COSMOLOGICAL SINGULARITY IN YILMAZ-TUPPER THEORY OF GRAVITATION

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The paper describes some aspects of the early evolution stage of cosmological models based on Yilmaz' theory of gravitation extended by Tupper. A possibility of another interpretation of Tupper's extension is also mentioned. Inspection of the obtained equations shows that at radiation era singular solutions cannot exist (this result is completely independent of the cosmical constant and of space-curvature). It seems to be connected with breaking of the energetic condition of the theorems concerning the singularity in cosmology.

The scalar theory of gravitation presented by Yilmaz (1958) is locally Lorentz invariant and therefore the theory was investigated mainly in non-cosmological approximation of the field of a point mass. It predicts the same values as the Schwarzschild solution for the three classical tests of general relativity. Recently Tupper (1971) presented an extension of the Yilmaz theory to the material case and discussed some cosmological models resulting from his research. The purpose of this note is a discussion of the Tupper extension of the Yilmaz theory — especially in the aspect of a singularity in cosmological solutions.

In Yilmaz' theory there appears a scalar field φ, defining the geometry of empty space. We can choose a reference system in which the line element is given by (relativistic units are used):

$$ds^2 = e^{2\nu(\varphi)}dt^2 - e^{2\lambda(\varphi)}[dr^2 + r^2 d\Omega^2]. \tag{1}$$

There is a general drawback of Yilmaz' theory, namely the lack of covariance in the form of metric (1). Therefore Tupper proposed another sense of Yilmaz' idea. He assumed that φ has only some effect on the geometry of space-time and then the metric tensor is not necessarily the function of φ only.

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In the material case (matter is represented by perfect fluid with density ϱ and pressure p) the condition imposed by Tupper is:

$$\nabla_i \varphi^i = 4\pi(\varrho - 3p). \tag{2}$$

The field equations in this case are given by:

$$G_i^j + \Lambda g_i^j - 8\pi C_i^j = -8\pi \mathring{T}_i^j, \tag{3}$$

with the condition:

$$\nabla_i(\mathring{T}_j^i - C_j^i) = 0, \tag{4}$$

where: G_i^j — Einstein tensor, $\mathring{T}_i^j$ — energy-tensor of perfect fluid, $C_i^j = \frac{1}{8\pi}(2\varphi_i\varphi^j - -\delta_i^j\varphi_s\varphi^s)$, Λ — cosmical constant.

Thus Tupper formally extended the idea of Yilmaz to the material case, but it seems that the above result can be interpreted simply as a variant of Einstein's theory of gravitation with energy-tensor $T_i^j = \mathring{T}_i^j - C_i^j$. From this point of view φ is a "potential" of a specific field modifying the geometry of space-time of perfect fluid.

If we put

$$\nu = 0, \qquad \lambda = \ln \frac{4R}{4 + kr^2}, \tag{5}$$

where $R = R(t)$ is the conventional radius of the Universe, and k — constant space-curvature, then metric (1) becomes the usual Robertson-Walker line element. Equations (3) and (4) are:

$$\frac{\partial \varphi}{\partial r} = 0,$$

$$\frac{d\varrho}{dt} + \frac{3}{R}\frac{dR}{dt}(p+\varrho) - \frac{\partial \varphi}{\partial t}(\varrho - 3p) = 0,$$

$$\frac{d^2\varrho}{dt^2} + \frac{3}{R}\frac{\partial \varphi}{\partial t}\frac{dR}{dt} = 4\pi(\varrho - 3p),$$

$$\frac{3}{R^2}\left[\left(\frac{dR}{dt}\right)^2 + k\right] + \left(\frac{\partial \varphi}{dt}\right)^2 - \Lambda = 8\pi\varrho,$$

$$-\frac{1}{R^2}\left[2R\frac{d^2R}{dt^2} + \left(\frac{dR}{dt}\right)^2 + k\right] + \left(\frac{\partial \varphi}{\partial t}\right)^2 - \Lambda = 8\pi p. \tag{6}$$

As it can be seen, the field φ is a function of t only, like the other parameters R, ϱ, p.

Tupper discussed solutions of equations (6) in case of dust (incoherent) matter with the equation of state $p = 0$. This is connected with a later stage of evolution of the Universe with great dilution of cosmic matter. In early stages of evolution cosmological models are obviously singular at the point $R = 0$ and $\varrho = \infty$. In order to explain this question in the theory discussed we accept the equation of state $p = \frac{1}{3}\varrho$ (radiation Universe).

Equations (6) become:

$$\frac{1}{\varrho}\frac{d\varrho}{dt} + \frac{4}{R}\frac{dR}{dt} = 0, \qquad (i)$$

$$\frac{d^2\varphi}{dt^2} + \frac{3}{R}\frac{d\varphi}{dt}\frac{dR}{dt} = 0, \qquad (ii)$$

$$\frac{3}{R^2}\left[R\frac{d^2R}{dt^2} + \left(\frac{dR}{dt}\right)^2 + k\right] - \left(\frac{d\varphi}{dt}\right)^2 - 2\Lambda = 0. \qquad (iii) \tag{7}$$

Integrating (7) (*i*) we have:

$$\varrho = \frac{\varrho_0}{R^4}, \quad \varrho_0 \text{ — constant,} \tag{8}$$

and with (7) (*ii*) one can obtain:

$$\frac{d\varphi}{dt} = \frac{C}{R^3}, \quad C \text{ — constant.} \tag{9}$$

Then, after some transformations the equation (7) (*iii*) can be brought to the form:

$$\left(\frac{dR}{dt}\right)^2 = \frac{\Lambda}{3}R^2 - k + \frac{C^*}{R^2} - \frac{C^2}{R^4}, \quad C^* \text{ — constant.} \tag{10}$$

The last equation leads to the conclusion that for $R \to 0$ the right-hand side is negative and real solution for $R(t)$ do not exist.

Hence: if equation (10) has a real and continuous solution $R(t)$ with the initial value $R(t_0) > 0$, then this solution cannot possess the singular point $R = 0$.

Thus, this theory in ultrarelativistic limit with scalar $\varphi(t)$ can produce cosmological models with no initial singularity. This conclusion is independent of the values k and Λ and its source is the field φ.

From the relations (8) and (9) it is found that:

$$\frac{d\varphi}{dt} \sim R\varrho. \tag{11}$$

This equation describes the modification of φ in the presence of matter.

We can compare the above results with the known theorems concerning singularity in general relativity presented recently by Hawking, Penrose (1970), Geroch (1967). As it was already pointed the field equations (3) are formally identical with the Einstein equations of general relativity with the right-hand side $[-8\pi(\mathring{T}_i^j - C_i^j)]$. Then one can use the energy-condition (the most essentially physical assumption of all theorems discussed) in the form ($\Lambda = 0$):

$$(\mathring{T}_{ij} - C_{ij}) \frac{dx^i}{ds} \frac{dx^j}{ds} \geqslant \tfrac{1}{2}(\mathring{T}_i^i - C_i^i). \tag{12}$$

In the co-moving system we have:

$$\varrho + 3p \geqslant \frac{1}{2\pi} \left(\frac{d\varphi}{dt} \right)^2. \tag{13}$$

If we accept for the equation of state $p = \frac{1}{3}\varrho$, the inequality (13) leads to the following:

$$R^2 \geqslant (4\pi)^{-1} \frac{C^2}{\varrho_0} > 0. \tag{14}$$

Thus we see a paradoxical fact: one of the conditions of singularity ($R = 0$) is that $R > 0$ always! Note that in Friedmann cosmology ($C = 0$) the inequality (14) is simply the identity $R^2 \geqslant 0$.

Hence, we can conclude that condition (14) is not valid in general and thus the theorems concerning the existence of a singularity with energy-condition (12) are not referred to the cosmological models discussed.

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