THE IMBEDDING METHOD OF FINDING THE MAXIMAL EXTENSIONS OF SOLUTIONS OF EINSTEIN FIELD EQUATIONS

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The problem of obtaining the maximal analytic extension of the solution of Einstein field equations is investigated. We propose a method of finding the maximal analytic extensions by imbedding the corresponding manifolds into a pseudoeuclidean spaces. The method is demonstrated for the Schwarzschild-like, Reissner-Nordström and Kerr solutions. All extensions which we obtain are maximal. We believe that the simplicity of the method will make it useful in physical applications.

1. *Introduction*

Exact solutions of the field equations are, in general, singular. Some of the singularities represent real singularities of the metric and have reasonable physical meaning. The existence of singularities of such a kind does not depend on the choice of parametrization and, in general, leads to an infinitely high curvature. Other singularities of the metric are nonphysical and occur because of poor parametrization. Most familiar examples of such singularities can be found in the static spherically symmetric solution (Schwarzschild solution) and in the Kerr solution.

The problem of completeness is closely related with singularities. The Riemannian manifold is g-complete if every geodesic has infinite length in both directions. If every geodesic has infinite length in both directions, or terminates on a real singularity, the manifold is called maximal. Solutions of Einstein equations are given in a fixed local map covering a rather accidental submanifold which is nonmaximal or not g-complete and is not suitable, if we try to study geometrical and physical features of the manifold. It is, therefore, necessary to look for the maximal analytic extension of the solution. The general method of finding such an extension consist in the analytical continuation of every geodesic.

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2. *An extension of the spherically and axially symmetric fields*

The maximal analytic extension of the Schwarzschild solution has been given by Kruskal [6]. It is summarized by transformation of the original parameters r, θ, φ, t to the new ones u, v, $\bar{\theta}$, $\bar{\varphi}$ given by the following equations

$$u \pm v = \left|\frac{r}{2m} - 1\right|^{\frac{1}{2}} \exp\left(\frac{r \pm t}{4m}\right); \ \bar{\theta} = \theta; \ \bar{\varphi} = \varphi.$$

In the $u, v, \bar{\theta}, \bar{\varphi}$ parametrization the Schwarzschild metric is regular everywhere outside of the really singular point $r=0$. In a similar way, the maximal extension for the Reissner--Nordström metric was given by Graves and Brill [5]. This metric in the case $m^2 - a^2 > 0$ has two pseudosingularities for $r_{\pm} = m \pm (m^2 - a^2)^{1/2}$ and a true singularity for $r = 0$. There exist two Kruskal-like transformations. The first eliminates the r_+ singularity, the second — the r_- singularity. By extension of the metric, first across the r_+ then across r_- and suitable identification of isometrical patches, Graves and Brill have constructed the maximal space for the Reissner-Nordström case. In the same manner Boyer and Lindquist [1] have constructed such an extension of the Kerr metric, describing the gravity field outside the rotating body. They have shown that maximal space can be obtained in another way, too. Instead of topological identification they construct an infinite sequence of Einstein-Rosen bridges.

3. *The imbedding method*

In this paper we investigate a different method of finding the maximal analytic extension for the solution of field equations. It is well known, that, accordingly to the Friedmann theorem [4], any Riemannian manifold $V_n(p, q)$ with analytic metric can be analytically and isometrically immersed in a pseudoeuclidean space $E_m(r, s)$ where $m = {}^1/_2 n(n+1)$ and r, s are any prescribed integers satisfying conditions $r \geqslant p; s \geqslant q$. Investigation of the imbedded curved space gives us a very simple and intuitive form of description of its geometry and topology. Many examples of physically important spaces immersed into flat space were given by Rosen [7]. If global imbedding exists, then its properties do not depend on the local map. We know that the wrong parametrization is the origin of difficulties connected with pseudosingularities. We expect therefore that imbedding, which consists in fact in the elimination of original parameters, may indicate how to remove pseudosingularities of the original parametrization. In this paper we construct the analytic extensions for the Schwarzschild-like, Reissner-Nordström and Kerr fields. All extensions obtained are equivalent up to a diffeomorphism with the extensions constructed by previous authors, and are maximal.

4. *Schwarzschild solution*

We shall start with the case of the Schwarzschild metric. In original parametrization it has the form

$$ds^2 = (1 - b/r)dt^2 - (1 - b/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$
$$b = 2m \tag{1}$$

where

$$t \in (-\infty, +\infty); \quad r \in (b, +\infty); \quad \theta \in [0, \pi); \quad \varphi \in [0, 2\pi); \tag{2}$$

the metric is singular for $r = b$ and $r = 0$. We shall construct an imbedding of the Schwarzschild space. Let us take a 7-dimensional flat space with the signature $(+, -, +, -, -, -, -)$ referred to a Cartesian coordinate system Z^A; $A = 1, \ldots, 7$. The imbedding is given by a set of functions $Z^A = Z^A(t, r, \theta, \varphi)$ satisfying the conditions:

$$g_{\mu\nu} = \eta_{AB} \frac{\partial Z^A}{\partial x^\mu} \frac{\partial Z^B}{\partial x^\nu}; \qquad \begin{matrix} A, B = 1, \ldots, 7 \\ \mu, \nu = 0, \ldots, 3 \end{matrix} . \tag{3}$$

Let us take

$$\begin{aligned} Z^1 &= 2b|1-b/r|^{1/2} \sinh t/2b \\ Z^2 &= 2b|1-b/r|^{1/2} \cosh t/2b \\ Z^3 &= b \ln |r/b| \\ Z^4 &= 2b \frac{1-b/r}{(b/r)^{\frac{1}{2}}} \\ Z^5 &= r \sin \theta \cos \varphi \\ Z^6 &= r \sin \theta \sin \varphi \\ Z^7 &= r \cos \theta. \end{aligned} \tag{4}$$

One can check by substitution, that conditions (3) are fulfilled. Because of the simplicity of the imbedding functions in 7-dimensional flat space we can give explicitly the formulas for Schwarzschild hypersurface.

$$\begin{aligned} &(Z^2/2b)^2 - (Z^1/2b)^2 = 1 - \exp(-Z^3/b) \\ &Z^4 = 4b \sinh(Z^3/4b) \\ &(Z^5)^2 + (Z^6)^2 + (Z^7)^2 = b^2 \exp(2Z^3/b). \end{aligned} \tag{5}$$

It is easy to see that the original Schwarzschild parametrization covers only a part of (5) for which $Z^3 > 0$.

We shall discuss now the mapping of the Kruskal manifold onto our hypersurface. Kruskal space can be illustrated by the following diagramme:

The region 1 (Fig. 1) corresponds to $u^2 - v^2 > 0$, $u > 0$
the region 1* corresponds to $u^2 - v^2 > 0$, $u < 0$
the region 0 corresponds to $u^2 - v^2 < 0$, $v > 0$
the region 0* corresponds to $u^2 - v^2 < 0$, $v < 0$.

The patch 1 ∪ 1* covers the standard Schwarzschild space and is joined with the region 0 ∪ 0* along the line $u^2 - v^2 = 0$ which corresponds to throat of the Einstein-Rosen bridge. By a simple change of parameters in (4) we see that

region 1 is mapped on the region $Z^3 > 0$, $Z^2 > 0$ in (5)
region 1* is mapped on the region $Z^3 > 0$, $Z^2 < 0$ in (5)
region 0 is mapped on the region $Z^3 < 0$, $Z^1 > 0$ in (5)
region 0* is mapped on the region $Z^3 < 0$, $Z^1 < 0$ in (5).

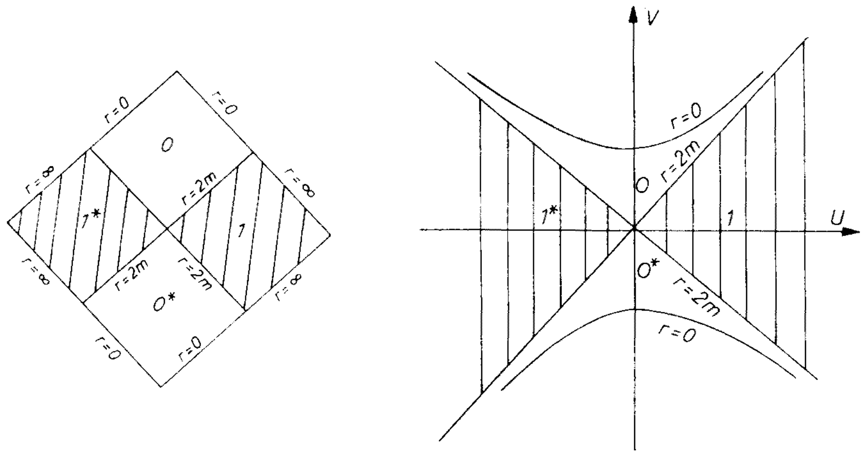

Fig. 1. The Kruskal manifold in u, v axes

This proves that the manifold (5) is indeed maximal.

Our results for other spherically symmetric spaces are presented in the Table I.

5. *The Reissner-Nordström and Kerr metric*

The pseudosingularities of the Reissner-Nordström and Kerr metrics are very similar. We shall consider the case $m^2 - a^2 > 0$ in which there exist two pseudosingularities $r_\pm = m \pm (m^2 - a^2)^{1/2}$. The difference with the Schwarzschild case is that there does not exist a single global map for the maximal space. By a similar procedure we have obtained results presented in Table II.

We shall discuss in detail the maximal extension for Reissner-Nordström and Kerr solutions. On Fig. 2 results obtained by Graves and Brill and by Boyer and Lindquist are illustrated.

The parameters are chosen as follows

$$\underset{(n)}{u} \pm \underset{(n)}{v} = \left(\frac{r-r_+}{2m}\right)^{\frac{1}{2}} \left(\frac{r-r_-}{2m}\right)^{-\frac{1}{2}\cdot r_-/r_+} \exp\left(\frac{r \pm t}{2\sigma_+}\right)$$

$$\underset{(n)}{u'} + \underset{(n)}{v'} = \exp\left(\frac{r-t}{2\sigma_-}\right); \quad \underset{(n)}{u'} - \underset{(n)}{v'} = \left(\frac{r_- - r}{2m}\right)\left(\frac{r_+ - r}{2m}\right)^{-r_+/r_-} \exp\left(\frac{t-r}{2\sigma_-}\right);$$

$$\sigma_\pm = m r_\pm (m^2 - a^2)^{-\frac{1}{2}}$$

TABLE I

Type	Metric	Imbedding functions	Hypersurface	Original domain
A2	$ds^2 = (b/Z-1)dt^2-(b/Z-1)^{-1}dZ^2-$ $-Z^2(dr^2+\sinh^2 r d\varphi^2)$	$(+,-,-,+,-,-,+)$ $Z^1 = 2b\lvert b/Z-1\rvert^{1/2}\sinh t/2b$ $Z^2 = 2b\lvert b/Z-1\rvert^{1/2}\cosh t/2b$ $Z^3 = b\ln\lvert Z/b\rvert$ $Z^4 = 2b(1-b/Z)(b/Z)^{-1/2}$ $Z^5 = Z\sinh r\sin\varphi$ $Z^6 = Z\sinh r\cos\varphi$ $Z^7 = Z\cosh r$	$(Z^2/2b)^2-(Z^1/2b)^2 = \exp(-Z^3/b)-1$ $Z^4 = 4b\sinh Z^3/2b$ $(Z^7)^2-(Z^6)-(Z^5)^2 = b^2\exp(2Z^3/b)$	$Z^3 > 0$
B1	$ds^2 = r^2(\sin^2\theta dt^2-d\theta^2)-$ $-(1-b/r)d\varphi^2-(1-b/r)^{-1}dr^2$	$(-,-,+,-,+,-,-)$ $Z^1 = 2b\lvert 1-b/r\rvert^{1/2}\sin\varphi/2b$ $Z^2 = 2b\lvert 1-b/r\rvert^{1/2}\cos\varphi/2b$ $Z^3 = b\ln\lvert r/b\rvert$ $Z^4 = 2b(1-b/r)(b/r)^{-1/2}$ $Z^5 = r\sin\theta\sinh t$ $Z^6 = r\sin\theta\cosh t$ $Z^7 = r\cos\theta$	$(Z^2/2b)^2+(Z^1/2b)^2 = 1-\exp(-Z^3/b)$ $Z^4 = 4b\sinh Z^3/2b$ $(Z^6)^2-(Z^7)^2+(Z^5)^2 = b^2\exp(2Z^3/b)$	$Z^3 > 0$
B2	$ds^2 = Z^2(\sinh^2 r dt^2-dr^2)-$ $-(b/Z-1)^{-1}dZ^2-(b/Z-1)\,d\varphi^2$	$(-,-,-,+,+,-,+)$ $Z^1 = 2b\lvert b/Z-1\rvert^{1/2}\sin\varphi/2b$ $Z^2 = 2b\lvert b/Z-1\rvert^{1/2}\cos\varphi/2b$ $Z^3 = b\ln\lvert Z/b\rvert$ $Z^4 = 2b(1-b/Z)(b/Z)^{-1/2}$ $Z^5 = Z\sinh r\sinh t$ $Z^6 = Z\sinh r\cosh t$ $Z^7 = Z\cosh r$	$(Z^2/2b)^2+(Z^1/2b)^2 = \exp(-Z^3/b)-1$ $Z^4 = 4b\sinh Z^3/2b$ $(Z^7)^2-(Z^6)^2+(Z^5)^2 = b^2\exp(2Z^3/b)$	$Z^3 > 0$

TABLE II

Type	Metric	Imbedding functions	Hypersurface	Original domain
R—N	$ds^2 = \left(1 - \frac{2m}{r} + \frac{a^2}{r^2}\right) dt^2 -$ $- \left(1 - \frac{2m}{r} + \frac{a^2}{r^2}\right)^{-1} dr^2 -$ $-r^3(\sin^2\theta d\varphi^2 + d\theta^2);$ $m^2 > a^2$	$(-, -, -, +, -, -, +, -)$ $Z^1 = r \sin\theta \cos\varphi$ $Z^2 = r \sin\theta \sin\varphi$ $Z^3 = r \cos\theta$ $Z^4 = \lvert 1-2m/r+a^2/r^2\rvert^{1/2} \sinh t$ $Z^5 = \lvert 1-2m/r+a^2/r^2\rvert^{1/2} \cosh t$ $Z^6 = \lvert 1-2m/r+a^2/r^2\rvert^{1/2}$ $Z^7 = r$ $Z^8 = \int \frac{rdr}{(r^2-2mr+a^2)}$	$(Z^1)^2+(Z^2)^2+(Z^3)^2-(Z^7)^2 = 0$ $[(Z^5)^2-(Z^4)^2]^2 = (Z^6)^4$ $[(Z^5)^2-(Z^4)^2]^2 = [1-2m/Z^7+a^2/(Z^7)^2]^2$ $Z^8 = f(Z^7)$	$Z^7 > r_+$
Kerr	$-ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) +$ $+(r^2+a^2)\sin^2\theta d\varphi^2 - dt^2 +$ $+ \frac{2mr}{\Sigma}(a \sin^2\theta d\varphi + dt)^2;$ $\Sigma = r^2 + a^2\cos^2\theta;$ $\Delta = r^2 - 2mr + a^2$	$(+, -, -, +, -, +, -, -, +,$ $-, -, -, +, -, -,)$ $Z^1 = \lvert A(r,\theta)\rvert \sinh t$ $Z^2 = \lvert A(r,\theta)\rvert \cosh t$ $Z^3 = \lvert B(r,\theta)\rvert \sinh(t-\varphi)$ $Z^4 = \lvert B(r,\theta)\rvert \cosh(t-\varphi)$ $Z^5 = \lvert C(r,\theta)\rvert \sinh(t+\varphi)$ $Z^6 = \lvert C(r,\theta)\rvert \cosh(t+\varphi)$ $Z^7 = r$ $Z^8 = a \sin\theta$ $Z^9 = A(r,\theta)$ $Z^{10} = B(r,\theta)$ $Z^{11} = C(r,\theta)$ $Z^{12} = r \sin\theta$ $Z^{13} = r \sinh[\alpha(r)] \cos\theta$ $Z^{14} = r \cosh[\alpha(r)] \cos\theta$ $Z^{15} = \alpha(r) \stackrel{\mathrm{df}}{=} \int \frac{rdr}{\Delta}$	$[(Z^2)^2-(Z^1)^2]^2 = \left\{\frac{(Z^7)^2+a^2}{a^2}(Z^8)^2 + 1 -\right.$ $- \left(1 + \frac{(Z^8)^2}{a} + \frac{(Z^8)^4}{a^2}\right) \times$ $\left.\times \frac{2mZ^7}{(Z^7)^2-(Z^8)^2+a^2}\right\}^2 \stackrel{\mathrm{df}}{=} A^4(Z^7, Z^8)$ $4[(Z^6)^2-(Z^5)^2]^2 = \left\{\frac{(Z^7)^2+a^2}{a^2}(Z^8)^2 +\right.$ $\left.+ \frac{(Z^8)^4}{a^2} \cdot \frac{2mZ^7}{(Z^7)^2-(Z^8)^2+a^2}\right\}^2 \stackrel{\mathrm{df}}{=} B^4(Z^7, Z^8)$ $4[(Z^4)^2-(Z^3)^2]^2 = \left\{\frac{(Z^7)^2+a^2}{a^2}(Z^8)^2 +\right.$ $+\left(\frac{(Z^8)^4}{a^2} - 2\frac{(Z^8)^2}{a}\right) \times$ $\left.\times \frac{2mZ^7}{(Z^7)^2-(Z^8)^2+a^2}\right\}^2 \stackrel{\mathrm{df}}{=} C^4(Z^7, Z^8)$ $2\,\mathrm{tgh}^{-1}(Z^1/Z^2) = \mathrm{tgh}^{-1}(Z^3/Z^4) + \mathrm{tgh}^{-1}(Z^5/Z^6)$	$Z > r_+$

The patch K_n represents the domain covered by $\underset{(n)}{u}$, $\underset{(n)}{v}$ Kruskal-like parametrization. It is an extension across the r_+ pseudosingularity. The K_n' represents the domain covered by the $\underset{(n)}{u'}$, $\underset{(n)}{v'}$ parametrization and forms an extension across the r_- pseudosingularity. The $K_n \cup K_n'$ formed the fundamental patch. It is a base for the construction of the maximal extension. In case (a) topological identification of the $(n-1)$ with the $(n+3)$

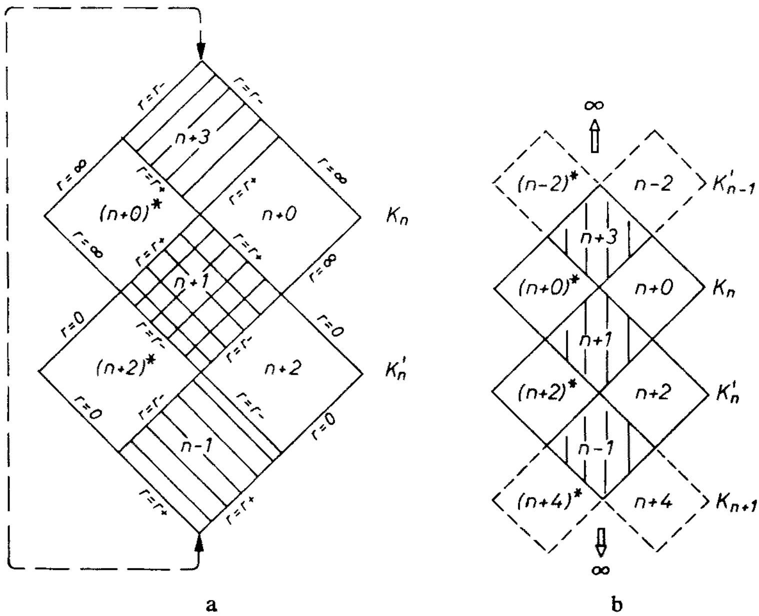

Fig. 2. Maximal extensions of Reissner-Nordström and Kerr fields, a) with topological identification, b) with an infinite sequence of Einstein-Rosen bridges

is possible because these two domain are isometric. The isomorphism between the related parts of the K_n, K_n' has the form

$$(\underset{(n)}{u}+\underset{(n)}{v})^{r_+}(-\underset{(n)}{u'}-\underset{(n)}{v'})^{r_-} = 1; \quad (-\underset{(n)}{u}+\underset{(n)}{v})^{r_+}(\underset{(n)}{u'}-\underset{(n)}{v'}) = 1.$$

It is easy to see that there exists a one-to-one mapping such that

		R−N field	Kerr field
Patch ..., $(n+0)$, ...	is mapped on	$Z^7 > r_+, Z^5 > 0$	or $Z^7 > r_+, Z^2 > 0$
Patch ..., $(n+0)^*$, ...	is mapped on	$Z^7 > r_+, Z^5 < 0$	or $Z^7 > r_+, Z^2 < 0$
Patch ..., $(n+2)$, ...	is mapped on	$Z^7 < r_-, Z^5 > 0$	or $Z^7 < r_-, Z^2 > 0$
Patch ..., $(n+2)^*$,...	is mapped on	$Z^7 < r_-, Z^5 < 0$	or $Z^7 < r_-, Z^2 < 0$
Patch ..., $(n+1)$, ...	is mapped on	$r_- < Z^7 < r_+, Z^4 > 0$	or $r_- < Z^7 < r_+, Z^1 > 0$
Patch ..., $(n-1)$, ...	is mapped on	$r_- < Z^7 < r_+, Z^4 < 0$	or $r_- < Z^7 < r_+, Z^1 < 0$

(the domain $r < 0$ is not connected whith the domain $r > 0$ and gives an unphysical case $m < 0$).

The maximal extensions defined by the infinite sequence of basic patches present no essential complication. The difficulty which arises from the fact that different patches are mapped on the same domain of our hypersurfaces can be resolved. We have to construct a sort of Riemann hypersurface with an infinity regular branches. Upon such a surface different patches are mapped on the different regular branches.

6. *Remarks*

All geometrical and physical properties of the obtained extensions are given in [1]. This work only presentes the imbedding method which gives a systematic and very simple way for the finding of analytic extensions of solutions of Einstein field equations.

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