Bethe Ansatz [28]. The corresponding equations for the Bethe roots u_k are given below:

$$\left(\frac{x_k^+}{x_k^-} \right)^2 = \prod_{r=1}^{j-2} \frac{x_k^- - x_r^+}{x_k^+ - x_r^-} \, \frac{1 - g^2/x_k^+ x_r^-}{1 - g^2/x_k^- x_r^+} \, \exp\left(2\, i \, \theta(u_k, u_r)\right) \, . \qquad (92)$$

Here we used the notations

$$x_k^{\pm} = \frac{u_k^{\pm}}{2} + \sqrt{\frac{(u_k^{\pm})^2}{4} - g^2}\,, \qquad u^{\pm} = u \pm \frac{i}{2} \tag{93}$$

and the dressing phase expansion [35]

$$\theta(u_k, u_j) = 4\zeta(3)\, g^6 (q_2(u_k)\, q_3(u_j) - q_3(u_k) q_2(u_j)) + \dots\,, \tag{94}$$

where q_2 and q_3 are integrals of motion. The solution for $u_k^{\pm}$ allows to find the anomalous dimensions

$$\gamma(g, M) = 2g^2 \sum_{k=1}^{M} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-} \right) . \tag{95}$$

In particular for four loops one can obtain [40]

$$\begin{aligned}\frac{\gamma_4}{256} \;=\; & \mathbf{4} S_{-7} + 6S_7 + 2(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) \\ & + \dots \\ & - 80 S_{1,1,-4,1} - \boldsymbol{\zeta}(3) S_1 (S_3 - S_{-3} + 2S_{-2,1})\,,\end{aligned} \tag{96}$$

where the harmonic sums depend on $j-2$ and dots mean the omitted terms (their number exceeds 200). All these terms satisfy the maximal transcendentality property. The last term appears from the dressing phase.

It turns out that after the analytic continuation of this expression in the complex j-plane we obtain from two first terms the pole $\sim 1/\omega^7$ for $\omega = j - 1 \to 0$, which does not agree with the singularity in this point predicted in 4 loops from the BFKL equation:

$$\lim_{j \to 1} \gamma_4(j) = -\frac{32}{\omega^4} \left(32\zeta_3 + \frac{\pi^4}{9}\omega \right) + \dots\,. \tag{97}$$

It means, that the asymptotic Bethe Ansatz should be modified starting from 4 loops. Namely, one should take into account the wrapping effects [40].

The interesting results were obtained also for the scattering amplitudes at $N = 4$ SUSY for particles on the mass shell [41]. These amplitudes were used in Ref. [42] for the construction of higher loop corrections to the BFKL kernel in this model. It was shown [42] that the BDS anzatz [41] does not satisfy correct factorization properties in the multi-Regge kinematics.

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