NUCLEAR INERTIAL MASS PARAMETER FOR DIFFERENT FORMS OF THE QUADRUPOLE FORCES

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The inertial mass parameter B for collective oscillations of the quadrupole types is calculated. The quadrupole forces are taken in the form $Q = F(r)\, Y_{20}(\Omega)$, where $F(r)$ assumes three different functional forms. Numerical calculations are performed in the Rare-Earth region.

The inertial mass parameter B plays an essential role in all microscopic calculations concerning the collective motion of nuclei, such as vibrations and fission process. The energy of the vibrational state is given by

$$E_{\text{vib}} = \hbar\omega_{\text{vib}} = \hbar\sqrt{C/B} \tag{1}$$

where C is the stiffness parameter (see, for example, Ref. [1]). When investigating the stability of the nucleus against spontaneous fission one gets in the WKB approximation the probability for the penetration of the fission barrier as:

$$P = \exp\left[-2\int_{\varepsilon_1}^{\varepsilon_2}\sqrt{\frac{2B}{\hbar}(V(\varepsilon)-E)}\,d\varepsilon\right], \tag{2}$$

where $V(\varepsilon)$ is the deformation dependent potential energy of the fissioning nucleus (see for example Ref. [2]).

It is easily seen that the determination of the inertial mass parameter is essential in finding the half-life for the spontaneous fission process; thus it plays an important role in the estimates of lifetimes of the nuclei in the new regions of stability (*cf.* Refs [2], [3]).

The microscopic calculations of the mass parameter B as the function of the deformation ε and the pairing forces strength were made in Ref. [3] with the assumption of the quadrupole forces in the form: $\hat{Q} \sim r^2 Y_{20}(\Omega)$ as responsible for the collective motion.

In this paper we shall investigate the dependence of the mass parameter on the radial form of the quadrupole forces.

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The collective Hamiltonian with the quadrupole forces is:

$$H = H_0 - \frac{\kappa}{2} \cdot \hat{Q} \cdot \hat{Q}. \tag{3}$$

One can introduce the so called "generating" one-body Hamiltonian:

$$H = H_0 - \kappa \langle \hat{Q} \rangle \cdot \hat{Q}, \tag{4}$$

where $\langle \hat{Q} \rangle$ is the expectation value of the $\hat{Q}$ operator. One usually assumes:

$$\hat{Q} = r^2 Y_{20}(\Omega). \tag{5}$$

When expressing the total energy of the nucleus in terms of the collective variable $Q = \langle \hat{Q} \rangle$, one gets:

$$E(Q) = \frac{B_Q}{2} \cdot Q^2 + \frac{C_Q}{2} \cdot Q^2, \tag{6}$$

where B_Q and C_Q are mass and stiffness parameters, respectively.

When adiabatic and harmonic approximations are applied to the problem, one gets for the mass parameter B_Q:

$$B_Q = h^2 \frac{2 \sum_3}{(2 \sum_1)^2} \tag{7}$$

where

$$\sum_i = \sum_{k \neq 0} \frac{|\langle k|\hat{Q}|0\rangle|^2}{(E_k - E_0)^i}. \tag{8}$$

E_i and $|i\rangle$ are the eigenvalues and eigenstates of Eq. (4), E_0 and $|0\rangle$ are the lowest eigenvalue and corresponding eigenstate.

Instead of the assumption (5) as regards the quadrupole forces, we shall follow the argumentation by Bohr and Mottelson (Ref. [5]) leading to the derivation of the radial form factor for the quadrupole forces when the potential and its deformation dependence are given.

Let us consider the Hamiltonian (4). We can write it in the form:

$$H = T + V(\bar{r}). \tag{9}$$

Next we shall impose a quadrupole deformation on the system, *i.e.* we transform the sphere $r = R_0$ so as to get the surface described by:

$$r = R_0 \left(1 + \beta Y_{20} - \frac{1}{4\pi} \cdot \beta^2\right) \tag{10}$$

where the last term follows from the volume conservation condition. If the interaction is of a sufficiently short range, the equipotential surfaces of the deformed average potential

$V(\bar{r})$ can be related to those of the spherical field $V_{\text{sph}}(r)$ that existed before the deformation was switched on:

$$V(\bar{r}) = V_{\text{sph}}(R_0) = V_{\text{sph}}\left(r \cdot \left(1+\beta Y_{20} - \frac{1}{4\pi}\beta^2\right)^{-1}\right). \tag{11}$$

Denoting the radial factor of $V_{\text{sph}}(r)$ by $f(r)$, we can write:

$$V_{\text{sph}}(r) = V_0 \cdot f(r) \tag{12a}$$

$$V(\bar{r}) = V_0 \cdot f\left(r\left(1+\beta Y_{20} - \frac{1}{4\pi}\beta^2\right)^{-1}\right). \tag{12}$$

For small deformations β we can expand (12) into Taylor series retaining only the terms up to β^2:

$$V(\bar{r}) = V_0 \cdot f(r) + V_0 \cdot \beta \cdot \left(\frac{\partial f}{\partial \beta}\right)_{\beta=0} + \tfrac{1}{2} V_0 \cdot \beta^2 \cdot \left(\frac{\partial^2 f}{\partial \beta^2}\right)_{\beta=0} + \ldots \tag{13}$$

Making use of the relation (10) we get:

$$V(\bar{r}) = V_0 \cdot f(r) - V_0 \cdot \beta \cdot r \cdot \frac{df}{dr} Y_{20} + \frac{V_0}{2} \cdot \beta^2 \cdot$$

$$\left[\frac{d^2 f}{dr^2} \cdot Y_{20}^2 + 2r \frac{df}{dr}\left(Y_{20}^2 + \frac{1}{4\pi}\right)\right]. \tag{14}$$

For the spherical harmonic function Y_{20} we have:

$$(Y_{20})^2 = \frac{1}{4\pi} + k_1 Y_{20} + k_2 Y_{40}. \tag{15}$$

In accordance with our previous assumption of small β, we treat the last term of (14) in the approximative way, replacing Y_{20}^2 by its average value: $\frac{1}{4\pi}$. This leads to the simpler expression for the Hamiltonian (9):

$$H(\bar{r}) = T + V_0 \cdot f(r) - V_0 \cdot \beta \cdot r \cdot \frac{df}{dr} Y_{20} + \frac{V_0}{8\pi} \cdot \beta^2 \cdot$$

$$\cdot \frac{1}{r^2} \frac{d}{dr}\left(r^4 \frac{df}{dr}\right). \tag{16}$$

The last term has no influence on the deformation dependence of the wave function, thus in practice we look for the eigenstates of the Hamiltonian:

$$H(\beta) = H_{\text{sph}} - V_0 \cdot \beta \cdot r \frac{df(r)}{dr} Y_{20} \tag{17}$$

or

$$H(\beta) = H_{\text{sph}} - V_0 \cdot \beta \cdot \hat{F}(\bar{r}), \tag{18}$$

if we define

$$\hat{F}(\bar{r}) = r \cdot \frac{df(r)}{dr} \cdot Y_{20}. \tag{19}$$

In that way we derived the radial dependence of the quadrupole force for the average field (12), namely instead of the standard form (5) we are left with (19). The usual form of the quadrupole force (5) is easily seen to result from the harmonic oscillator potential:

$$V_{\text{h.o.}}(r) = V_0 \cdot r^2/2 \tag{20}$$

and is consistent with the Nilsson model potential which is basically a deformed oscillator field with some additional terms like $\bar{l} \cdot \bar{s}$ and $\bar{l}^2$ (see Ref. [6]).

We shall investigate the quadrupole forces resulting from two other potentials, namely the square-well potential:

$$V_W(r) = \begin{cases} -V_0 & \text{for } r < R_0 \\ 0 & \text{for } r \geqslant R_0 \end{cases} \tag{21}$$

or

$$V_W(r) = -V_0 \cdot \theta(R_0 - r) \tag{22}$$

with

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leqslant 0 \end{cases}$$

and the Woods-Saxon potential:

$$V_{\text{w.s.}}(r) = -V_0 \frac{1}{1+\exp((r-R_0)/a)}. \tag{23}$$

We shall derive the radial form factor $F(r)$ for the quadrupole forces (19) in those cases and compute the mass parameter B as follows from the formulae (7) and (8).

For the single-particle energies and wave functions we shall take those resulting from the Nilsson model potential with the pairing forces included in the BCS approximation. The fully consistent treatment would require the use of the single-particle energies and wave functions derived from the square-well and Woods-Saxon potentials, respectively. However we shall use the energies and functions appropriate in the case (20) in all three cases: (20), (22) and (23) because of the computational facilities offered by the Nilsson model.

The quadrupole forces related to the potentials (20), (22) and (23) are respectively:

$$F_{\text{h.o.}}(r) = r^2 Y_{20} \tag{24}$$

$$F_W(r) = r \cdot \delta(r - R_0) Y_{20} \tag{25}$$

$$F_{\text{w.s.}}(r) = \frac{1}{a} \cdot \frac{r \exp((r-R_0)/a)}{(1+\exp((r-R_0)/a))^2} Y_{20}. \tag{26}$$

We shall compare the mass parameter B resulting in those three cases changing in each case B_F to B_ε according to the formula:

$$B_\varepsilon = B_F \cdot \left(\frac{dF}{d\varepsilon}\right)^2 \tag{27}$$

where

$$F = \sum_\nu 2v_\nu^2 f_{\nu\nu} \tag{28}$$

$$f_{\mu\nu} = \langle \nu|\hat{F}|\mu\rangle \tag{29}$$

and v_ν^2 is an occupation factor for the level $|\nu\rangle$ resulting from the BCS approximation. When the pairing forces are included B_F is given by (7) with

$$\sum\nolimits_i = \sum_{\mu,\nu} \frac{|f_{\mu\nu}|^2(u_\nu v_\mu + u_\mu v_\nu)^2}{(E_\mu + E_\nu)^i} \tag{8a}$$

where $E_\nu = \sqrt{(e_\nu - \lambda)^2 + \Delta^2}$.

The basic wave functions in the Nilsson model are of the form:

$$|Nl\Lambda(\varrho)\rangle \sim \varrho^l \cdot e^{-\varrho^2/2} \cdot \mathscr{L}_n^{l+1/2}(\varrho^2) Y_{l\Lambda}(\Omega_\varrho) \tag{30}$$

where

$$n = (N-l)/2$$

$$\varrho^2 = \frac{M\omega_0}{\hbar}\left[(1+\tfrac{1}{3}\varepsilon)(x^2+y^2) + (1-\tfrac{2}{3}\varepsilon)z^2\right] \tag{31}$$

and ε is a deformation parameter.

In order to evaluate the matrix elements of the operators (25) and (26) between the states of the type (30) we express the Nilsson wave function in terms of the spherical oscillator wave functions:

$$|Nl\Lambda(\varrho)\rangle = \sum_i c_i \times |n_i l_i m_i(r)\rangle, \quad \sum_i |c_i|^2 = 1 \tag{32}$$

and making use of the spherical radial matrix elements of (25) and (26):

$$\langle N'l'|r \cdot \delta(r-R_0)|Nl\rangle = N_{Nl}^{N'l'} \cdot e^{-\alpha^2 R_0^2} \cdot (\alpha R_0)^{l+l'+3} \cdot$$

$$\cdot \sum_{i=0}^{n} \sum_{j=0}^{n'} a_{ij}(\alpha^2 R_0^2)^{i+j}, \tag{33}$$

$$\langle N'l'|\frac{r}{a}\frac{\exp\left(\frac{r-R_0}{a}\right)}{\left[1+\exp\left(\frac{r-R_0}{a}\right)\right]^2}|Nl\rangle =$$

$$= N_{Nl}^{N'l'} \cdot \frac{1}{a} \cdot \sum_{i=0}^{n} \sum_{j=0}^{n'} a_{ij} \cdot I_{ij} \tag{34}$$

with

$$N_{Nl}^{N'l'} = (-1)^{n+n'} 2^{l+l'-n-n'} \sqrt{\frac{5}{\pi} \frac{n!(n')!(N+l+1)!(N'+l'+1)!}{(n+l)!(n'+l')!}} \tag{35}$$

$$a_{ij} = (-4)^{i+j} \frac{(i+l)!(j+l')!}{i!j!(n-i)!(n'-j)!(2i+2l+1)!(2j+2l'+1)!} \tag{36}$$

$$I_{ij} = \int_0^\infty \frac{\exp\left(\frac{r-R_0}{a}\right)}{\left[1+\exp\left(\frac{r-R_0}{a}\right)\right]^2} e^{-\alpha^2 r^2} (\alpha r)^{l+l'+3+2(i+j)} dr \tag{37}$$

and

$$\alpha = \sqrt{\frac{\hbar}{M\omega_0}}. \tag{38}$$

In case of Eq. (24) this procedure is not necessary because the matrix elements of

$$2r^2 Y_{20}(\bar{r}) = \frac{M\omega_0}{\hbar} (2\varrho^2 Y_{20}(\hat{\varrho}) + \varepsilon/3 \cdot \varrho^2) \tag{39}$$

can be evaluated directly with the functions of the type (30).

The main difference between the matrix elements of (24) and those of (25) and (26) lies in the selection rules for the oscillator quantum number N. Namely in the case of the operator proportional to r^2 only the states with $\Delta N = 0$ and $\Delta N = \pm 2$ lead to non-zero matrix elements. For both (35) and (34) the only limitation is $\Delta N = \pm 2p$, $p = 0, 1, 2, \ldots$ entier $(N_{\max}/2)$.

Numerical calculations were performed for a few nuclei in the Rare-Earth region, namely $A = 164\,(Z = 66, 68)$, $A = 168$ $(Z = 68, 70)$ and $A = 174$ $(Z = 70, 72)$. The Nilsson model parameters were taken as usually in this region, *i.e.*

$$\kappa_p = 0.0637 \qquad \mu_p = 0.60 \tag{40}$$

$$\kappa_n = 0.0637 \qquad \mu_n = 0.42$$

and the oscillator shells were taken up to $N_{\max} = 6$ for protons and $N_{\max} = 7$ for neutrons.

The BCS equations for the energy gap Δ and the Fermi energy λ

$$\frac{2}{G} = \sum_\nu \frac{1}{\sqrt{(e_\nu - \lambda)^2 + \Delta^2}}$$

$$n = \sum_\nu 2v_\nu^2 = \sum_\nu \left[1 - \frac{e_\nu - \lambda}{\sqrt{(e_\nu - \lambda)^2 + \Delta^2}}\right] \tag{41}$$

where n = number of the particles (*i.e.* Z or N), were solved with the number of levels included in the summation (41) equal to the number of particles (Z or N) and the pairing forces strength was assumed as

$$G_p = 20.8 \text{ MeV/A} \qquad G_n = 15.6 \text{ MeV/A} \tag{42}$$

for protons and neutrons respectively.

In (34) we assumed the nuclear radius R_0 to be

$$R_0 = 1.2 \, A^{1/3} \text{ (fm)} \tag{43}$$

and in (34) the diffuseness parameter a was taken as

$$a = 0.67 \text{ fm.} \tag{44}$$

The numerical results for B_ε for $\varepsilon = 0.1$ and $\varepsilon = 0.2$ are presented in Table I. One can see that for the given set of the parameters involved in the calculation, B_ε for the usual quadrupole forces lies in between the B_ε values for the square-well and those for the

TABLE I

Inertial mass parameter B_ε for different quadrupole operators

ε	A	Z	B_ε [$\hbar^2$/MeV]		
			$\hat{F}_W$	$\hat{Q} = F_{\text{h.o.}}$	$\hat{F}_{\text{WS}}$
0.1	164	66	143.05	166.69	171.63
	164	68	135.20	152.92	161.90
	168	68	148.62	165.87	178.92
	168	70	142.49	163.21	171.08
	174	70	149.17	174.69	180.17
	174	72	156.51	170.14	177.90
0.2	164	66	109.94	129.83	130.71
	164	68	111.48	128.71	144.15
	168	68	109.97	135.39	144.43
	168	70	125.65	148.25	160.07
	174	70	115.54	156.43	—
	174	72	126.04	155.28	—

Woods-Saxon potentials. One can look at the square-well potential as the $a \to 0$ limit of the Woods-Saxon potential. In order to investigate the sensitivity of this calculation to the changes of the diffuseness parameter a, B_ε was calculated for the forces (26) with $a = 0.60$ and 0.50 fm (*i.e.* decreased by 10% and 25% with respect to the initial value $a = 0.67$ fm) and the results are presented in Table II together with the relative change in the B_ε value.

The comparison between the matrix elements of the usual quadrupole force and those of the force resulting from the Woods-Saxon potential, where the radial derivative was

TABLE II

The dependence of B_{WS} on the diffuseness parameter a

A	Z	B_{WS} ($a = 0.67$)	B_{WS} ($a = 0.60$)	ΔB [%]	B_{WS} ($a = 0.50$)	ΔB [%]
164	66	171.63	168.07	−2.1	144.47	−15.8
164	68	161.90	158.55	−2.1	136.07	−15.9

taken from the full potential including $\vec{l}.\vec{s}$ and Coulomb terms, was made by Kumar and Sorensen in Ref. [7]. They found that if the matrix elements were calculated consistently on the corresponding wave functions (h.o. for the forces (5) and Woods-Saxon in the other case), the results were very similar. They did not find as good an agreement in the "mixed" cases, *i.e.*, the matrix elements of the $\hat{Q}$-force on the Woods-Saxon states and *vice versa*, were sometimes found to differ considerably. When investigating the mass parameter it is not sufficient to get similar values of $q_{\mu\nu}$ in order to reproduce the B-values because of the dependence of B on the single-particle energies e_ν. The differences in the single-particle spectra, especially in the region close to the Fermi level λ, may affect the results for B. The additional non-zero matrix elements for $|\Delta N| \geqslant| 2$ are not so important because the energy denominators in (8a) are much bigger. The effects of the couplings between the levels with $\Delta N = \pm 4, \pm 6$ can be deduced from Table III, where the $2\Sigma_1$ and $2\Sigma_3$ values are listed for the nucleus $A = 164$ ($Z = 66$) and $\varepsilon = 0.10$.

TABLE III

The influence of different ΔN couplings on B_ε values

ΔN	$\hat{F}_W$		$\hat{F}_{WS}$	
	$2\Sigma_1$	$2\Sigma_3$	$2\Sigma_1$	$2\Sigma_3$
0	18.523	95.496	8.562	45.877
±2	12.987	3.787	6.077	1.887
±4	0.419	0.023	0.069	0.004
±6	0.002	$4 \cdot 10^{-5}$	0.002	$5 \cdot 10^{-5}$
Total	31.510	98.993	14.709	47.768

The first column specifies the ΔN value, the second and third columns give the values of $2\Sigma_1$ and $2\Sigma_3$ for the interaction (25), the fourth and fifth give $2\Sigma_1$ and $2\Sigma_3$ for the interaction (26).

We investigated the inertial mass parameter B_ε for three different kinds of the radial shape of the quadrupole forces. The one usually used is characterized by the quadratic increase with r. The other two are centered at the surface of the nucleus $r = R_0$; one in a delta-like fashion, the other acts in a broader region around $r = R_0$ with the parameter a describing its range.

When comparing the numerical results obtained with the reasonable values of all

entering parameters (diffuseness parameter a, nuclear radius R_0, Nilsson model parameters κ, μ) we get the discrepancies in the B_ε values not bigger than 20%.

The B-values for $\hat{F} \sim r \cdot \delta(r - R_0)$ are smaller than those for $\hat{Q} \sim r^2$ by less than 15%. The B-values for $\hat{F}$ derived from the Woods-Saxon potential with $a = 0.67$ fm are bigger than those for $\hat{Q}$ by at most 5%. The mass parameter B for this kind of forces depends on the value of the diffuseness parameter a: when decreasing it one gets decreasing values for B. One can expect to reproduce any value of B_ε from the range between B (square-well) and B (W-S, $a = 0.67$) by changing a. In particular $a = 0.60$ fm gives the values of B (W-S) nearly the same as those derived from the ordinary quadrupole forces.

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