ON AFFINE PROPERTIES OF THE LIGHT CONE AND THEIR APPLICATION IN THE QUANTUM ELECTRODYNAMICS

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It is shown that there exists on the light cone an affine connection which is metric, semisymmetric and totally integrable. The connection is used to displace parallelly polarization vectors of the photon; a new gauge for the electromagnetic field is thus introduced and the commutator of two potentials in the new gauge is calculated. The commutator is invariant with respect to the four-parameter group which leaves invariant a fixed null direction.

1. *Introduction*

Polarization of a photon is determined by a unit vector e^μ orthogonal to the energy-momentum vector k^μ. We shall investigate the following question: is there any natural way to fix polarization of all photons, if polarization of one, arbitrarily chosen photon, has been fixed? Clearly, to answer this question we have to investigate a general problem of displacement of polarization vectors. The polarization vectors are to be displaced within the set of all energy-momentum vectoıs *i. e.* within the light cone

$$g_{\mu\nu}k^\mu k^\nu \equiv (k^0)^2-(k^1)^2-(k^2)^2-(k^3)^2 = 0, \quad k^0 > 0. \tag{1.1}$$

2. *Affine connection within the light cone*

Any four functions $k^\mu(u^i)$, $i = 1, 2, 3$, such that $g_{\mu\nu}k^\mu k^\nu \equiv 0$ and $\partial(k^1, k^2, k^3)/\partial(u^1, u^2, u^3) > 0$, determine a parametrization of the light cone. There are two well-known invariants upon the cone: the metric

$$ds^2 = -g_{\mu\nu}dk^\mu dk^\nu = -g_{\mu\nu}\frac{\partial k^\mu}{\partial u^i}\frac{\partial k^\nu}{\partial u^k}du^i du^k \stackrel{\mathrm{df}}{=} g_{ik}du^i du^k, \tag{2.1}$$

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and the volume

$$dv = \frac{dk^1 dk^2 dk^3}{k^0} = \frac{1}{k^0}\frac{\partial(k^1, k^2, k^3)}{\partial(u^1, u^2, u^3)} du^1 du^2 du^3 \overset{\mathrm{df}}{=} w du^1 du^2 du^3. \tag{2.2}$$

Since det $(g_{ik}) = 0$, the invariants (2.1) and (2.2) are independent. Usually one introduces an affine connection as the solution of equations

$$\nabla_i g_{kl} \equiv \partial_i g_{kl} - \Gamma^n_{ik} g_{nl} - \Gamma^n_{il} g_{kn} = 0 \tag{2.3}$$

and

$$\Gamma^i_{kl} - \Gamma^i_{lk} = 0. \tag{2.4}$$

If, however, det $(g_{ik}) = 0$, equations (2.3) and (2.4) are in general incompatible [1]; for the light cone they are incompatible indeed. It is necessary to reject one of them. Lemmer [2] and Daŭtcourt [3] reject (2.3); we prefer to reject (2.4) or, more precisely, to replace (2.4) by a weaker condition. We cannot reject the condition (2.3) because the condition means that a parallelly displaced vector preserves its length; it is clear that displacement of polarization vectors should have this property. It turns out that the strongest condition compatible with (2.3) may be formulated as follows: the affine connection should be semisymmetric *i. e.* there should exist a vector S_k such that

$$\Gamma^i_{kl} - \Gamma^i_{lk} = S_k \delta^i_l - S_l \delta^i_k. \tag{2.4a}$$

There is a wide class of connections satisfying conditions (2.3) and (2.4a); we add therefore two further conditions:

$$\nabla_i w \equiv \partial_i w - \Gamma^n_{in} w = 0 \tag{2.5}$$

and

$$R^k_{mnl} \equiv \partial_n \Gamma^k_{ml} - \partial_m \Gamma^k_{nl} + \Gamma^k_{nt}\Gamma^t_{ml} - \Gamma^k_{mt}\Gamma^t_{nl} = 0. \tag{2.6}$$

Strictly speaking, the condition (2.5) follows from (2.6) but it is convenient to introduce (2.5) explicitly, because (2.5) is an algebraic condition while (2.6) a differential one. The condition (2.6) implies that a vector displaced parallelly along a closed curve returns back to its original direction; the condition (2.5) means that a volume determined by three linearly independent vectors is preserved in the process of parallel displacement.

Since all the conditions (2.3), (2.4a), (2.5) and (2.6) are covariant, we may solve them in a particular coordinate system. We shall use in this paper the stereographic parametrization of the light cone; denoting $u^1 = \omega$, $u^2 = x$, $u^3 = y$, we have

$$k^0 = \omega, k^1 = \omega f x, k^2 = \omega f y, k^3 = \omega(2f-1), \tag{2.7}$$

where

$$\frac{1}{f} = 1 + \tfrac{1}{4}(x^2+y^2). \tag{2.8}$$

In the stereographic coordinates $ds^2 = \omega^2 f^2 (dx^2+dy^2)$ and $dv = \omega f^2 d\omega dx dy$.

3. *Fields of parallel vectors upon the cone*

To find the connection one has to solve Eq. (2.3), (2.4a), (2.5) and (2.6) with respect to 27 unknown functions Γ^i_{kl}. We omit details of calculations which are essentially trivial. It turns out that the vector S_i is a gradient: $S_i = \partial_i G$, where

$$G(\omega, x, y) = \ln(\omega f) + G_0(x, y) \tag{3.1}$$

and G_0 is an arbitrary harmonic function:

$$\frac{\partial^2 G_0}{\partial x^2} + \frac{\partial^2 G_0}{\partial y^2} = 0. \tag{3.2}$$

Since the connection is totally integrable, there exist three linearly independent fields $\underset{s}{e_i}$ of parallel vectors; they have the form

$$\underset{1}{e_i} = S_i, \quad \underset{2}{e_i} = \omega f(0, \cos F, \sin F), \quad \underset{3}{e_i} = \omega f(0, -\sin F, \cos F), \tag{3.3}$$

where F is a harmonic function conjugate with G_0: $F_x = -G_{0y}$, $F_y = G_{0x}$. The connection Γ^i_{kl} may be obtained as the algebraic solution of equations $\nabla_i \underset{s}{e_k} = 0$.

It follows from (3.3) that a vector displaced parallelly along a closed curve may rotate; to avoid this we shall assume that

$$\oint dF = 2n\pi, n = 0, \pm 1, \pm 2, \ldots, \tag{3.4}$$

for every closed curve. It should be remembered that this global integrability condition holds for smooth curves only, because only for smooth curves the process of parallel displacement is determined.

4. *Lorentz invariance of the affine connection*

We have determined the connection up to an arbitrary harmonic function; our procedure has been Lorentz invariant. Is it possible to impose further Lorentz invariant conditions and to determine the connection uniquely? The answer is negative: each particular choice of G_0 breaks the Lorentz invariance. In fact, the Lorentz transformation may be written in the form

$$z = \frac{\alpha z' + \beta}{\gamma z' + \delta}, \text{ where } z = x + iy \text{ and } \alpha\delta - \beta\gamma = 1. \tag{4.1}$$

Since $G(\omega, z)$ is a scalar, $G(\omega', z') = G(\omega(\omega', z'), z(z'))$ and therefore

$$\ln[\omega' f(z')] + G_0'(z') = \ln[\omega(\omega', z') f(z(z'))] + G_0(z(z')). \tag{4.2}$$

Hence

$$G_0'(z') = G_0(z(z')) + \ln\frac{\omega(\omega', z') f(z(z'))}{\omega' f(z')} = G_0(z(z')) - \ln\left|\frac{dz}{dz'}\right|. \tag{4.3}$$

The connection will be Lorentz invariant if for all homographic substitutions (4.1), $G'_0(z') = G_0(z')$; this condition, however, can never be fulfilled and therefore the affine connection invariant with respect to the Lorentz group does not exist. It is interesting to find how the Lorentz symmetry is broken for the simplest choice of G_0, namely for $G_0 = 0$. We see from (4.3) that this particular choice of G_0 is invariant with respect to the group of substitutions of the form

$$z = az' + b, \ |a| = 1. \tag{4.4}$$

It is well known that the subgroup (4.4) is the subgroup which preserves a null vector; our parametrization is chosen in such a way that substitutions (4.4) preserve the vector $k^\mu = (1, 0, 0, -1)$.

5. *The polarization vectors*

Having established an integrable connection within the light cone we can fix polarization of all photons from the following principle: polarization vectors should form a parallel field upon the cone. This principle is neither covariant nor unique since it involves an arbitrary choice of the function G_0. Nevertheless, the original arbitrariness in the choice of polarization vectors is substantially reduced. If no conditions are imposed, any vector $e^\mu(k^\lambda)$ such that $g_{\mu\nu}k^\mu e^\nu = 0$ and $g_{\mu\nu}e^\mu e^\nu = -1$ may be chosen as a polarization vector; hence $e^\mu(k^\lambda)$ is determined up to two functions of three variables; in our construction it is determined up to one harmonic function of two variables. The parallel vectors (3.3) are referred to an internal coordinate system upon the cone. In applications it is more convenient to treat polarization vectors as vectors in the Minkowski space. We shall therefore project the internal vectors (3.3) to the fourdimensional space. From the covariant triad (3.3) we construct the inverse triad $\overset{t}{e}{}^i$ as the solution of equations

$$\underset{s}{e}{}_i \overset{t}{e}{}^i = \delta^t_s, \qquad s, t = 1, 2, 3, \tag{5.1}$$

and project the parallel vectors $\overset{t}{e}{}^i$ to the fourdimensional space:

$$\overset{t}{e}{}^\mu = \overset{t}{e}{}^i \frac{\partial k^\mu}{\partial u^i}. \tag{5.2}$$

One easily finds that $\overset{1}{e}{}^\mu = k^\mu$; $\overset{2}{e}{}^\mu$ and $\overset{3}{e}{}^\mu$ are unit space-like vectors orthogonal to k^μ and to each other; we shall use them as polarization vectors.

6. *Rigging of the light cone*

It is impossible to rig[1] the light cone in a Lorentz invariant way. If, however, polarization vectors have been chosen, an invariant rigging becomes possible. We shall introduce a rigging of the light cone by means of the future oriented null vector m^μ which satisfies conditions: $m_\mu k^\mu = 1$, $m_\mu \overset{2}{e}{}^\mu = 0$, $m_\mu \overset{3}{e}{}^\mu = 0$.

[1] We use this word after J. A. Schouten [4].

7. *The commutator* $[A_\mu(x), A_\nu(y)]$

The above constructed affine connection has some remarkable applications in the classical electrodynamics [5]; in this paper we shall apply the connection to find a new gauge for the quantized electromagnetic field.

Let $A_\mu(x)$ denote components of the electromagnetic vector potential. If only transverse degrees of freedom are taken into account, the commutator $[A_\mu(x), A_\nu(y)] = iD_{\mu\nu}(x-y)$ may be written in the form

$$D_{\mu\nu}(x) = \frac{1}{(2\pi)^3}\int dv\,(-\sum_A \overset{A}{e}_\mu \overset{A}{e}_\nu)\sin(k_\lambda x^\lambda); \tag{7.1}$$

here dv is the invariant volume (2.2), $\overset{A}{e}_\mu$ are polarization vectors, $k^0 = \sqrt{(k^1)^2+(k^2)^2+(k^3)^2}$. Usually one chooses the polarization vectors in this way: in a fixed inertial frame of reference one puts $\overset{A}{e}_0 = 0$. This condition determines the sum over polarization vectors uniquely and the commutator takes on the form [6]:

$$D_{00}(x) = 0, \quad D_{0i}(x) = 0, \quad D_{ik}(x) = \left(g_{ik} + \frac{1}{\Delta}\partial_i\partial_k\right)D(x), \tag{7.2}$$

$$i, k = 1, 2, 3,$$

where Δ is the Laplace operator and D is the Pauli-Jordan function. Let us see what happens if polarization vectors are chosen as parallel fields upon the light cone; we shall investigate only the simplest case $G_0 = 0$. Taking into account the algebraic identity

$$-\sum_A \overset{A}{e}_\mu \overset{A}{e}_\nu \equiv g_{\mu\nu} - k_\mu m_\nu - k_\nu m_\mu, \tag{7.3}$$

where m_μ is the rigging vector, we may write

$$D_{\mu\nu}(x) = g_{\mu\nu}D(x) + \partial_\mu \Lambda_\nu(x) + \partial_\nu \Lambda_\mu(x), \tag{7.4}$$

where

$$D(x) = \frac{1}{(2\pi)^3}\int dv\,\sin(k_\lambda x^\lambda) \tag{7.5}$$

is the Pauli-Jordan function and

$$\Lambda_\mu(x) = \frac{1}{(2\pi)^3}\int dv m_\mu \cos(k_\lambda x^\lambda). \tag{7.6}$$

For $G_0 = 0$, $m_0 = m_3 = 1/2\omega f$, $m_1 = m_2 = 0$; consequently $\Lambda_1 = \Lambda_2 = 0$ and

$$\Lambda_0 = \Lambda_3 = \frac{1}{2(2\pi)^3}\int_0^\infty d\omega \iint dxdyf\cos\omega[x^0-(xx^1+yx^2)f-(2f-1)x^3], \tag{7.7}$$

where $1/f = 1+\frac{1}{4}(x^2+y^2)$. Unfortunately, the integral (7.7) is logarithmically divergent for $\sqrt{x^2+y^2} \to \infty$; the divergence is not connected with the fundamental divergencies of quantum electrodynamics, which occur for $\omega \to \infty$. In (7.7) the angular part of the integral diverges in the result of our choice of polarization vectors. We shall calculate the angular part of (7.7) as follows:

$$\iint \frac{dxdy}{1+\frac{1}{4}(x^2+y^2)} \cos \omega[x^0-(xx^1+yx^2)f-(2f-1)x^3] =$$

$$= \iint \frac{dxdy}{1+\frac{1}{4}(x^2+y^2)} \{\cos \omega[x^0-(xx^1+yx^2)f-(2f-1)x^3]-\cos \omega(x^0+x^3)\}+$$
$$+\cos \omega(x^0+x^3) \iint \frac{dxdy}{1+\frac{1}{4}(x^2+y^2)} \,. \tag{7.8}$$

The first integral is convergent; the second is an infnite constant. Integrating over ω we get

$$\Lambda_0 = \frac{1}{4\pi}\left\{\frac{\theta(-x_\mu x^\mu)}{|x^0+x^3|} - \delta(x^0+x^3) \ln(-\omega_0^2 x_\mu x^\mu) + \text{const. inf. } \delta(x^0+x^3)\right\}; \tag{7.9}$$

here

$$\theta(x) = \begin{cases} 1 & \text{for} \quad x>0; \\ 0 & \text{for} \quad x \leqslant 0; \end{cases} \tag{7.10}$$

ω_0 is an arbitrarily chosen frequency introduced for dimensional reasons and const. inf. is the infinite constant. It is clear from (7.9) that the infinite constant may be interpreted as an arbitrary additive constant in the logarithmic potential.

8. Properties of the commutator

The procedure applied in the previous Section is certainly objectionable. If, however, the infinite constant in (7.9) is replaced by a finite one, the expression (7.9) has a definite meaning and we can forget that it has been defined by a divergent integral. The finite constant may be absorbed into ω_0 and the vector Λ_μ takes on the form

$$\Lambda_0 = \Lambda_3 = \frac{1}{4\pi}\left\{\frac{\theta(-x_\mu x^\mu)}{|x^0+x^3|} - \delta(x^0+x^3) \ln(-\omega_0^2 x_\mu x^\mu)\right\},$$
$$\Lambda_1 = \Lambda_2 = 0. \tag{8.1}$$

The vector Λ_μ has the following properties: (a) $\Box \Lambda_\mu = 0$; (b) $\partial^\mu \Lambda_\mu = -D$ where D is the Pauli-Jordan function; (c) the vector Λ_μ is invariant with respect to the four-parameter subgroup of the Lorentz group, which preserves the null direction $k^1/k^0 = k^2/k^0 = 0$, $k^3/k^0 = -1$. It is seen from (7.4) that these properties imply the following properties of the commutator $D_{\mu\nu}(x)$: (a) $\Box D_{\mu\nu}(x) = 0$; (b) $\partial^\mu D_{\mu\nu}(x) = 0$, $\partial^\nu D_{\mu\nu}(x) = 0$; (c) the tensor $D_{\mu\nu}(x)$ is invariant with respect to the Lorentz transformations which preserve the null direction $k^1/k^0 = k^2/k^0 = 0$, $k^3/k^0 = -1$. It is clear that the particular gauge introduced above is similar to the Coulomb gauge: the role of the rotation group is taken over by the subgroup preserving a null direction. (The affine connection with $G_0 = 0$ is invariant with respect to the three-parameter subgroup which preserves a null vector; the commutator contains only the sum over polarization vectors and this gives rise to an additional symmetry.)

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