MANY-PARTICLE INCLUSIVE CROSS-SECTIONS AND CORRELATIONS IN HADRONIC PRODUCTION PROCESSES AT VERY HIGH ENERGY*

BY Z. KOBA

The Niels Bohr Institute, University of Copenhagen**

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Theoretical studies on many-particle inclusive cross-sections and correlations in hadronic collision are reviewed. General properties of cross-sections and correlations, fluid analogy, kinematical constraints (sum rules), coexistence of two mechanisms, and asymptotic predictions of correlations in various models are discussed.

(These contents are summaries, paraphrases or illustrations of already known results, except the statement 2 at the end of Section 8 and the simple derivation of sum rules in Section 9, which are so far not found, to my knowledge, in the existing literature or circulated preprints.)

Introduction

Even within the framework of hadronic multiparticle production processes, the word "correlation" can be used in various ways. For instance, the well-known properties of transverse momenta of secondary particles can be regarded as correlations between the incident and secondary momenta.

In this report, however, we limit ourselves only to correlations in many particle inclusive cross-sections. Typical questions we are going to ask are thus the following: Suppose we have detected a particle with momentum $\vec{p}$. What is, then, our expectation of finding another particle with momentum $\vec{p}'$ in the same event? How much is this expectation different from the case where we have not made the first detection? The first question concerns essentially the two particle distribution, and the second essentially the two particle correlation.

* This article was originally prepared as a material for the rapporteur's talk at the Third International Colloquium on Multiparticle Reactions, held on June 20–24, 1972, at Zakopane, Poland. This colloquium was dedicated to the memory of our late colleagues Dr O. Czyżewski and Dr L. Michejda. So is the present report.

** Address: The Niels Bohr Institute, University of Copenhagen, Copenhagen Ø, Blegdamsvej 17, Denmark.

In general, such a correlation presents an alternative description of many particle distribution. The description in terms of correlations will be often — but not always — suitable for extracting new features included in many-particle distributions and is expected to reveal dynamical properties of the production process more clearly.

The subjects we plan to cover in this report are:

A. Cross-sections, correlations, and fluid analogy[1]

1. * General view
2. Generating functional
3. Correlations
4. Generating function and multiplicity distribution
5. Associated multiplicity and semi-inclusive cross-sections
6. Fluid analogy

B. Kinematical constraints

7. Direct derivation
8. * Energy conservation for correlation functions
9. Derivation from generating functional
10. Leading particles and clustering

C. Coexistence of two (or more) mechanisms

11. * Joint correlation
12. * Some examples
13. Separation of diffraction and pionization

D. Specific predictions of various models

14. * Asymptotic behaviour of integrated correlation
15. * Longitudinal correlations
16. * Uncorrelated jet model
17. * Short range correlation models
18. * Diffractive excitation model

Appendices

Table I shows an incomplete list of recent reviews and articles of general character which treat problems of correlations. The present report owes very much to the latest three reviews by Caneschi[2], by Peccei[2] and by Chan[3].

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[1] In the talk at the Zakopane meeting, contents of the sections with*, together with a summary of contributed, theoretical works concerning analyses and interpretations of experimental data, were presented. See the Proceedings of the colloquium to be published very soon.

[2] I am grateful to Dr L. Caneschi for showing me these articles.

[3] I am grateful to Dr H. Miettinen for showing me this article.

TABLE I

K. Zalewski [1]	(1970)	presented at the colloquia of this series
L. Van Hove [2]	(1971)	
E. L. Berger [3]	(1971)	
O. Czyżewski [4]	(1971)	
K. G. Wilson [5]	(1970)	
D. Horn [6]	(1971)	
L. Van Hove [7]	(1971)	
C. E. De Tar [8]	(1971)	
H. D. Abarbanel [9]	(1971)	
A. H. Mueller [10]	(1971)	
J. D. Bjorken [11]	(1971)	
A. P. Bassetto, M. Toller, L. Sertorio [12] (1971)		
E. Predazzi, G. Veneziano [13] (1971)		
W. R. Frazer, L. Ingber, C. H. Mehta, C. H. Poon, D. Silverman, K. Stowe, P. D. Ting, H. J. Yesian [14]	(1972)	
L. S. Brown [15]	(1972)	
R. C. Hwa [16]	(1972)	
H. T. Nieh, J. M. Wang [17] (1972)		
A. Białas, K. Fiałkowski, R. Wit, K. Zalewski [18] (1972)		
L. Caneschi [19]	(1972)	
R. D. Peccei [20]	(1972)	
Chan H. M. [21]	(1972)	

A. CROSS-SECTIONS, CORRELATIONS, AND FLUID ANALOGY

1. General view

In order to describe multiparticle production processes appropriately, we need to introduce a number of quantities, and various sets of notation are currently used by various authors. To avoid confusion, let us first of all specify some of notations adopted in this report. We take firstly the case of a single kind of neutral scalar particles. Extension to more general cases will be made later when it becomes necessary.

Inclusive cross-sections, normalized and invariant form, are also called distribution functions[4] $f^{(k)}$

$$\frac{\omega_1 \ldots \omega_k}{\sigma} \frac{d^{3k}\sigma_{\text{incl}}}{d^3\vec{p}_1 \ldots d^3\vec{p}_k} = f^{(k)}(\vec{p}_1, \ldots, \vec{p}_k),$$

$$k = 1, 2, \ldots \tag{1.1}$$

with

$$\omega = (p^2 + m^2)^{\frac{1}{2}}. \tag{1.2}$$

The normalization factor σ represents the sum (and integral) of all the cross-sections we are concerned with. (It can be taken equal to the cross-section, or the inelastic total

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[4] Some people call this k-particle density or spectrum, reserving the word "distribution function" for the exclusive k-particle cross-section.

cross-section, or the sum of a special set of cross-sections, depending on the problem we are interested in.) It is sometimes convenient to define also

$$f^{(0)} = 1. \tag{1.3}$$

Integral of the distribution function over the whole phase space is denoted by $F^{(k)}$

$$\int \dots \int f^{(k)}(p_1, \dots, p_k) \frac{d^3 \vec{p}_1 \dots d^3 \vec{p}_k}{\omega_1 \dots \omega_k} = F^{(k)}. \tag{1.4}$$

These integrals are, as is well known, related to the moments of multiplicity distribution

$$F^{(k)} = \langle n(n-1)\dots(n-k+1)\rangle. \tag{1.5}$$

Correlations $\varrho^{(k)}$ are defined [5], [10], [13] following the pattern of the cluster expansion [22], [23], where the k-th equation defines $\varrho^{(k)}$ in terms of $\varrho^{(1)}, \dots \varrho^{(k-1)}$ and $f^{(k)}$.

$$f^{(1)}(p_1) = \varrho^{(1)}(p_1), \tag{1.6a}$$

$$f^{(2)}(p_1, p_2) = \varrho^{(1)}(p_1)\varrho^{(1)}(p_2)+\varrho^{(2)}(p_1, p_2), \tag{1.6b}$$

$$f^{(3)}(p_1, p_2, p_3) = \varrho^{(1)}(p_1)\varrho^{(1)}(p_2)\varrho^{(1)}(p_3)+ + \sum_{\text{perm}} \varrho^{(1)}(p_1)\varrho^{(2)}(p_2, p_3)+\varrho^{(3)}(p_1, p_2, p_3). \tag{1.6c}$$

In general

$$f^{(k)}(\vec{p}_1, \dots, \vec{p}_k) = \sum_{\{m_l\}} \sum_{\text{perm}} \underbrace{\varrho^{(1)}(\ )\dots\varrho^{(1)}(\ )}_{m_1 \text{ factors}} \ \underbrace{\varrho^{(2)}(,)\dots\varrho^{(2)}(,)}_{m_2 \text{ factors}} \dots\dots \underbrace{\varrho^{(k)}(, , \dots,)}_{m_k \text{ factors}}$$

$$k = 1, 2, \dots \tag{1.6}$$

where m_l is zero or positive integer and the set of integers $\{m_l\}$ satisfies

$$\sum_{l=1}^{k} l m_l = k. \tag{1.7}$$

Also we make the convention

$$\varrho^{(0)} = 0. \tag{1.8}$$

Integrals of the correlation functions over the phase space we denote by $R^{(k)}$

$$\int \dots \int \varrho^{(k)}(\vec{p}_1, \dots, \vec{p}_k) \frac{d^3 \vec{p}_1 \dots d^3 \vec{p}_k}{\omega_1 \dots \omega_k} = R^{(k)}. \tag{1.9}$$

These can be expressed, through (1.6) and (1.5), in terms of the moments of multiplicity distribution. For instance,

$$R^{(1)} = F^{(1)} = \langle n \rangle, \tag{1.10a}$$

$$\begin{aligned} R^{(2)} &= \langle n(n-1)\rangle - \langle n \rangle^2 = \\ &= D^2 - \langle n \rangle, \end{aligned} \tag{1.10b}$$

$$\begin{aligned} R^{(3)} &= \langle n(n-1)(n-2)\rangle - 3\,\langle n(n-1)\rangle\,\langle n \rangle + 2\,\langle n \rangle^3 = \\ &= \langle (n-\langle n \rangle)^3 \rangle - 3D^2 + 2\langle n \rangle, \end{aligned} \tag{1.10c}$$

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with

$$D^2 = \langle (n-\langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2. \tag{1.11}$$

So far with the notation. We shall now enumerate the quantities we have to do with in multiparticle production. (Here we are limiting ourselves to the simplest, unrealistic case of a single kind of neutral scalar particles.)

i) exclusive differential cross-sections
ii) semi-inclusive cross-sections[5]
iii) topological cross-sections[6]
iv) total cross-section
v) inclusive differential cross-sections
vi) partially integrated inclusive cross-sections, which are related to the moments of associated multiplicities [19][7]
vii) full integral of inclusive cross-sections, which are related to the moments of multiplicity distribution
viii) correlation functions
ix) partially integrated correlation functions
x) full integrals of correlation functions

These quantities are of course connected with each other through certain relations, some of which are fairly complicated. It is thus desirable to formulate a unifying scheme which can summarize these kinematical relations into a compact form.

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[5] Originally this term was introduced in a realistic case of hydrogen bubble chamber experiment, where both charged and neutral particles are present in the final state and the neutral one escape detection [24]. In the idealized simple case being discussed here, however, the semi-inclusive cross-section describes the case, where a part of the final particles (for instance, j out of k final particles) are measured precisely so that their momenta are specified, while the rest ($(k-j)$ out of k final particles) are only known to be present, *i.e.*, their number is definite, but their momenta are not specified. In other words, the semi--inclusive cross-sections here are partially integrated exclusive cross-sections.

[6] A similar argument as in the foregoing footnote applies here too. In our idealized case, the topological cross-sections represent full integrals of exclusive cross-sections of a specified number of particles. Some people call them partial cross-sections.

[7] See Section 5 for more details.

An important step towards this direction was taken by Mueller [10] when he introduced the generating function, which combines the topological cross-sections, the moments of multiplicity distribution and the full integrals of correlation functions into a single expression[8]. Moreover, this generating function has a close formal resemblance to the grand partition function of statistical mechanics[9] and leads to a more precise and explicit form of the fluid analogy, which is often utilized in treating inclusive cross-sections and correlations.

A further generalization, so as to include the more differential cross-sections and correlations as well, has been achieved by constructing a generating functional [15], [27]–[29]. This includes, as a special case, the generating function of Mueller, and also can be utilized in order to derive kinematical constraints in a quite general way[10] [13], [15]. It should be remarked that such a technique has been known to statistical physicists since a long time ago: A generating functional was introduced by Bogolubov in 1945 and applied to the molecular distribution function [30]. Thus the fluid analogy is effective on this level, too.

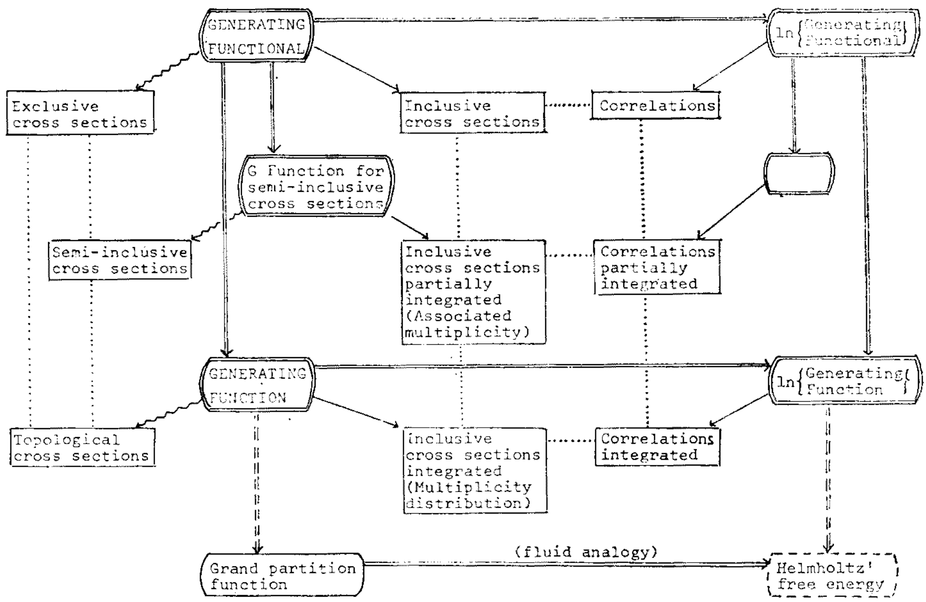

Fig. 1

Fig. 1 summarizes logical connections of the generating functional and all the other quantities mentioned in this Section. In the following Sections 2–6 we shall discuss them in more details.

[8] Similar relations were discussed in somewhat less explicit forms by other authors too [25], [26].

[9] This was already referred to in Ref. [10], but with careful reservation. Later it has been pursued further and has led to some interesting results. See Sections 6, 13.

[10] See Section 9.

2. *Generating functional*

2.1. Definition

In the case of a single kind of neutral scalar particles the generating functional (hereafter abridged as GF) can be defined by [27]

$$F[s;\, h(\vec{p})] = \left\langle :\exp\left\{\int \frac{d^3\vec{p}}{\omega} N(\vec{p})h(\vec{p})\right\}: \right\rangle, \tag{2.1}$$

where

$$s = \text{CMS energy squared},$$

$$h(\vec{p}) = \text{variable function defined over } \vec{p},$$

$N(\vec{p})$ is the number density operator with the usual commutation relations, namely

$$N(\vec{p}) = a^{\dagger}(\vec{p})a(\vec{p}),$$

$$[a(\vec{p}),\, a^{\dagger}(\vec{p}\,')] = \omega\delta^{(3)}(\vec{p}-\vec{p}\,'),$$

$$[a(\vec{p}),\, a(\vec{p}\,')] = [a^{\dagger}(\vec{p}),\, a^{\dagger}(\vec{p}\,')] = 0 \tag{2.2}$$

and : denotes normal product.

The average symbol means

$$\langle A\rangle = \langle\psi_s|A|\psi_s\rangle/\langle\psi_s|\psi_s\rangle \tag{2.3}$$

with

$$|\psi_s\rangle = (S-1)|\psi_{\text{initial}}\rangle.$$

In the general case we introduce a discrete suffix β, which specifies different kinds of particles, helicities, charges, *etc.*, and generalize (2.1) by the following procedure

$$h(\vec{p}) \to h_\beta(\vec{p}), \qquad N(\vec{p}) \to N_\beta(\vec{p}),$$

$$\int \frac{d^3\vec{p}}{\omega} \to \sum_\beta \int \frac{d^3\vec{p}}{\omega}. \tag{2.4}$$

Also the final state will be in general a statistical mixture; then (2.3) is to be replaced by

$$\langle A\rangle = \mathrm{Tr}(\varrho_s A)/\mathrm{Tr}(\varrho_s)$$

with

$$\varrho_s = \sum_j |\psi_{s,j}\rangle w_j \langle\psi_{s,j}|. \tag{2.5}$$

The definition (2.1) can be also expressed without the normal product symbol (see Appendix A)

$$F[s;\, h(\vec{p})] = \left\langle \exp\left[\int \frac{d^3\vec{p}}{\omega} N(\vec{p}) \ln\{1+h(\vec{p})\}\right]\right\rangle, \tag{2.6}$$

which is, crudely speaking, $\langle \Pi\{1+h(\vec{p})\}^{N(\vec{p})}\rangle$ where Π stands for a continuously multiple product, and reveals more explicitly its formal analogy to the grand partition function. This form is also useful when deriving kinematical constraints (see Section 9).

2.2. Exclusive and inclusive cross-sections

The GF introduced above has the following property. (For proof see Appendix B.)

a) When expanded in the power series of $\{1+h(\vec{p})\}$, the expansion coefficients give normalized exclusive cross-sections.

b) When expanded in the power series of $h(\vec{p})$, the expansion coefficients give normalized inclusive cross-sections, *i. e.*, distribution functions.

That is to say

$$F[s; h(\vec{p})] = \frac{1}{\sigma}\sum_k \frac{1}{k!}\int\cdots\int \frac{d^{3k}\varrho_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k}\prod_{j=1}^{k}\{1+h(\vec{p}_j)\}d^3p_j= \tag{2.7}$$

$$= \frac{1}{\sigma}\sum_k \frac{1}{k!}\int\cdots\int \frac{d^{3k}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k}\prod_{j=1}^{k} h(\vec{p}_j)d^3\vec{p}_j, \tag{2.8}$$

$$= \sum_k \frac{1}{k!}\int\cdots\int f^{(k)}(\vec{p}_1, \ldots, \vec{p}_k)\prod_{j=1}^{k} h(\vec{p}_j)\frac{d^3\vec{p}_j}{\omega_j}, \tag{2.9}$$

with

$$\sigma = \sum_k \frac{1}{k!}\int\cdots\int \frac{d^{3k}\sigma_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k}\, d^3\vec{p}_1\ldots d^3p_k. \tag{2.10}$$

In (2.7) and (2.10) the definition of the exclusive cross-section includes a four-dimensional delta-function of the energy-momentum conservation.

The expansion coefficients, *i. e.*, the normalized cross-sections can be obtained by the usual procedure,

$$\frac{1}{\sigma}\frac{d^{3k}\sigma_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \left.\frac{\delta^k F}{\delta h(\vec{p}_1)\ldots\delta h(\vec{p}_k)}\right|_{h(\vec{p})=-1} \tag{2.11}$$

and

$$\frac{1}{\sigma}\frac{d^{3k}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \left.\frac{\delta^k F}{\delta h(\vec{p}_1)\ldots\delta h(\vec{p}_k)}\right|_{h(\vec{p})=0.} \tag{2.12}$$

Thus the GF includes complete information on cross-sections, and also establishes mutual relations between inclusive and exclusive cross-sections. They are namely given by [27, 28]

$$\frac{d^{3k}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \sum_{l=0}^{\infty}\frac{1}{l!}\int\cdots\int\frac{d^{3(k+l)}\sigma_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}}\,d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}, \tag{2.13}$$

$$\frac{d^{3k}\sigma_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\int\cdots\int\frac{d^{3(k+l)}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}}\,d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}. \tag{2.14}$$

The equation (2.13) is the usual definition of inclusive cross-sections. The physical meaning of the inverse relation (2.14) is as follows. Denoting the total 4-momentum by $\boldsymbol{P}$,

$$\frac{d^{3k}\sigma_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \frac{d^{3k}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} \quad \text{if} \quad \sum_{j=1}^{k}\boldsymbol{p}_j = \boldsymbol{P}, \tag{2.14a}$$

$$0 = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\int\cdots\int\frac{d^{3(k+l)}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}}\,d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}$$
$$\text{if} \sum_{j=1}^{k}\boldsymbol{p}_j \neq \boldsymbol{P}. \tag{2.14b}$$

That is to say, the inclusive cross-section has in general an exclusive component in it and, when the measured final particles saturate the energy-momentum conservation, only this exclusive component (boundary value) survives, while in the case where measured 4-momenta are not equal to the total, the alternating series of the r. h. s. sums up to zero[11].

When there are more than two kinds of particles, one can easily define a partially inclusive and partially exclusive cross-sections by putting some of the $h_\beta(\vec{p})$ equal to -1 and others equal to 0 after functional differentiation of GF.

3. *Correlations*

Correlations have been defined in (1.6). It is based on the fluid analogy, the correlation functions having the same property as the cluster functions when correlations are of short range. (See Refs [10], [22], [23].) Thus this definition may not always be the most suitable one for describing actual situation; but we can at least take it as a working definition and explore its consequences.

It is convenient to construct a generating functional R such that its expansion coefficients in power series of $h(\vec{p})$ yield the correlation functions $\varrho^{(k)}$

$$R[s; h(\vec{p})] = \sum \frac{1}{k!}\int\cdots\int \varrho^{(k)}(\vec{p}_1, \ldots, \vec{p}_k)\prod_{j=1}^{k}\left\{h(\vec{p}_j)\frac{d^3\vec{p}_j}{\omega_j}\right\} \tag{3.1}$$

[11] For more details on this point, see the note added in proof of Ref. [28].

or

$$\varrho^{(k)}(\vec{p}_1, \ldots, \vec{p}_k) = \omega_1 \ldots \omega_k \frac{\delta^k R}{\delta h(\vec{p}_1) \ldots \delta h(\vec{p}_k)} \bigg|_{h(\vec{p})=0.} \tag{3.2}$$

Then the whole set of relations (1.6), defining $\varrho^{(k)}$ in terms of $f^{(k)}$ or *vice versa*, can be summarized as a simple relation between the two functionals

$$F[s; h(\vec{p})] = \exp \{R[s; h(\vec{p})]\}. \tag{3.3}$$

One can easily verify, by equating the expansion coefficients of both sides in powers of $h(\vec{p})$, that (1.6) is reproduced. The well-known fact that the choice of normalization factor σ is very essential for evaluation of correlations is reflected in the non-linearity of the above relation (3.3).

Further, one sees immediately from (3.3) that at a finite energy an infinite number $\varrho^{(k)}$ will be non-vanishing. This is because the distribution function $f^{(k)}$ vanishes for $k > N$, where $N = \sqrt{s}/m$ represents the maximal number of final particles allowed by energy--momentum conservation, thus making F a "polynomial" in $h(\vec{p})$, which requires that R should be an "infinite series" in $h(\vec{p})$ since the relation (3.3) is transcendental. (This will be verified by a more precise argument in Section 8.)

4. Generating function and multiplicity distribution

The GF in its full generality represents a complete information of cross-section and correlations, but when we are interested in less detailed aspects of the many-body final states, we can reduce the GF into a correspondingly simplified form. For example, we put $h_{\beta_1}(\vec{p}) = 0$ when we do not detect particles of the kind β_1 (this is often the case with neutral particles), or we put $h_{\beta_2}(\vec{p}) = h_{\beta_3}(\vec{p})$ when we not distinguish between particles of the kind β_2 and those of β_3.

The most global features of many-particle distributions and correlations are obtained from the multiplicity distribution. For the purpose of studying them, we put

$$h(\vec{p}) \to h \tag{4.1}$$

in the GF, (2.7), and get the generating function of Mueller [10]

$$F[s; h(\vec{p})] \to F(s; h) = \sum (1+h)^k \sigma_k/\sigma = $$

$$= \sum \frac{h^k}{k!} F^{(k)} =$$

$$= \exp \sum \frac{h^k}{k!} R^{(k)}. \tag{4.2}$$

Here $F^{(k)}$ and $R^{(k)}$ are the integrals of distribution and correlation functions, respectively, as have been defined by (1.4) and (1.9), while σ_k denotes the cross-section of producing k particles [12] (in our simple case it is the "topological cross-section).

[12] Sometimes it is called "partial cross-section".

It can be readily checked that (4.2) reproduces (1.5) and the integrals of (1.6), *i. e.*,

$$
\begin{aligned}
F^{(1)} &= R^{(1)}, \\
F^{(2)} &= R^{(1)2}+R^{(2)}, \\
F^{(3)} &= R^{(1)3}+3R^{(1)}R^{(2)}+R^{(3)}, \\
&\ldots\ldots
\end{aligned}
\tag{4.3}
$$

and consequently (1.10).

5. *Associated multiplicity and semi-inclusive cross-sections*

The multiplicity distribution has thus given us the total integrals of many-particle distributions and correlations. Further, an insight into their structure can be obtained from the distribution of the "associated multiplicity" which is related to their partial integrals [17], [31], [32], [18]. The associated multiplicity $n_B(\vec{p}_A)$ of particle B means the number of particle B in an event where a particle A of momentum $\vec{p}_A$ has been detected. When A and B are the same kind of particles, we replace $n_B(\vec{p}_A)$ by $(n-1)(\vec{p})$ since in this case we have already counted one by the first detection.

By the definition of distribution functions, the average of associated multiplicity is given by

$$\langle n_B\rangle_{\vec{p}_A} = \frac{1}{f_A^{(1)}(\vec{p}_A)}\int f_{AB}^{(2)}(\vec{p}_A,\vec{p}_B)\frac{d^3\vec{p}_B}{\omega_B}, \tag{5.1}$$

or

$$\langle n-1\rangle_{\vec{p}} = \frac{1}{f^{(1)}(\vec{p})}\int f^{(2)}(\vec{p},\vec{p}\,')\frac{d^3\vec{p}\,'}{\omega'}. \tag{5.2}$$

If there is no correlation between particles A and B, then the two-particle distribution function factorizes

$$f_{AB}^{(2)}(\vec{p}_A,\vec{p}_B) = f_A^{(1)}(\vec{p}_A)f_B^{(1)}(\vec{p}_B) \tag{5.3}$$

and one gets

$$\langle n_B\rangle_{\vec{p}_A} = \langle n_B\rangle, \tag{5.4}$$

or

$$\langle (n-1)\rangle_{\vec{p}} = \langle n\rangle, \tag{5.5}$$

that is to say, the average associated multiplicity will be the same as the (unconditional) average multiplicity.

In general, however, the relation (5.4) or (5.5) will not hold, and the difference of l. h. s. from r. h. s. gives, when multiplied by $f_A^{(1)}(\vec{p}_A)$, the partial integral of correlation $\varrho^{(2)}$

$$\langle n_B\rangle_{\vec{p}_A}-\langle n_B\rangle = \frac{1}{f_A^{(1)}(\vec{p}_A)}\int \varrho_{AB}^{(2)}(\vec{p}_A,\vec{p}_B)\frac{d^3\vec{p}_B}{\omega_B}. \tag{5.6}$$

One can go further and evaluate partial integrals of higher distribution and correlation functions from the moments of associated multiplicity distribution. In order to derive general formulas, we can utilize the GF for semi-inclusive cross-sections [28], [33], [34].

The semi-inclusive cross-sections are defined[13], for instance, as follows:

$$n\frac{d^3\sigma_n}{d^3\vec{p}} \quad \text{for} \quad a+b\to \underbrace{C(\vec{p})+C+\to+C}_{n}+(\text{anything not } C). \tag{5.7}$$

That is to say, we measure a particle C with momentum $\vec{p}$ and, besides, we count the number n of the particle C.

After once functionally differentiating, we put in GF

$$h(\vec{p})\to h$$

and get

$$\omega\frac{\delta F}{\delta h(\vec{p})}\bigg|_{h(\vec{p})=h} = \frac{1}{\sigma}\sum_{n=1}^{\infty}(1+h)^{n-1}n\frac{d^3\sigma_n}{d^3\vec{p}/\omega} \tag{5.8}$$

from the exclusive expansion (1.7), and also

$$= \sum_{n=1}^{\infty}\frac{h^{n-1}}{(n-1)!}\int\cdots\int f^{(n)}(\vec{p},\vec{p}_2,\ldots,\vec{p}_n)\frac{d^3\vec{p}_2\ldots d^3\vec{p}_n}{\omega_2\ldots\omega_n} \tag{5.9}$$

from the inclusive expansion (1.9).

Comparing the expansion coefficients of (5.8) and (5.9) in powers of h, we get firstly

$$\frac{1}{\sigma}\sum_{n=1}^{\infty}n\frac{d^3\sigma_n}{d^3\vec{p}/\omega} = f^{(1)}(\vec{p}), \tag{5.10}$$

which shows that the inclusive cross-section is subdivided into semi-inclusive cross-

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[13] The notation used in Refs [24, [28] is related to the one in (5.7), which is used in Ref. [32], in the following way

$$\sigma_n g^{(n)}(\vec{p})/\omega = n\frac{d^3\sigma_n}{d^3\vec{p}}.$$

The normalization is given by

$$\int g^{(n)}(\vec{p})\frac{d^3\vec{p}}{\omega} = n, \quad \int\frac{d^3\sigma_n}{d^3\vec{p}}d^3\vec{p} = \sigma_n,$$

respectively.

-sections according to the numbers of the particle C. Secondly, we see from the linear term in h

$$\frac{1}{\sigma}\sum_{n=2} (n-1)n \frac{d^3\sigma_n}{d^3\vec{p}/\omega} = \langle(n-1)\rangle_{\vec{p}} f^{(1)}(\vec{p}) =$$

$$= \int f^{(2)}(\vec{p}, \vec{p}_2) \frac{d^3\vec{p}_2}{\omega_2}, \tag{5.11}$$

which reproduces (5.2). Further comparison yields the general formula

$$\frac{1}{\sigma}\sum_{n=k+1}^{\infty} (n-k)(n-k+1)\dots(n-1)n \frac{d^3\sigma_n}{d^3\vec{p}/\omega} =$$

$$= \langle(n-k)\dots(n-1)\rangle_{\vec{p}} f^{(1)}(\vec{p}) =$$

$$= \int\dots\int f^{(k+1)}(\vec{p}, \vec{p}_2, \dots, \vec{p}_{k+1}) \frac{d^3\vec{p}_2\dots d^3\vec{p}_{k+1}}{\omega_2\dots\omega_{k+1}}. \tag{5.12}$$

Generalization to cases where two or more kinds of particles are present [32] is straightforward. For instance, we can consider a slightly different type of semi-inclusive cross-section

$$n_A \frac{d^3\sigma n_B}{d^3p_A} \text{ for } a+b \to A(\vec{p}_A) + \underbrace{B+\dots+B}_{n_B} + (\text{anything not } B). \tag{5.13}$$

Correspondingly, we construct the following generating function

$$\omega_A \frac{\delta F[s; h_A(\vec{p}), h_B(\vec{p})]}{\delta h_A(\vec{p}_A)}\Bigg|_{\substack{h_A(\vec{p})=0\\ h_B(\vec{p})=h}} = F(s; h; \vec{p}_A). \tag{5.14}$$

Equating its exclusive expansions, we get

$$F(s; h; \vec{p}_A) = \sum_{n_B=0}^{\infty} (1+h)^{n_B} n_A \frac{d^3\sigma_{n_B}}{d^3\vec{p}_A/\omega_A} =$$

$$= \sum_{k=0}^{\infty} \frac{h^k}{k!} \int\dots\int f^{(k+1)}_{A\underbrace{B\dots B}_{k}}(\vec{p}_A, \vec{p}_2, \dots, \vec{p}_{k+1}) \frac{d^3\vec{p}_2\dots d^3\vec{p}_{k+1}}{\omega_2\dots\omega_{k+1}}. \tag{5.15}$$

From (5.15) we obtain firstly, putting $h = 0$,

$$\sum_{n_B=0}^{\infty} n_A \frac{d^3\sigma_{n_B}}{d^3\vec{p}_A/\omega_A} = f^{(1)}_A(\vec{p}_A), \tag{5.16}$$

and further, putting $h = 0$ after differentiation, the relation (5.1) and its generalization,

$$\langle (n_B-k+1)\dots(n_B-1)n_B\rangle_{\vec{p}_A} f_A^{(1)}(\vec{p}_A) = $$
$$= \underbrace{\int\dots\int}_{k} f^{(k+1)}_{AB\dots B}(\vec{p}_A, \vec{p}_2, \dots, \vec{p}_{k+1}) \frac{d^3\vec{p}_2\dots d^3\vec{p}_{k+1}}{\omega_2\dots\omega_{k+1}} . \tag{5.17}$$

It is also not difficult to extend the semi-inclusive cross-sections (5.7), (5.13) to cases where momenta of two or more particles are measured. Then a problem concerning such many-particle semi-inclusive cross-sections arises: how to define "correlations" among these particles consistently with the definition of correlations in inclusive cross-sections. This is not trivial because of non-linearity of the definition of correlation. A proposal has been made to answer this question [35]. (See Appendix C.)

6. *Fluid analogy*

Feynman's fluid analogy has been the essential background of introducing correlations in inclusive many-particle cross-sections [5]. On the basis of Mueller's generating function and its cluster-function-like expansion [10], Bjorken [11] has made a further step of taking the "thermodynamical limit" and in this way he has given a more explicit formulation of the fluid analogy, which can incorporate specific features of a given model as an "equation of state". Below we sketch this argument.

Here it is more convenient to regard the generating function (4.2) as a function of

$$z = 1+h \tag{6.1}$$

$$Y = 2\sinh^{-1}\left(\frac{(s-4m^2)^{\frac{1}{2}}}{m}\right) \approx \ln\frac{s}{m^2} \tag{6.2}$$

instead of h and s. Thus one writes

$$F(Y; z) = \sum z^k \frac{\sigma_k(Y)}{\sigma(Y)} . \tag{6.3}$$

The formal analogy to a system in statistical mechanics is:

$Y \leftrightarrow$ volume,

$z \leftrightarrow$ fugacity (*i. e.* $\exp(-\mu/kT)$, with μ being the chemical potential),

$F(Y, z) \leftrightarrow$ grand partition function.

Now the assumption of short range correlation requires that all the correlation functions, when they are averaged over transverse momenta and are expressed as a function of longitudinal rapidities,

$$y_j = \tfrac{1}{2}\ln\frac{\omega_j + p_{j,\parallel}}{\omega_j - p_{j,\parallel}} , \tag{6.4}$$

should have the following property

$$\varrho^{(k)}(y_1, \dots, y_k) \approx 0$$
$$\text{if } |y_i - y_j| > \lambda \text{ for any } 1 \leqslant i < j \leqslant k, \tag{6.5}$$

λ being a constant (usually taken to be ~ 2). This implies

$$\lim_{Y\to\infty} R^{(k)} = \lim \int_0^Y \dots \int_0^Y \varrho^{(k)}(y_1, \dots, y_k) dy_1 \dots dy_k = $$
$$= a^{(k)} Y + b^{(k)}, \tag{6.6}$$

where $a^{(k)}$, $b^{(k)}$ are constants. The relation (6.6) can be understood if one changes integration variables to the CM coordinate and (k—1) relative coordinates in the rapidity space, since only the integration over the CM coordinate will yield a value proportional to Y, all others giving some finite (*i. e.*, Y-independent) values.

As a consequence of (6.6), one gets[14]

$$\lim_{Y\to\infty} R(Y; z) = p(z)Y + w(z). \tag{6.7}$$

Here the analogy is

$$\begin{aligned} R(Y, z) = \ln F(Y, z) &\leftrightarrow \text{free energy,} \\ p(z) &\leftrightarrow \text{pressure,} \\ w(z) &\leftrightarrow \text{surface tension,} \end{aligned}$$

and the relation (6.7) expresses existence of the "thermodynamical limit".

One can also derive the "density" ϱ defined by[15]

$$\varrho(z) = \lim_{Y\to\infty} z \frac{\partial}{\partial z} \frac{1}{Y} R(Y, z) =$$
$$= \lim_{Y\to\infty} \frac{1}{Y} \frac{\sum n z^n \sigma_n}{\sum z^n \sigma_n}, \tag{6.8}$$

and eliminating the "fugacity" z, obtain a relation between "pressure" and "density", *i. e.*, an "equation of state".

Thus the conclusion is that when a multiplicity distribution is specified as a function of energy in such a way as to satisfy (6.7), for which the short range correlation is a sufficient condition, there is a corresponding "fluid" characterized by a "pressure-fugacity" relation or by a "pressure-density" relation ("equation of state").

As a simple example [11] let us take the Poisson distribution with the average multiplicity proportional to $\ln s = Y$ (*e. g.* Chew-Pignotti model),

$$\frac{\sigma_n}{\sigma} = e^{-gY} \frac{(gY)^n}{n!}. \tag{6.9}$$

This gives

$$F(Y; z) = \exp gY(z-1), \; R(Y, z) = gY(z-1), \tag{6.10}$$

---

[14] The letter p is used to emphasize the analogy to "pressure". I hope there will be no confusion with momentum.

[15] The letter ϱ is used to emphasize the analogy to "density", hoping that there will be no confusion with the correlation functions.

and consequently, from (6.7), (6.8),

$$p(z) = g(z-1), \tag{6.11}$$

$$\varrho(z) = gz. \tag{6.12}$$

Thus the "equation of state" is

$$p = \varrho - g.$$

At the actual value $z = 1$, one gets $p = 0$, $\varrho = g$.

Another example will be discussed in Section 13. For further applications and examples, see Refs [11], [36], [33], [34].

B. KINEMATICAL CONSTRAINTS

7. *Direct derivation* [37], [38]

When discussing properties of many particle distribution and correlation function, it is very important to separate, as far as possible, "kinematical effects" due to conservation laws from genuine dynamical features.

In this section simple direct derivation of such kinematical constraints ("sum rules") is described. Here we are concerned with additively conserved quantities (*e. g.*, any component of fourmomentum, electric charge, baryon number, hypercharge, *etc.*) (A more general derivation with the help of GF, which will hopefully allow an extension to more complicated cases of isospin, is given in Section 9.) Usually the particles of final state are described as eigenstates of such an additive quantity q. Then an obvious constraint on the exclusive cross-section in that the latter should vanish unless the sum of q_j (eigenvalue of q for the particle j) is equal to its initial value Q,

$$Q = \sum_j q_j. \tag{7.1}$$

Its implication for inclusive cross-sections is less straightforward, but can be also understood in a fairly simple way.

Take, for example, the case of charge conservation and for simplicity assume that the final particles carry only ± 1 or 0 charge. Then for each single collision we have

$$Q = n_+ - n_-, \tag{7.2}$$

where n_+ and n_- are the number of positive and negative charge particles, respectively. After averaging over the whole set of events,

$$Q = \langle n_+ \rangle - \langle n_- \rangle =$$

$$= \int f_+^{(1)}(\vec{p}) \frac{d^3\vec{p}}{\omega} - \int f_-^{(1)}(\vec{p}) \frac{d^3\vec{p}}{\omega}; \tag{7.3}$$

$f_\pm^{(1)}(\vec{p})$ denotes the distribution function of positive and negative particles, respectively. This is the first constraint.

Next consider the case where we have detected a positively charged particle with momentum $\vec{p}_+$. Then we count the remaining plus and minus charge. Evidently

$$Q-1 = (n_+-1)-n_-. \tag{7.4}$$

We collect all those events where a positive particle has momentum $\vec{p}_+$ and take the average over this subset of events.

$$Q-1 = \langle n_+-1\rangle_{\vec{p}_+} - \langle n_-\rangle_{\vec{p}_+} \tag{7.5}$$

where $\langle n_+-1\rangle_{\vec{p}_+}$ and $\langle n_-\rangle_{\vec{p}_+}$ are the average associated multiplicities discussed in Section 5, *i. e.*,

$$\langle n_+-1\rangle_{\vec{p}_+} = \frac{1}{f_+^{(1)}(\vec{p}_+)}\int f_{++}^{(2)}(\vec{p}_+,\vec{p}'_+)\frac{d^3\vec{p}'_+}{\omega'_+},$$

$$\langle n_-\rangle_{\vec{p}_+} = \frac{1}{f_+^{(1)}(\vec{p}_+)}\int f_{+-}^{(2)}(\vec{p}_+,\vec{p}_-)\frac{d^3\vec{p}_-}{\omega_-}. \tag{7.6}$$

Thus we get the second constraint relation

$$(Q-1)f_+^{(1)}(\vec{p}_+) =$$
$$= \int f_{++}^{(2)}(\vec{p}_+,\vec{p}'_+)\frac{d^3\vec{p}'_+}{\omega'_+} - \int f_{+-}^{(2)}(\vec{p}_+,\vec{p}_-)\frac{d^3\vec{p}_-}{\omega_-}. \tag{7.7}$$

The same argument can be repeated to derive similar charge conservation constraint connecting $f^{(n)}$ to a single integral of $f^{(n+1)}$

$$(Q-\sum_{j=1}^{n} q_j)f_{q_1q_2\ldots q_n}^{(n)}(\vec{p}_1,\ldots,\vec{p}_n) =$$
$$= \sum_{q_{n+1}=\pm} q_{n+1}\int f_{q_1,\ldots,q_n,q_{n+1}}^{(n+1)}(\vec{p}_1,\ldots,\vec{p}_n,\vec{p}_{n+1})\frac{d^3\vec{p}_{n+1}}{\omega_{n+1}}. \tag{7.8}$$

We can express (7.3), (7.7), (7.8) also in terms of correlation functions. Simply by inserting the definition of $\varrho^{(k)}$, we get

$$Q = \int \varrho_+^{(1)}(\vec{p})\frac{d^3\vec{p}}{\omega} - \int \varrho_-^{(1)}(\vec{p})\frac{d^3\vec{p}}{\omega}, \tag{7.9}$$

$$-\varrho_+^{(1)}(\vec{p}) = \int \varrho_{++}^{(2)}(\vec{p},\vec{p}')\frac{d^3\vec{p}'}{\omega'} - \int \varrho_{+-}^{(2)}(\vec{p},\vec{p}')\frac{d^3\vec{p}'}{\omega'}, \tag{7.10}$$

in general,

$$(-\sum_{j=1}^{k} q_j)\varrho_{q_1\ldots q_k}^{(k)}(\vec{p}_1,\ldots,\vec{p}_k) =$$
$$= \sum_{q=\pm} q\int \varrho_{q_1\ldots q_kq}^{(k+1)}(\vec{p}_1,\ldots,\vec{p}_k,\vec{p})\frac{d^3\vec{p}}{\omega}. \tag{7.11}$$

In this set of relations for correlation functions, the initial value Q appears in the first relation (7.9) but not in the second and further relations. This is in sharp distinction[16] from the case of constraints on $f^{(n)}$.

Suppose, for a moment, that the correlation $\varrho^{(2)}_{++}$ and $\varrho^{(2)}_{+-}$ are short range[17], *i.e.*, they are non-vanishing only within a limited region

$$\vec{p}' \in \Omega_{\vec{p}}. \tag{7.12}$$

Then equation (7.10) tells us that when we detect a positive charge with momentum $\vec{p}$, the two functions

$$\frac{1}{\varrho^{(1)}_{+}(\vec{p})}\,\varrho^{(2)}_{++}(\vec{p},\vec{p}'), \qquad \frac{1}{\varrho^{(1)}_{+}(\vec{p})}\,\varrho^{(2)}_{+-}(\vec{p},\vec{p}') \tag{7.13}$$

which are the analogues to integrands of associated multiplicity (5.2), (5.1), are distributed in this small region is such a way that when integrated over $\vec{p}'$, the total charge just cancels the positive charge at $\vec{p}$. (See Fig. 2.)

So far we have discussed the case of charge conservation. The argument has been very simple owing to the fact that the value of charge is independent of momentum, as

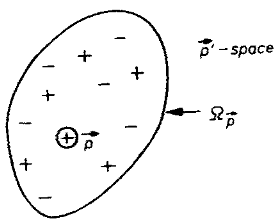

Fig. 2

is the case also with other "internal" additive quantum numbers. In the case of energy--momentum conservation, however, we need a slight generalization of the foregoing arguments. For instance, we have, in the case of a single kind of particles, for each event

$$P^{\mu}_{\text{tot}} = \sum p^{\mu} \tag{7.14}$$

and the average over all the events yields the first constraint,

$$P^{\mu}_{\text{tot}} = \int p^{\mu} f^{(1)}(\vec{p})\,\frac{d^3\vec{p}}{\omega} \tag{7.15}$$

and so on.

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[16] When $Q = 0$, of course, the constraints on $f^{(n)}$ and $\varrho^{(n)}$ are formally identical.

[17] For a realistic definition of short range correlations, see Section 17.

For one conserved quantity, there exists one set of kinematical constraints for distribution functions, of the general form [38–42]

$$Q = \sum_{\beta} \int q_\beta(\vec{p}) f_\beta^{(1)}(\vec{p}) \frac{d^3\vec{p}}{\omega},$$

$$\{Q - \sum_{j=1}^{n} q_{\beta_j}(\vec{p}_j)\} f^{(n)}_{\beta_1 \dots \beta_n}(\vec{p}_1, \dots, \vec{p}_n) =$$

$$= \sum_{\beta} \int q_\beta(\vec{p}) f^{(n+1)}_{\beta_1 \dots \beta_n \beta}(\vec{p}_1, \dots, \vec{p}_n, \vec{p}) \frac{d^3\vec{p}}{\omega},$$

$$(n \geqslant 1) \tag{7.16}$$

or equivalently one set for correlation functions,

$$Q = \sum_{\beta} \int q_\beta(\vec{p}) \varrho_\beta^{(1)}(\vec{p}) \frac{d^3\vec{p}}{\omega},$$

$$\{- \sum_{j=1}^{n} q_{\beta_j}(\vec{p}_j)\} \varrho^{(n)}_{\beta_1 \dots \beta_n}(\vec{p}_1, \dots, \vec{p}_n) =$$

$$= \sum_{\beta} \int q_\beta(\vec{p}) \varrho^{(n+1)}_{\beta_1 \dots \beta_n \beta}(\vec{p}_1, \dots, \vec{p}_n, \vec{p}) \frac{d^3\vec{p}}{\omega}. \tag{7.17}$$

8. Energy conservation for correlation functions

A particularly interesting role is played by the set of energy-conservation constraints on correlation functions. For simplicity we take again the case of a single kind of particles. Then the constraints (sum rules) are the following (see (7.17))

$$\sqrt{s} = \int \varrho^{(1)}(\vec{p}')d^3\vec{p}', \tag{8.1}$$

$$-\omega\varrho^{(1)}(\vec{p}) = \int \varrho^{(2)}(\vec{p}, \vec{p}')d^3\vec{p}', \tag{8.2}$$

$$(-\sum_{j=1}^{k} \omega_j)\varrho^{(k)}(\vec{p}_1, \dots, \vec{p}_k) = \int \varrho^{(k+1)}(p_1, \dots, p_k, p')d^3\vec{p}'. \tag{8.3}$$

An important point, which distinguishes this set of constraints from all others, is that the coefficient on the l.h.s. of (8.2), (8.3) never vanishes, so that we can always "solve" it and get

$$\varrho^{(k)}(\vec{p}_1, \dots, \vec{p}_k) = (-\sum_{j=1}^{k} \omega_j)^{-1} \int \varrho^{(k+1)}(\vec{p}_1, \dots, \vec{p}_k, \vec{p}')d^3\vec{p}'. \tag{8.4}$$

This should be compared with the corresponding case of distribution function (inclusive cross-section)

$$f^{(k)}(\vec{p}_1, \ldots, \vec{p}_k) = (\sqrt{s} - \sum_{j=1}^{k} \omega_j)^{-1} \int f^{(k+1)}(\vec{p}_1, \ldots, \vec{p}_k, \vec{p}')d^3\vec{p}' + \\ + \xi\delta(\sqrt{s} - \sum_{j=1}^{k} \omega_j). \tag{8.5}$$

Here the coefficient ξ (which is essentially the normalized exclusive cross-section of k particle production) us not determined by $f^{(k+1)}$.

The key relation (8.4) leads immediately to the following statements.

1) Correlation functions of any order n can never identically vanish [40–42]. For, if $\varrho^{(n)} = 0$ all over the phase space, then (8.4) yields successively

$$0 = \varrho^{(n-1)} = \varrho^{(n-2)} = \ldots = \varrho^{(2)} = \varrho^{(1)} \tag{8.6}$$

which contradicts with (8.1)[18]. To be more explicit, it is easy to derive from (8.1), (8.2), (8.3),

$$\int \cdots \int \varrho^{(k+1)}(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k, \vec{p}') \frac{d^3\vec{p}_1}{\omega_1} \cdots \frac{d^3\vec{p}_k}{\omega_k} \frac{d^3\vec{p}'}{\sqrt{s}} = (-1)^k k!. \tag{8.7}$$

2) Denoting by N the maximum number of final particles allowed by energy-momentum conservation

$$N = \sqrt{s}/m, \tag{8.8}$$

the correlation function $\varrho^{(N)}$ together with the normalization factor σ give in a simple way complete information of cross-sections and correlations.[19] From $\varrho^{(N)}$ one can namely derive, with the help of (8.4), the whole set of $\varrho^{(N-1)}$, $\varrho^{(N-2)}$, ..., $\varrho^{(2)}$, $\varrho^{(1)}$, which in turn determine $f^{(1)}$, $f^{(2)}$, ... $f^{(N)}$. Since it is known that $f^{(N+1)} = f^{(N+2)} = \ldots = 0$, we have obtained all the $f^{(n)}$ and consequently all the $\varrho^{(n)}$. This circumstance is not trivial since the same argument does not work if we take $f^{(N)}$ and σ instead of $\varrho^{(N)}$ and σ, as is seen from (8.5).

9. *Derivation from generating functional*

The general form of kinematical constraints (7.17), (7.18) have been discussed by many authors [38, 39], [13], [40–42]. In particular, Brown [15] has obtained a compact expression by application of the GF technique[20].

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[18] Putting higher order correlations equal to zero (a simple example being the case of Poisson distribution) is thus an approximation which will be allowed when the number of particles is large and addition or removal of one particle does not affect the state to any appreciable extent.

[19] This is an outcome of discussion with H. B. Nielsen and P. Olesen.

[20] See further Refs [43, [44].

Here we give a simple derivation of these constraints utilizing the GF formalism. This method appears to have a potentiality of being generalized to isospin conservation constraints, such as recently discussed by Di Giacomo [45][21].

We start from the second expression (2.6) of GF. To be more explicit, it is given by

$$F[s; h_\beta(\vec{p})] =$$

$$= \langle\psi_s|\exp\left[\sum_\beta \int \frac{d^3\vec{p}}{\omega} N_\beta(\vec{p}) \ln\{1+h_\beta(\vec{p})\}\right]|\psi_s\rangle \Big/ \langle\psi_s|\psi_s\rangle \tag{9.1}$$

with

$$|\psi_s\rangle = (S-1)|\psi_{\text{initial}}\rangle. \tag{9.2}$$

Consider the opertor $\boldsymbol{Q}$ for a certain additively conserved quantity

$$\boldsymbol{Q} = \sum_{\beta\beta'} \int \frac{d^3\vec{p}}{\omega} a^\dagger_\beta(\vec{p}) \boldsymbol{q}_{\beta\beta'}(\vec{p}) a_{\beta'}(\vec{p}) \tag{9.3}$$

where $\boldsymbol{q}(\vec{p})$ is the corresponding one-particle operator. Since we have assumed that the final state particles are eigenstates of $\boldsymbol{q}$, we can put

$$\boldsymbol{q}_{\beta\beta'}(\vec{p}) = q_\beta(\vec{p})\delta_{\beta\beta'} \tag{9.4}$$

so that

$$\boldsymbol{Q} = \sum_\beta \int \frac{d^3\vec{p}}{\omega} N_\beta(\vec{p}) q_\beta(\vec{p}). \tag{9.5}$$

The conservation law requires

$$\boldsymbol{Q}|\psi_s\rangle = (S-1)\boldsymbol{Q}|\psi_{\text{initial}}\rangle = Q|\psi_s\rangle, \tag{9.6}$$

where Q is the initial value (c-number) of the conserved quantity. Thus we get from (9.1), (9.5) and (9.6),

$$QF[s; h_\beta(\vec{p})] =$$

$$= \left\langle \exp\left[\sum_\beta \int \frac{d^3\vec{p}}{\omega} N_\beta(\vec{p}) \ln\{1+h_\beta(\vec{p})\}\right] \cdot \sum_{\beta'} \int \frac{d^3\vec{p}'}{\omega'} N_{\beta'}(\vec{p}') q_{\beta'}(\vec{p}') \right\rangle =$$

$$= \sum_{\beta'} \int \frac{d^3\vec{p}'}{\omega'} q_{\beta'}(\vec{p}')\{1+h_{\beta'}(\vec{p}')\}\omega' \frac{\delta F}{\delta h_{\beta'}(\vec{p}')}. \tag{9.7}$$

This is the same expression as obtained by Brown, and by equating the coefficients of power series expansion of both sides in $h_\beta(\vec{p})$, one derives the set of constraints (7.16).

[21] This reference has been added after the Colloquium in Zakopane.

From (9.7) we have also

$$Q = \sum_{\beta'} \int \frac{d^3\vec{p}'}{\omega'} q_{\beta'}(\vec{p}') \{1+h_{\beta'}(\vec{p}')\} \frac{\omega'}{F} \frac{\delta F}{\delta h_{\beta'}(\vec{p}')} =$$

$$= \sum_{\beta'} \int \frac{d^3\vec{p}'}{\omega'} q_{\beta'}(\vec{p}') \{1+h_{\beta'}(\vec{p}')\}\omega' \frac{\delta R}{\delta h_{\beta'}(\vec{p}')} \tag{9.8}$$

which reproduces, when expanded in power series of $h_\beta(\vec{p})$, the whole set (7.17) of constraints on the correlation functions.

10. Leading particles and clustering

So far we have not made any distinction among the produced particles, but it is empirically known that in many cases the so-called leading particles carry away a large fraction of the available energy. When combined with the energy-momentum conservation, this can lead to non-trivial effects of clustering of the other particles. In this section we briefly outline arguments of Berger, Krzywicki and Petersson on this point [46–48].

Let the scaling variable

$$x = \frac{p_{||}}{\sqrt{s}/2} \tag{10.1}$$

of the forward- and backward-going leading particles [22] be denoted by x_1 and x_2, respectively (x_1 is close to $+1$, and x_2 is close to -1). Then the CMS energy and longitudinal momentum of the rest of the particles are

$$E \approx \frac{\sqrt{s}}{2}\{(1-x_1)+(1+x_2)\}, \quad P_{||} \approx \frac{\sqrt{s}}{2}\{(1-x_1)-(1+x_2)\}, \tag{10.2}$$

so that the invariant mass and the average CMS rapidity of these particles are given by

$$M = (E^2-P^2)^{\frac{1}{2}} \approx \{s(1-x_1)(1+x_2)\}^{\frac{1}{2}}, \tag{10.3}$$

$$\bar{y} = \tfrac{1}{2}\ln\frac{E+P_{||}}{E-P_{||}} \approx \tfrac{1}{2}\ln\frac{1-x_1}{1+x_2}. \tag{10.4}$$

Two cases can be now considered.

1) The multiplicity depends strongly on the values of x_1, x_2, so that for small M the multiplicity is also small.

2) The multiplicity depends only weakly on x_1 and x_2, so that for small M a large multiplicity can also take place.

The multiperipheral model corresponds to the first case, leading to no appreciable clustering of the remaining particles. If one makes a specific assumption ("bootstrap

[22] They are final state nucleons in the case of nucleon-nucleon collision.

condition") on the single particle distribution of the other-than-leading particles, one can set up an integral equation[23] for the overall single particle distribution $f[1]$.

In the second case, for small M and larger multiplicity, the produced particles other than the leading ones will be confined in a small region of rapidity, thus forming a cluster. The position of the cluster in the rapidity space fluctuates from event to event so that the single particle distribution will be flat in the central region, but the clustering leads to certain correlations, which could sometimes simulate other dynamical mechanisms.

C. COEXISTENCE OF TWO (OR MORE) MECHANISMS

11. Joint correlation

Let us now discuss the following problem. Suppose there are two mechanisms of particle production with different dynamical characters, for instance, pionization and diffraction dissociation. Denote them by A and B, respectively. For each mechanism, taken by itself, we can construct GF and correlation functional F_A, F_B and R_A, R_B, which of course satisfy

$$F_A = \exp R_A, \qquad F_B = \exp R_B. \tag{11.1}$$

The "total" cross-section for each mechanism will be denoted by σ_A and σ_B. What happens, then, if the two mechanisms coexist?

For simplicity we shall assume that the two mechanisms are effective in different regions of the phase space, so that the interference can be neglected. Then the resultant distribution functions are given by the weighted mean of those for the two mechanisms,

$$f^{(n)} = \frac{\sigma_A}{\sigma_A+\sigma_B} f^{(n)}_{(A)} + \frac{\sigma_B}{\sigma_A+\sigma_B} f^{(n)}_{(B)}. \tag{11.2}$$

Consequently the same relation holds for GF

$$F = \frac{\sigma_A}{\sigma_A+\sigma_B} F_A + \frac{\sigma_B}{\sigma_A+\sigma_B} F_B. \tag{11.3}$$

Thus the joint distribution is simply expressed in terms of the constituent distributions.

On the other hand, the joint correlation takes a complicated (though elementary) form when expressed in terms of the constituent correlations,

$$R = \ln\left\{\frac{\sigma_A}{\sigma_A+\sigma_B} \exp R_A + \frac{\sigma_B}{\sigma_A+\sigma_B} \exp R_B\right\}. \tag{11.4}$$

---

[23] See also Ref. [49]. (This reference has been added after the Colloquium.)

From this one gets, for example[24],

$$\varrho^{(1)}(\vec{p}) = \frac{\sigma_A}{\sigma_A+\sigma_B}\,\varrho_A^{(1)}(\vec{p}) + \frac{\sigma_B}{\sigma_A+\sigma_B}\,\varrho_B^{(1)}(\vec{p}), \tag{11.5}$$

$$\varrho^{(2)}(\vec{p}_1,\vec{p}_2) = \frac{\sigma_A}{\sigma_A+\sigma_B}\,\varrho_A^{(2)}(\vec{p}_1,\vec{p}_2) + \frac{\sigma_B}{\sigma_A+\sigma_B}\,\varrho_B^{(2)}(\vec{p}_1,\vec{p}_2) +$$

$$+\frac{\sigma_A\sigma_B}{(\sigma_A+\sigma_B)^2}\{\varrho_A^{(1)}(\vec{p}_1)-\varrho_B^{(1)}(\vec{p}_1)\}\{\varrho_A^{(1)}(\vec{p}_2)-\varrho_B^{(1)}(\vec{p}_2)\}. \tag{11.6}$$

The third term on the r.h.s. of (11.6) is non-negative when $\vec{p}_1 \approx \vec{p}_2$ and also when integrated over the whole phase space. The latter case gives

$$R^{(2)} = \frac{\sigma_A}{\sigma_A+\sigma_B}\,R_A^{(2)} + \frac{\sigma_B}{\sigma_A+\sigma_B}\,R_B^{(2)} + \frac{\sigma_A\sigma_B}{(\sigma_A+\sigma_B)^2}\,(\langle n_A\rangle - \langle n_B\rangle)^2. \tag{11.7}$$

That is to say, the integral of joint 2-particle correlation is larger than or equal to the weighted mean of the integrals of constituent 2-particle correlations, the equality holding only when the average number of two mechanisms taken separately are equal to each other.

Białas, Fiałkowski, Wit and Zalewski [18], [32], [51] give a physical interpretation to this apparent increase of correlation in the following way: If a slow pion is detected, it indicates that the event is non-diffractive and this increases the probability of finding another slow pion.

Another interpretation for (11.7) is obtained by recalling the relation (1.10b) and noting

$$\langle n\rangle = \frac{\sigma_A}{\sigma_A+\sigma_B}\,\langle n_A\rangle + \frac{\sigma_B}{\sigma_A+\sigma_B}\,\langle n_B\rangle. \tag{11.8}$$

Then (11.7) can be rewritten in the form

$$D^2 = \frac{\sigma_A}{\sigma_A+\sigma_B}\,D_A^2 + \frac{\sigma_B}{\sigma_A+\sigma_B}\,D_B^2 + \frac{\sigma_A\sigma_B}{(\sigma_A+\sigma_B)^2}\,\{\langle n_A\rangle - \langle n_B\rangle\}^2, \tag{11.9}$$

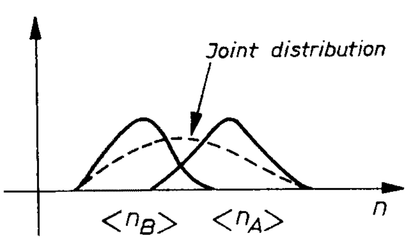

Fig. 3

[24] The relation (11.6) has been pointed out in Ref. [51]. Its special case is found already in Ref. [5].

showing that the squared dispersion of the joint distribution is larger than the weighted mean of those of constituent distributions unless the average numbers of constituent distributions coincide. This is easily understandable. See Fig. 3.

12. Some examples

The relation (11.9) appears, explicitly or implicitly, in many cases.

As an example we refer to the work of Giovannini [48] who superposes, with a weight e^{-m}, a continuously infinite number of Poisson distributions with the average value $m\langle n\rangle$, where $\langle n\rangle$ is a constant and m is a positive parameter varying between 0 and ∞. That is,

$$\frac{\sigma_k}{\sigma_{\text{tot}}} = P_k = \int dm e^{-m} p_k^{(m)} \tag{12.1}$$

with

$$P_k^{(m)} = e^{-m\langle n\rangle} \frac{(m\langle n\rangle)^k}{k!}, \quad k = 0, 1, 2, \ldots. \tag{12.2}$$

This leads to a Furry distribution (geometric series distribution)

$$\frac{\sigma_k}{\sigma_{\text{tot}}} = \frac{1}{\langle n\rangle+1}\left(1-\frac{1}{\langle n\rangle+1}\right)^k \tag{12.3}$$

which has the following properties,

$$\sum kP_k = \langle n\rangle \tag{12.4}$$

(the notation $\langle n\rangle$ has been used in (12.2) and (12.3) in anticipation of this result), and

$$D^2 = \sum k^2 P_k - (\sum kP_k)^2 = \langle n\rangle^2 + \langle n\rangle. \tag{12.5}$$

This large dispersion should be contrasted to the small one which each constituent Poisson distribution possesses.

Another example can be found in a simple case of LeBellac's relation[25] [53], [21].

---

[25] LeBellac's work [53] is of more general validity. He has shown that if

$$\frac{1}{\sigma}\frac{d^k\sigma_{\text{incl}}}{dy_1\ldots dy_k} \to \left[\frac{1}{\sigma}\frac{d^j\sigma_{\text{incl}}}{dy_1\ldots dy_j}\right]\left[\frac{1}{\sigma}\frac{d^{k-j}\sigma_{\text{incl}}}{dy_{j+1}\ldots dy_k}\right] \tag{i}$$

when

$$|y_i - y_g| \to \infty, \quad 1 \leqslant i \leqslant j, \quad j+1 \leqslant g \leqslant k, \tag{ii}$$

then the ratio σ_n/σ of n-particle production cross-section σ_n to $\sigma = \sum \sigma_n$ decreases faster than any power of $(\ln s)$. Here y is the rapidity variable

$$y = \frac{1}{2}\ln\left(\frac{\omega+p_{\|}}{\omega-p_{\|}}\right) \tag{iii}$$

and $d^k\sigma_{\text{incl}}/dy_1 \ldots dy_k$ represents the inclusive cross-section integrated over transverse momenta. The condition (*i*), (*ii*) leads to the short range correlation result,

$$R^{(2)} \sim \ln s. \tag{iv}$$

In the text we illustrate that if there is an energy-independent part in any σ_k then $R^{(2)}$ must behave as $\sim (\ln s)^2$ and not as $\sim (\ln s)$.

One takes a mechanism A of short range correlation (for instance, multiperipheral model) which yields

$$\langle n_A \rangle \sim \ln s, \quad D_A^2 \sim \ln s \tag{12.6}$$

on one hand, and an energy-independent component B with

$$\langle n_B \rangle = \text{const}, \quad D_B^2 = \text{const} \tag{12.7}$$

on the other. Then (11.9) indicates that for the joint distribution,

$$D^2 \sim (\ln s)^2, \tag{12.8}$$

i.e.

$$R^{(2)} \sim (\ln s)^2. \tag{12.9}$$

Fig. 4 illustrates the situation.

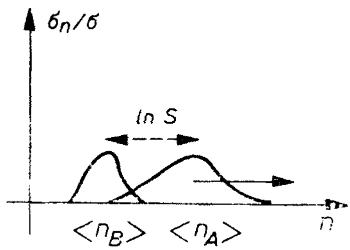

Fig. 4

13. Separation of diffraction and pionization

From the foregoing arguments it is obvious that, if the hadronic production process consists of diffraction and pionization, one has to separate the former from the latter or somehow eliminate effects of the former before one can study the short range correlation character of the latter mechanism.

One of the emphasized points in Wilson's program [5] concerns this problem. He proposes as the first experiment to measure the multiplicity distribution of charged particles at a sufficiently high energy. As is seen from the exaggerated Figure 4 of the preceding section, one can then expect a dip to appear between the energy-independent low-multiplicity part due to diffraction and the pionization part which moves with increasing energy towards the higher multiplicity.

Applying the fluid analogy described in Section 6, Bander [36] has proposed a method of evaluating this possibility in more detail. Denoting as before the pionization and diffraction by A and B, respectively, and for simplicity[26] taking the Poisson distribution (6.9) for A,

$$F_A(Y; z) = \exp\{g(z-1)Y\} \tag{13.1}$$

[26] In Ref. [36] a more general expression of the multiperipheral model is used.

and a set of energy-independent topological cross-sections for B,

$$F_B(Y;z) = \frac{1}{\sigma_B}\sum z^n \sigma_{B,n} = F_B(z) \tag{13.2}$$

one gets a "grand partition function" for the joint distribution

$$F(Y;z) = \frac{\sigma_A}{\sigma_A+\sigma_B}\exp\{g(z-1)Y\} + \frac{\sigma_B}{\sigma_A+\sigma_B}F_B(z), \tag{13.3}$$

where σ_A and σ_B are assumed to be independent of energy.

Consequently, the "pressure-fugacity" relation turns out to be

$$p(z) = \lim_{Y\to\infty}\frac{1}{Y}\ln F(Y;z) = \begin{cases} g(z-1) & \text{for } z \geqslant 1 \\ 0 & \text{for } z < 1 \end{cases} \tag{13.4}$$

reminiscent of a "phase transition" at the actual value $z = 1$. (See Fig. 5.)

The analysis proposed is thus to plot $\frac{1}{Y}\ln\{\sum Z^n\sigma_n/\sigma_{\text{tot}}\}$ *vs* $\frac{1}{Y}$ and, for various values of z, make an extrapolation to $1/Y = 0$. This yields the function $p(z)$, and one will

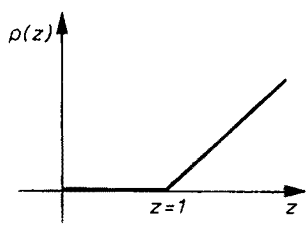

Fig. 5

see firstly whether the expected behaviour (13.4) really takes place, and, if so, one will be able to determine the unknown parameters involved in $F(Y, z)$. In this way one will predict the multiplicity distribution at any energy, and, in particular, shape of the dip at a sufficiently high energy, value of the energy where one can separate (at least partially) the two mechanisms, *etc.* (The experimental data available at present do not seem to allow a definite extrapolation. Two possible extrapolations and conclusions therefrom are given in Ref. [36].)

For other possibilities and further discussions on the problem of separating the two mechanisms see Refs [5], [36], [51].

D. SPECIFIC PREDICTIONS OF VARIOUS MODELS

So far we have been mostly[27] discussing rather general, model-independent aspects of many-particle distributions and correlations. Now we turn to predictions of various models of particle production. In fact one of the main motivations of studying many--particle distributions and correlations lies in the expectation that here clear-cut distinc-

[27] Except Sections 6, 10, 13 and part of Section 12.

tions will be made between a number of models, all of which yield almost the same results as far as the total cross-section and the single particle distributions are concerned. We shall be thus mainly concerned with qualitative features which characterize various models at asymptotic energies.

14. Asymptotic behaviour of integrated correlations

Different models predict indeed quite different behaviour of two particle correlation. This can be readily seen by considering its full integral over the phase space, which is given by the average and width of the multiplicity distribution [12], [55], [56], $R^{(2)} = D^2 - \langle n \rangle$, as has been already shown in (1.10b).

Let us inquire the energy dependence of $R^{(2)}$ when[28]

$$s \to \infty, \ \langle n \rangle \sim \ln s. \tag{14.1}$$

The answers of various models are enumerated in Table II. Notice that $R^{(2)}$ is bounded from below[29],

$$R^{(2)} = D^2 - \langle n \rangle \geqslant -\langle n \rangle. \tag{14.2}$$

TABLE II

UJM (uncorrelated jet model) [12], [57]–[59] with transverse momentum cut off	$R^{(2)} \sim$ const. < 0
UJM in general [60] Short range correlation models [10], [14] MPM (multiperipheral model) [5], [8], [10]–[12], [61], [62] DRM (dual resonance model) [63]–[69]	$R^{(2)} \sim a \ln s$ $a > 0$ $a < 0$
MPM with absorption [70] Diffraction and pionization [50], [5], [32], [51], [18], [53], [54] (Unitary isospin model [71]) (Non-equilibrium model [72], [52])	$R^{(2)} \sim b(\ln s)^2$ $b > 0$
DEM (diffractive excitation model) [16], [73], [74], [77] or LFH (limiting fragmentation hypothesis) [75], [76]	$R^{(2)} \sim c\, s^{1/2}$ $c > 0$

From (14.1) and (14.2), $R^{(2)}$ can be either positive or negative when it behaves like $\sim \ln s$ (or weaker than this), but it can be only positive when it behaves as $\sim (\ln s)^2$ or $s^{1/2}$.

The distinction is quite obvious. It becomes less effective if one considers the integral $F^{(2)}$ of two-particle distribution function, instead of correlation. All the models except the last group (DEM) lead to

$$F^{(2)} \sim (\ln s)^2 \tag{14.3}$$

while the DEM yields

$$F^{(2)} \sim s^{1/2}. \tag{14.4}$$

---

[28] Empirically the relation $\langle n \rangle \sim \ln s$ is not yet really established. But since nearly all the current theoretical models yield this behaviour, we shall here assume its asymptotic validity.

[29] This simple fact must have been known to many people, but I owe this remark to K. Zalewski, who, in turn, has been notified by I. Białynicki-Birula.

15. Longitudinal correlations

Transverse correlations (and spin correlations) are reviewed in a very illuminating way in the reviews of Caneschi [19] and Abarbanel [78]. So I shall not enter this topic here, but go over directly to the problem of longitudinal correlations. Hereafter I shall use the same symbols $f^{(k)}$, $\varrho^{(k)}$ to denote distribution and correlation functions integrated over the transverse momenta and thus depending only on longitudinal variables.

For the longitudinal variable, one can use either the scaling variable

$$x = \frac{p_{||}}{(p_{||})_{\max}} \approx \frac{2p_{||}}{\sqrt{s}}, \tag{15.1}$$

sometimes together with

$$\bar{x} = \left(x^2 + \frac{m^2+p_\perp^2}{s/4}\right)^{\frac{1}{2}} \underset{s\to\infty}{\to} |x|, \tag{15.2}$$

in order to specify more explicitly how the limit is approached, or the rapidity variable

$$y = \tfrac{1}{2} \ln \frac{\omega + p_{||}}{\omega - p_{||}}. \tag{15.3}$$

In the following we shall be mainly concerned with the two-particle correlation $\varrho^{(2)}$ (or distribution $f^{(2)}$ in some cases). They can be represented either in x_1-x_2-plot or in y_1-y_2-plot. (We shall also use a modified x_1-x_2-plot in Section 18.) The advantage and disadvantage of x and y variables are well known: The x_1-x_2-plot is suitable for describing the asymptotic behaviour ($s \to \infty$) but the central region is compressed into a single point ($x_1 \approx x_2 \approx 0$). The y_1—y_2-plot can be made only for a finite energy[30], but explicitly shows how much contribution comes from various parts of the phase space to the full integral of the correlation function,

$$R^{(2)} = \int_{-1}^{1}\int_{-1}^{1} \varrho^{(2)}(x_1, x_2) \frac{dx_1 dx_2}{\bar{x}_1 \bar{x}_2} = \iint \varrho^{(2)}(y_1, y_2) dy_1 dy_2. \tag{15.4}$$

All these are natural extension of the case of single particle distribution. There is, however, one more point which favours the x_1-x_2-plot of $\varrho^{(2)}$. The energy conservation sum rule takes namely the following form

$$-\bar{x}_1 \varrho^{(1)}(x_1) = \int \varrho^{(2)}(x_1, x_2) dx_2 \tag{15.5}$$

$$\left(\text{not } \frac{dx_2}{\bar{x}_2}\right)$$

whose r.h.s. can be directly read on the x_1-x_2-plot.

---

[30] For an infinite energy one can use a normalized rapidity variable $y/\ln s$, but this is not suitable for our purpose, because the finite correlation lengths that characterize the short range correlation models (Section 17) are not distinguished.

In the x_1-x_2-plot $(-1 \leqslant x_1, x_2 \leqslant 1)$ the kinematical boundary for 2-particle distribution is given by

$$\begin{aligned} 1-|x_1|-|x_2| &= 0 \quad \text{for} \quad x_1x_2 > 0, \\ (1-|x_1|)(1-|x_2|) &= 0 \quad \text{for} \quad x_1x_2 < 0. \end{aligned} \tag{15.6}$$

See Fig. 6. Consequently $f^{(2)}(x_1, x_2)$ vanishes outside this boundary (shaded regions in Fig. 6) but the product of $f^{(1)}(x_1)f^{(1)}(x_2)$ survives, in general, and gives rise to negative

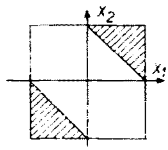

Fig. 6

values of correlation in these regions. This is, so to speak, trivial negative correlation of an obvious origin, but often plays an important role in saturating the sum rule. See examples in the following sections.

16. UJM (with transverse momentum cut off)

The UJM is the simplest model[31] which incorporates the energy momentum conservation. Following Caneschi [19] we assume that the total cross-section is constant when $s \to \infty$, and take only the leading terms in the results of UJM[32], and get

$$f^{(1)}(x) = \varrho^{(1)}(x) = 2(1-|x|), \tag{16.1}$$

$$f^{(2)}(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1x_2 > 0,\ |x_1+x_2| > 1, \\ 4\,(1-|x_1|-|x_2|) & \text{if} \quad x_1x_2 > 0,\ |x_1+x_2| \leqslant 1, \\ 4\,(1-|x_1|)(1-|x_2|) & \text{if} \quad x_1x_2 < 0. \end{cases} \tag{16.2}$$

Consequently

$$\varrho^{(2)}(x_1, x_2) = \begin{cases} -4(1-|x_1|)(1-|x_2|) & \text{if} \quad x_1x_2 > 0,\ |x_1+x_2| > 1, \\ -4\,|x_1| \cdot |x_2| & \text{if} \quad x_1x_2 > 0,\ |x_1+x_2| \leqslant 1, \\ 0 & \text{if} \quad x_1x_2 < 0. \end{cases} \tag{16.3}$$

Fig. 7 illustrates the function $\varrho^{(2)}(x_1, x_2)$. In this approximation, the correlation vanishes when two particles go in the opposite directions in CMS (the second and the fourth quadrants) and is everywhere negative when two particles go into the same hemisphere in CMS (the first and the third quadrants.)

---

[31] *P*-factorizable model in Bassetto-Toller-Sertorio's classification [12].

[32] For instance Eq. (3.9) or Eq. (4.11) of Ref. [57], where $\lambda = 2$ in order that the total cross-section is asymptotically constant.

The distribution of negative correlation in the region $0 \leqslant x_1, x_2 \leqslant 1$ is, in this case, symmetric with respect to the kinematical boundary of $f^{(2)}$, *i.e.*, $1-x_1-x_2 = 0$. The least favoured point for simultaneous compared to individual existence is $x_1 = x_2 = \frac{1}{2}$, as may have been anticipated, where $\varrho^{(2)} = -1$.

The energy conservation constraint requires that for any fixed x_1, the line integral over $-1 \leqslant x_2 \leqslant 1$ should yield a value just to cancel $|x_1|\ \varrho^{(1)}(x_1)$, the curve of which is also drawn in Fig. 7 at the bottom.

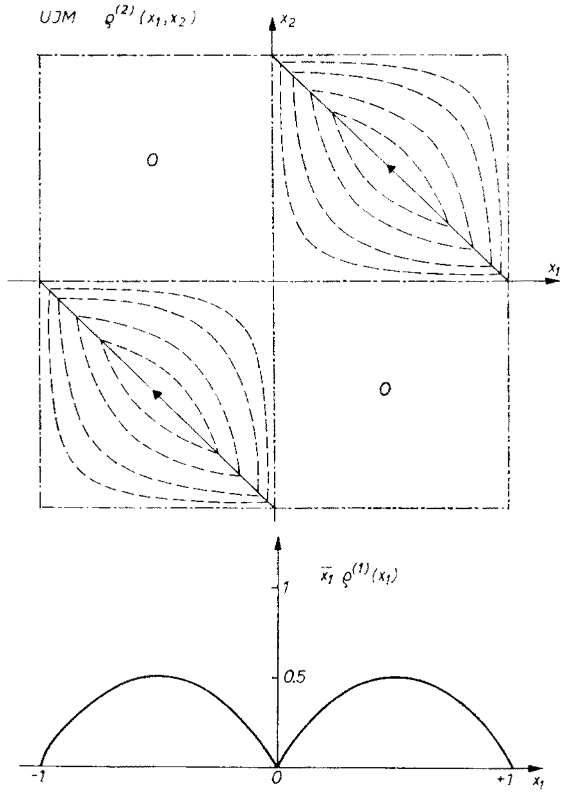

Fig. 7. The upper figure shows $\varrho^{(2)}$ (x_1, x_2) of UJM at $s \to \infty$. The chain contour lines (·—·—·—) represent the value 0, while the broken contour lines (— — — — —) represent negative values. In the regions $x_1 x_2 > 0$ the value of $\varrho^{(2)}$ is negative and its minimum is indicated by ▲, the value being −1. The contour lines surrounding the minimum stand, in the order from inside to outside, for $\varrho^{(2)} = -0.75, -0.5, -0.25, -0.1$. The solid line in these regions are the kinematical boundaries of $f^{(2)}(x_1, x_2)$. In the region $x_1 x_2 < 0$, $0 \leqslant |x_1|, |x_2| \leqslant 1$ the value of $\varrho^{(2)}$ vanishes everywhere. The lower figure represents the l.h.s. of the following sum role

$$\bar{x}_1\ \varrho^{(1)}(x_1) = -\int \varrho^{(2)}(x_1, x_2)\, dx_2$$

From this figure it looks as if the effect of anticorrelation due to the energy-momentum conservation, which is, apart from the transverse momentum cut-off, the only ingredient of the model, were quite overwhelming. This is of course not true; this impression simply

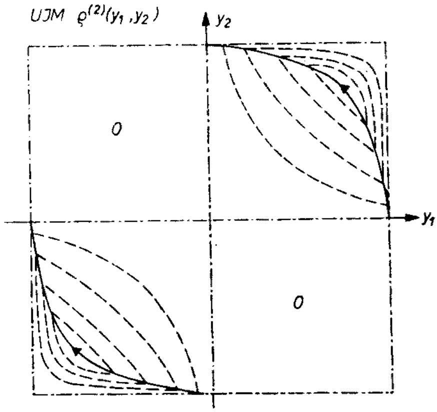

Fig. 8

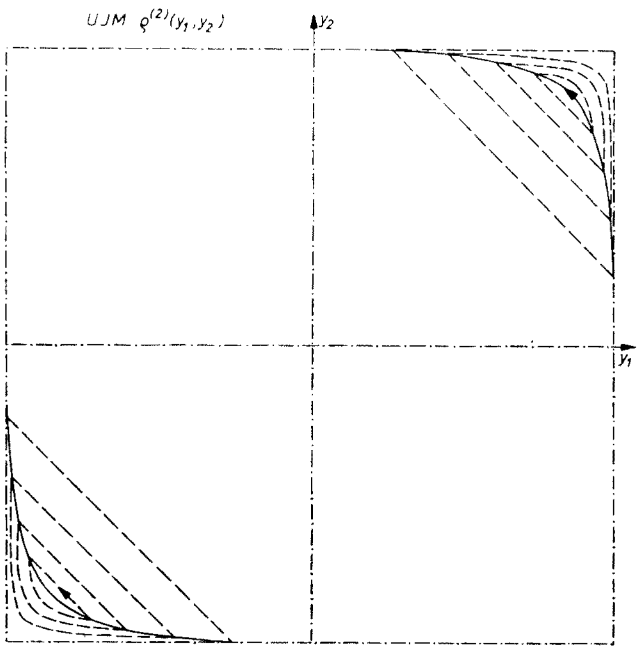

Fig. 9

Figs 8, 9. The function $\varrho^{(2)}(y_1, y_2)$ of UJM for $s = 54\ \text{GeV}^2$ ($p_{\text{lab}} = 28\ \text{GeV}/c$) (Fig. 8) and for $s = 2820\ \text{GeV}^2$ ($p_{\text{lab}} = 1500\ \text{GeV}/c$) (Fig. 9). The asymptotic expression for $\varrho^{(2)}$ is used here without any finite energy corrections. The values represented by the symbol ▲ and the contour lines are the same as in Fig. 7 except that the outer part of the contour $\varrho^{(2)} = -0.1$ is not drawn due to lack of space reflects the property of x-variable which overemphasizes the fragmentation regions. In order to see relative contributions to the full integral $R^{(2)}$, we should go over to the y_1-y_2--plot. See Figs 8, 9. With increasing energy the regions of appreciable negative correlation are farther pushed to the corner[33], leaving a vast, ever-growing central region of practically no correlation.

Although the UJM itself produces no positive correlation (at least in this approximation), it shows that at higher energy there is an ample space in the central region for occurrence of positive correlation of dynamical origin without violating the energy sum rule.

17. Short range correlation models

For MPM or DRM, in their simplest forms[34] at least, the characteristic prediction is the short range correlation in the rapidity space. That is to say, the correlation in the central region is appreciable only when

$$|y_1-y_2| \lesssim \lambda \tag{17.1}$$

and dies away for large $|y_1-y_2|$ as

$$\varrho^{(2)}(y_1, y_2) \sim \exp\left\{-\frac{1}{\lambda}|y_1-y_2|\right\}. \tag{17.2}$$

Both y_1 and y_2 are assumed to be much different from the boundary values, *i.e.*,

$$-\tfrac{1}{2}\ln\frac{s}{m^2}+\lambda_1 < y_1, y_2 < \tfrac{1}{2}\ln\frac{s}{m^2}-\lambda_1. \tag{17.3}$$

The correlation length λ (and λ_1) is usually estimated to be ~ 2.

Thus the most appropriate way to represent $\varrho^{(2)}$ in these models is to use y_1-y_2-plot [12]. See Figs 10, 11. Here the regions where correlations are significant are indicated. In the central region, $\varrho^{(2)}$ will depend only on $|y_1-y_2|$ because of the homogeneity, so that the contours are parallel to $y_1 = y_2$. Presumably its value is positive[35] [61] near $y_1 \sim y_2$ and change sign somewhere outside [54].

When we express the same regions in x_1-x_2-plot we get Fig. 12 [74]. The straight lines

$$y_2 = y_1 \pm 2 \tag{17.4}$$

---

[33] Since we are using the asymptotic expression (3) for $\varrho^{(2)}$, it is not really consistent to plot it for a finite energy. Corrections are, however, small, particularly at higher energy, and in any case it suffices for our purpose of qualitative description.

[34] MPM without any unitarity or absorption corrections, DRM without loop corrections.

[35] Private communication from C. I. Tan and from M. Toller.

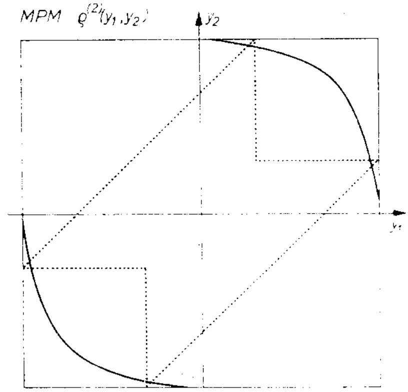

Fig. 10

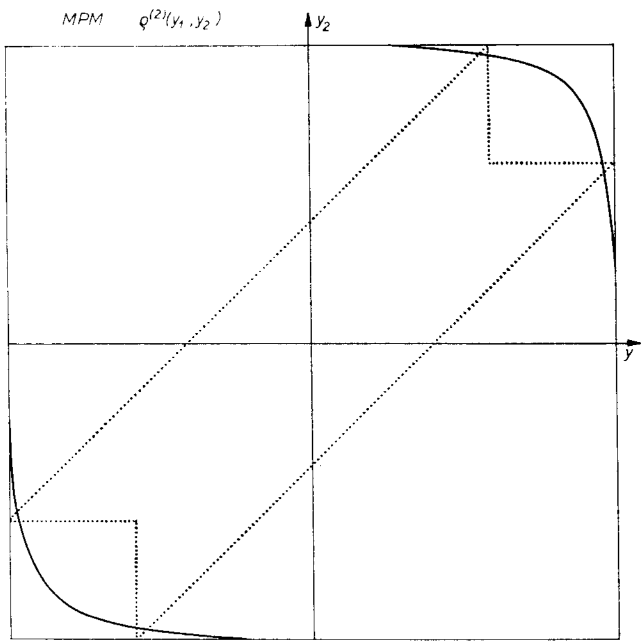

Fig. 11

Figs 10, 11. The function $\varrho^{(2)}(y_1, y_2)$ of short range correlation model. The dotted lines (.........) show the region where coııelations can be appreciable. Fig. 10 is for $s = 54$ GeV^2 and Fig. 11 for $s = 2820$ GeV^2. The regions which are within influence of boundaries are also indicated are mapped into straight lines

$$\frac{x_2}{x_1} = \exp(\pm 2) \tag{17.5}$$

when $s \to \infty$. If s is finite we get a hyperbola with these lines as asymptotes.

Again the effect of boundaries (fragmentation regions) becomes conspicuous, indicating that these regions are essential in saturating the energy sum rules[36], although they contribute little to the full integral $R^{(2)}$.

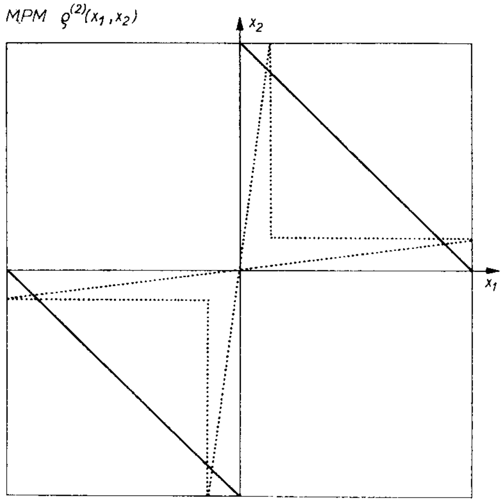

Fig. 12. The function $\varrho^{(2)}(x_1, x_2)$ of short range correlation model at $s \to \infty$. As in Figs 10, 11, the region of short range correlations and the regions influenced by the boundaries are indicated by dotted lines

18. DEM

This group of models can be regarded as a natural consequence of the original limiting fragmentation hypothesis (without pionization) [75].

As a simple example we take here the mathematical model of Quigg, Wang and Yang [76], which illustrates characteristic features of DEM (including more realistic versions [73], [74], [77]) and suffices for our purpose. The details of this model are described in Appendix D.

In order to present salient features of DEM, we first plot the normalized non-invariant two-body inclusive cross-sections

$$\frac{1}{\sigma_{\text{tot}}} \frac{d^2\sigma_{\text{incl}}}{dx_1 dx_2} = f^{(2)}(x_1, x_2)/(\bar{x}_1 \bar{x}_2). \tag{18.1}$$

[36] The single particle distribution of MPM in the x-variable is qualitatively similar to that of UJM.

See Fig. 13. For contrast we have also plotted the same quantity for the asymptotic UJM (*cf.* (16.2)), which will presumably resemble MPM outside the two lines, $x_2/x_1 = \exp(\pm 2)$, mentioned before, (17.5). See Fig. 14.

A conspicuous difference is found in the first and the third quadrants ($x_1 x_2 > 0$) in the region close to the axes $x_1 = 0$, $x_2 = 0$. At a fixed value $x_1 > 0$ let $x_2 \to 0_+$. Then

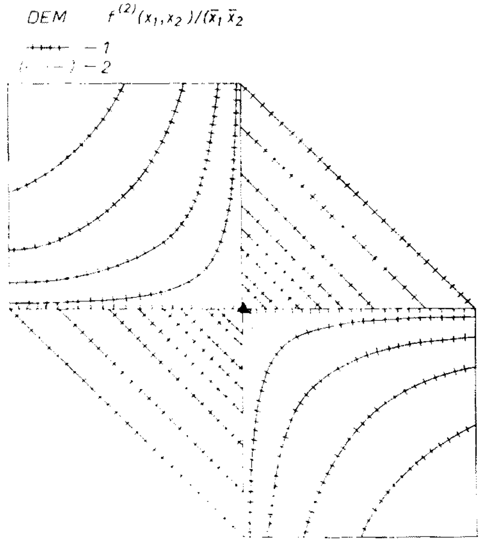

Fig. 13. The function $f^{(2)}(x_1, x_2)/(\bar{x}_1\bar{x}_2) = \dfrac{1}{\sigma_{\rm tot}} \dfrac{d^2\sigma_{\rm inc}}{dx_1, dx_2}$ of the mathematical model of Quigg, Wang and Yang [76]. The contour lines *1* represent positive values while the chain line *2* represents zero. The ▲ at the origin represents a singularity growing like $\sim s^{3/2}$. The values of the drawn contour lines in the regions $x_1 x_2 > 0$ are, in the order from outside to inside, ∞ (delta function along $|x_1| + |x_2| = 1$), 4, 10, 20, 40, 100, 200, 2000, and those in the regions $x_1 x_2 < 0$ are, from outside to inside, 2, 4, 12, 40

the function $f^{(2)}/\bar{x}_1\bar{x}_2$ of UJM blows up (because of the denominator), giving rise to $\sim \ln s$ dependence when integrated over x_2. The function $f^{(2)}/\bar{x}_1\bar{x}_2$ of DEM, on the other hand, remains finite[37].

This property leads to the prediction [76] that the associated multiplicity of right-going particles in DEM

$$\langle n^{(R)} - 1 \rangle_{x_1} = \frac{1}{\dfrac{d\sigma_{\rm incl}}{dx_1}} \int_0^1 \frac{d^2\sigma_{\rm incl}}{dx_1 dx_2}\, dx_2 \tag{18.2}$$

remains constant and does not grow with s if the specified value of x_1 is positive[38]. This is evidently in sharp contrast to the case of UJM or MPM, where this associated multi-

[37] This property is shared by Hwa's (more realistic) model [74]. In fact this will be common to all of DEM, as can be guessed from the physical argument given below.

[38] This can be directly verified by inserting (D.10), (D.11) into (18.2).

plicity blows up like $\sim \ln s$. (If x_1 is negative, $\langle n^{(R)} \rangle_{x_1}$ will behave like $\sim \ln s$ in both DEM[38] and UJM or MPM. This can be also read from Figs 13, 14.)

These authors give a clear physical argument for this remarkable behaviour of the associated multiplicity: when a particle with a large CMS momentum (corresponding

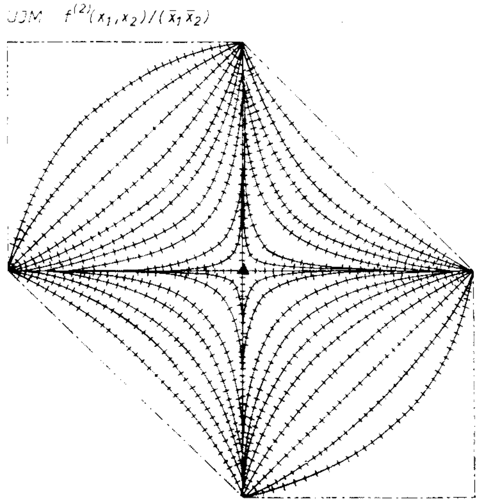

Fig. 14. The function $f^{(2)}(x_1, x_2)/(\bar{x}_1\bar{x}_2) = \frac{1}{\sigma^{\text{tot}}} \frac{d^2\sigma_{\text{incl}}}{dx_1, dx_2}$ of UJM at $s \to \infty$. The contour lines +++++++++ represent positive values. The ▲ at the origin represents a singularity growing like $\sim s$. In the regions $x_1x_2 > 0$ the values of the drawn contours are, in the order from outside to inside, 0, 4, 10, 20, 40, 100, 400, 1600. In the regions $x_1x_2 < 0$ they are, in the same order, 0, 1, 2, 4, 8, 15, 25, 45, 100, 400, 1600

to a finite $x_1 > 0$) has been detected, the cross-section $\sigma_n^{(R)}$ for producing n right-going particles is reduced by a factor $\chi(n, x_1)$

$$\sigma_n^{(R)} \to \sigma_n^{(R)}(x_1) = \chi(n; x_1)\sigma_n^{(R)} \tag{18.3}$$

because the available energy is now reduced to $(1-x_1)$. The reduction factor in this specific model can be evaluated to be[39]

$$\chi(n; x_1) = n(n-1)(1-x_1)^{n-2}. \tag{18.4}$$

Thus the cross-sections for large multiplicities will be strongly suppressed as far as x_1 is finite. Since the divergence of average multiplicity $\langle n \rangle \sim \ln s$ in the DEM originates from slow decrease of σ_n with increasing n (*i.e.* $\sigma_n \sim 1/n^2$), this suppression of σ_n for

[39] From (D. 3)

$$\sigma_n^{(R)'}(x_1) = \frac{1}{(n-1)!} \int \cdots \int \frac{d^n \sigma_{\text{excl}}^{(R)}}{dx_1\, dx_2 \ldots dx_n}\, dx_2 \ldots dx_n = K(1-x_1)^{n-2},$$

$$\chi(n; x_1) = \frac{\sigma_n^{(R)'}}{\sigma_n} = n(n-1)(1-x_1)^{n-2}.$$

large n can essentially change the behaviour of $\langle n \rangle$. Indeed one finds a convergent value,

$$\langle n^{(R)} \rangle_{x_1} = \frac{\sum n\sigma_n^{(R)\prime}(x_1)}{\sum \sigma_n^{(R)\prime}(x_1)} = \frac{\sum n(1-x_1)^{n-2}}{\sum (1-x_1)^{n-2}} = \frac{1+x_1}{x_1}\,. \tag{18.5}$$

Another decisive difference is the much stronger singularity at the origin. When approaching the origin along the direction with a finite fixed angle to both axes (*i.e.*, not along the x_1- or x_2-axis), the function $f^{(2)}/(x_1x_2)$ of DEM increases as $\sim r^{-3}$ while that

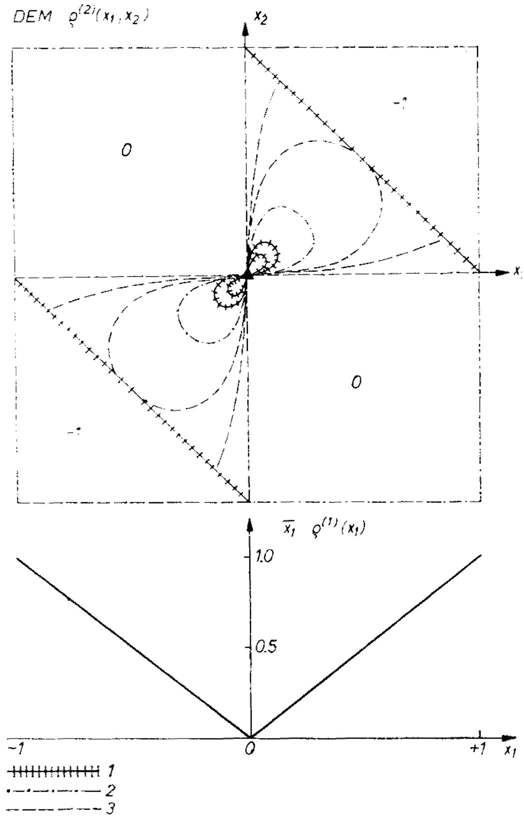

Fig. 15. The function $\varrho^{(2)}(x_1, x_2)$ of Quigg-Wang-Yang's mathematical model [76]. The three kinds of contour lines *1*, *2*, *3* represent positive, zero and negative values, respectively. In the regions $x_1x_2 > 0$, $|x_1|+|x_2| > 1$, this model gives $\varrho^{(2)} = -1$ everywhere, and along the line $|x_1|+|x_2| = 1$ there is a delta-function-like singularity of positive sign. The ▲ at the origin stands for a positive singularity $\sim s^{1/2}$. The values of the drawn contour lines are, in the order from inside to outside, 2, 1, 0, −0.5, −0.75. The lower figure shows the function $\bar{x}_1\varrho^{(1)}(x_1) = \bar{x}_1$, which is the l.h.s. of the sum rule (*cf.* the caption of Fig. 7)

of UJM increases as $\sim r^{-2}$, where r denotes the distance from the origin. This distinction can be more clearly seen in the plot of $f^{(2)}$ or $\varrho^{(2)}$.

Fig. 15 shows $\varrho^{(2)}(x_1, x_2)$ of this mathematical model of DEM[40]. No correlation is present in the regions $x_1x_2 < 0$, since independent fragmentation of two hadrons is assumed.

[40] We take here $\varrho^{(2)}$ for the purpose of comparing with the other models.

Outside the kinematical boundary of 2-particle distribution, correlation is of course negative (everywhere -1 in this particular model). At these boundaries themselves, there exist δ-function type positive correlations, which represent exclusive 2-particle cross--section remaining in $\varrho^{(2)}$. Directly inside these boundaries the correlations are again

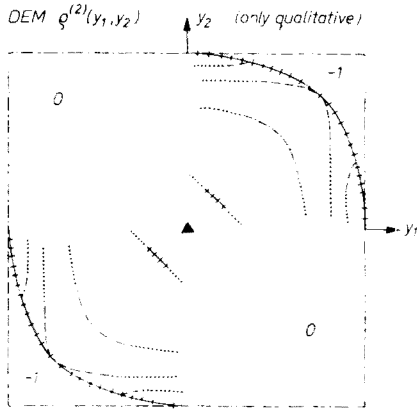

Fig. 16

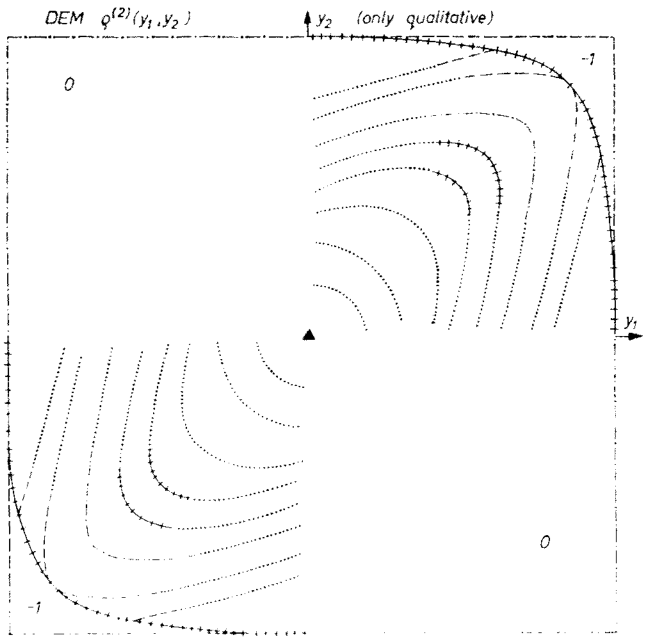

Fig. 17

Figs 16, 17. The function $\varrho^{(2)}(y_1, y_2)$ of the mathematical model [76]. These have only very crude, qualitative meaning, since finite energy corrections are ambiguous. Three kinds of contour lines are used as in Fig. 15, but the dotted parts (.........) are more uncertain. A tentative estimation gives the value of maximum at the origin 1.7 and those of drawn contours, in the order from inside to outside, 1, 0, -0.5, -0.75 for $s = 54\ \text{GeV}^2$ (Fig. 16), and the maximum 23 and the contours, in the same order, 20, 10, 5, 2, 1, 0, -0.5, -0.75, for $s = 2820\ \text{GeV}^2$ (Fig. 17) negative but near the origin there is a region of positive correlation, with a singularity $\sim s^{\frac{1}{2}}$ at the origin.

We can also verify that the energy sum rule is exactly satisfied (*cf.* the curve $\bar{x}\varrho^{(1)}(x)$ at the bottom of Fig. 15).

Finally we try to illustrate the situation in $y_1 - y_2$-plot. (Figs 16, 17.) These figures can have only very qualitative validity because this mathematical model is defined for asymptotic energy and how to make finite-energy corrections is ambiguous. These figures are just meant to indicate characteristic features such as *i*) rapid increase of the region of positive correlation, *ii*) rapid increase of the height of maximum at the origin[41], *iii*) dependence of the positive correlation on $y_1 + y_2$. The last point is to be compared with the case of MPM, where the positive correlation in the central region will be a function of $|y_1 - y_2|$ only. This difference is of course to be expected: in the DEM, where the correlation is long-range, the value of correlation will depend not only on the relative distance of two particles in the rapidity space but also on the location of the centre of the pair.

APPENDIX A

Equivalence of (2.1) *and* (2.6)

We shall prove equivalence of the two operators in the case of a single degree of freedom, *i.e.*,

$$:\exp(Nh): = \exp\{N \ln(1+h)\} \tag{A.1}$$

where

$$N = a^\dagger a, \tag{A.2}$$

$$[a, a^\dagger] = 1, \tag{A.3}$$

and h is an arbitrary c-number. Generalization to the case of a number (a continuously infinite number in our case) of degrees of freedom is straightforward.

For this purpose we introduce an orthonormal complete set of eigenfunctions of the number operator N,

$$|n\rangle = \frac{1}{(n!)^{\frac{1}{2}}} (a^\dagger)^n |0\rangle, \tag{A.4}$$

for which

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad n \geqslant 1, \tag{A.5}$$

$$a|0\rangle = 0, \tag{A.6}$$

[41] The origin is proportional to $s^{3/2}$ and the effective region is proportional to s^{-1}, so that the integral gives $\sim s^{1/2}$.

and thus

$$N|n\rangle = n|n\rangle \tag{A.7}$$

$$\langle m|n\rangle = \delta_{mn}. \tag{A.8}$$

Then for arbitrary m and n

$$\langle m|:\exp(Nh):|n\rangle = \sum_{j=0}^{\infty} \frac{h^j}{j!} \langle m|\,(a^\dagger)^j(a)^j|n\rangle =$$

$$= \sum_{j=0}^{m} \frac{h^j}{j!} \langle m-j|\,\{m(m-1)\ldots(m-j+1)\}^{\frac{1}{2}}\{n(n-1)\ldots(n-j+1)\}^{\frac{1}{2}}|n-j\rangle =$$

$$= \delta_{mn} \sum_{j=0}^{m} \frac{h^j}{j!}\,\frac{m!}{(m-j)!} =$$

$$= \delta_{mn}(1+h)^m =$$

$$= \langle m|\,(1+h)^N|n\rangle, \quad \text{q.e.d.} \tag{A.9}$$

APPENDIX B

Proof of (2.7), (2.8) *or* (2.11), (2.12)

We shall prove (2.11) and (2.12) which are equivalent to (2.7) and (2.8), respectively. Firstly we notice that the projection operator onto the vacuum state is given by

$$|0\rangle\langle 0| = :\exp\left\{-\int N(\vec{p})\,\frac{d^3\vec{p}}{\omega}\right\}:. \tag{B.1}$$

This is seen, for the case of a single degree of freedom, from the result of Appendix A. Putting $h = -1$ in (A.9) we get namely

$$\langle m|:\exp(-N):|n\rangle = 0 \quad \text{if} \quad m \neq 0, \quad \text{or} \quad n \neq 0, \tag{B.2}$$

while for $m = n = 0$, one obtains directly from the definition

$$\langle 0|:\exp(-N):|0\rangle = \langle 0|1|0\rangle = 1. \tag{B.3}$$

Thus

$$:\exp(-N): = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |m\rangle\langle m|:\exp(-N):|n\rangle\langle n| =$$

$$= |0\rangle\langle 0|. \tag{B.4}$$

Generalizing (B.4) to the case of a continuously infinite degrees of freedom, we get (B.1).

Now, using the notation given in (2.2), (2.3), we can write

$$\frac{\omega_1\ldots\omega_k}{\sigma}\frac{d^{3k}\sigma_{\text{excl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \frac{\langle\psi_s|a^\dagger(\vec{p}_k)\ldots a^\dagger(\vec{p}_1)|0\rangle\langle 0|a(\vec{p}_1)\ldots a(\vec{p}_k)|\psi_s\rangle}{\langle\psi_s|\psi_s\rangle} =$$

$$= \langle E_{\text{excl}}(\vec{p}_1, \ldots, \vec{p}_k)\rangle \tag{B.5}$$

with

$$E_{\text{excl}}(\vec{p}_1, \ldots, \vec{p}_k) = |a^\dagger(\vec{p}_k)\ldots a^\dagger(\vec{p}_1)|0\rangle\langle 0|a(\vec{p}_1)\ldots a(\vec{p}_k)|. \tag{B.6}$$

Here we apply (B.1) and get

$$E_{\text{excl}}(\vec{p}_1, \ldots, \vec{p}_k) = :N(\vec{p}_1)N(\vec{p}_2)\ldots N(\vec{p}_k)\exp\left\{-\int N(\vec{p})\frac{d^3p}{\omega}\right\}:. \tag{B.7}$$

Consequently

$$\langle E_{\text{excl}}(\vec{p}_1, \ldots, \vec{p}_k)\rangle = \omega_1\ldots\omega_k\left[\frac{\delta^k}{\delta h(\vec{p}_1)\ldots\delta h(\vec{p}_k)}\left\langle :\exp\left\{\int N(\vec{p})h(\vec{p})\frac{d^3p}{\omega}\right\}:\right\rangle\right]_{h(\vec{p})=-1} \tag{B.8}$$

(B.5) and (B.8) yield (2.11).

For the inclusive cross-sections, we have similarly

$$\frac{\omega_1\ldots\omega_k}{\sigma}\frac{d^{3k}\sigma_{\text{incl}}}{d^3\vec{p}_1\ldots d^3\vec{p}_k} = \langle E_{\text{incl}}(\vec{p}_1, \ldots, \vec{p}_k)\rangle \tag{B.9}$$

with

$$E_{\text{incl}}(\vec{p}_1, \ldots, \vec{p}_k) = \sum_X |a^\dagger(\vec{p}_k)\ldots a^\dagger(p_1)|X\rangle\langle X|a(\vec{p}_1)\ldots a(\vec{p}_k)| =$$

$$= |a^\dagger(\vec{p}_k)\ldots a^\dagger(p_1)a(p_1)\ldots a(p_k)| =$$

$$= :N(\vec{p}_1)\ldots N(\vec{p}_k):. \tag{B.10}$$

Consequently

$$\langle E_{\text{incl}}(\vec{p}_1, \ldots, \vec{p}_k)\rangle =$$

$$= \omega_1\ldots\omega_k\left[\frac{\delta^k}{\delta h(\vec{p}_1)\ldots\delta h(\vec{p}_k)}\left\langle :\exp\left\{\int N(\vec{p})h(\vec{p})\frac{d^3\vec{p}}{\omega}\right\}:\right\rangle\right]_{h(\vec{p})=0}. \tag{B.11}$$

(B.9) and (B.11) yield (2.12).

APPENDIX C

Correlations in semi-inclusive cross-sections

In a general semi-inclusive experiment one counts the number n of the particles C in the final state and for k out of them ($1 \leqslant k \leqslant n$) one measures the momenta $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$,

$$a+b \to \overbrace{\underbrace{C(\vec{p}_1)+\ldots+C(\vec{p}_k)}_{k}+C+\ldots+C}^{n}+(\text{anything not } C). \tag{C.1}$$

Let us denote this semi-inclusive cross-section (invariant form) by

$$\binom{n}{k}\omega_1\ldots\omega_k\,\frac{d^{3k}\sigma_n}{d^3\vec{p}_1\ldots d^3\vec{p}_k}=\sigma_n g^{(n,k)}(\vec{p}_1,\,\ldots,\,\vec{p}_k).\qquad\text{(C.2)}$$

The normalization condition

$$\frac{1}{k!}\int\cdots\int\frac{d^{3k}\sigma_n}{d^3\vec{p}_1\ldots d^3\vec{p}_k}\,d^3\vec{p}_1\ldots d^3\vec{p}_k=\sigma_n\qquad\text{(C.3)}$$

implies

$$\int\cdots\int g^{(n,k)}(\vec{p}_1,\,\ldots,\,\vec{p}_k)\,\frac{d^3\vec{p}_1\ldots d^3\vec{p}_k}{\omega_1\ldots\omega_k}=\frac{n!}{(n-k)!}\,.\qquad\text{(C.4)}$$

Differentiating GF (2.7–9) k times functionally and then putting $h(p)=h$, and multiplying by $(k!\,\omega_1\,\ldots\,\omega_k)$, we get from exclusive and inclusive expansions, respectively [28][42],

$$k!\omega_1\ldots\omega_k\,\frac{\delta^k F}{\delta h(\vec{p}_1)\ldots\delta h(\vec{p}_k)}=\sum_{n=k}^{\infty}(1+h)^{n-k}\,\frac{\sigma_n}{\sigma}\,g^{(n,k)}(\vec{p}_1,\,\ldots,\,\vec{p}_k)=\qquad\text{(C.5)}$$

$$=\sum_{n=k}^{\infty}\frac{h^{n-k}}{(n-k)!}\int\cdots\int f^{(n)}(\vec{p}_1,\,\ldots,\,\vec{p}_k,\,\vec{p}_{k+1},\,\ldots,\,\vec{p}_n)\,\frac{d^3\vec{p}_{k+1}\ldots d^3\vec{p}_n}{\omega_{k+1}\ldots\omega_n}\,.\qquad\text{(C.6)}$$

From (C.5) and (C.6), equating the coefficients of expansion in h,

$$\sum_{n=k}^{\infty}\frac{\sigma_n}{\sigma}\,g^{(n,k)}(\vec{p}_1,\,\ldots,\,\vec{p}_k)=f^{(k)}(\vec{p}_1,\,\ldots,\,\vec{p}_k),\qquad\text{(C.7)}$$

$$\sum_{n=k+l}(n-k)\,(n-k-1)\ldots(n-k-l+1)\,\frac{\sigma_n}{\sigma}\,g^{(n,k)}(\vec{p}_1,\,\ldots,\,\vec{p}_k)=$$

$$=\int\cdots\int f^{(k+l)}(\vec{p}_1,\,\ldots,\,\vec{p}_k,\,\vec{p}_{k+1}\ldots\vec{p}_{k+l})\,\frac{d^3\vec{p}_{k+1}\ldots d^3\vec{p}_{k+l}}{\omega_{k+1}\ldots\omega_{k+l}}\,.\qquad\text{(C.8)}$$

$$l=1,2,\ldots$$

[42] In Ref. [28] we have used the same symbol without the factor $\binom{n}{k}$ to represent (C.2). To be more precise, $\omega_1\,\ldots\,\omega_k\,\dfrac{d^{3k}\,\sigma_n}{d^3\vec{p}_1\,\ldots\,d^3\,\vec{p}_k}$ in Ref. [28] $=\binom{n}{k}\,\omega_1\,\ldots\,\omega_k\,\dfrac{d^{3k}\sigma_n}{d^3\vec{p}_1\,\ldots\,d^3\vec{p}_k}$ in (C.2) $=$

$$=\frac{\omega_1\,\ldots\,\omega_k}{(n-k)!}\int\cdots\int\frac{d^{3n}\,\sigma_{\text{excl}}}{d^3\vec{p}_1\,\ldots\,d^3\vec{p}_k\,d^3\vec{p}_{k+1}\,\ldots\,d^3\vec{p}_n}\,d^3\vec{p}_{k+1}\,\ldots\,d^3\vec{p}_n.$$

As is explicitly shown in (C.7), the inclusive cross-section on r.h.s. is subdivided into semi-inclusive cross-sections on l.h.s., each of which is labelled by the number n of particles C. Assuming that a single dynamical mechanism dominates and the correlations defined in terms of inclusive cross-sections have physical significance, it is proposed [35] to define "semi-inclusive correlation function $\pi_n^{(j)}(\vec{p}_1, ..., \vec{p}_j)$" in such a way that the following conditions are satisfied.

1) By adding "semi-inclusive correlation" with appropriate weight factors, one reproduces "inclusive correlation".

2) If all the correlation functions $\varrho^{(j)}$ except $\varrho^{(1)}$ vanish, then all the semi-inclusive correlations $\pi_n^{(j)}$ except $\pi_n^{(1)}$ vanish[43].

Such semi-inclusive correlations $\pi_n^{(j)}$ can be introduced by

$$g^{(n,1)}(\vec{p}_1) = \pi_n^{(1)}(\vec{p}_1), \tag{C.9}$$

$$g^{(n,2)}(\vec{p}_1, \vec{p}_2) = \frac{n(n-1)}{\langle n(n-1)\rangle} f^{(1)}(\vec{p}_1) f^{(1)}(\vec{p}_2) + \pi_n^{(2)}(\vec{p}_1, \vec{p}_2), \tag{C.10}$$

$$g^{(n,3)}(\vec{p}_1, \vec{p}_2, p_3) = \frac{n(n-1)(n-2)}{\langle n(n-1)(n-2)\rangle} \{f^{(1)}(\vec{p}_1) f^{(1)}(\vec{p}_2) f^{(1)}(\vec{p}_3) + \\ + \sum_{\text{perm}} f^{(1)}(\vec{p}_1) \varrho^{(2)}(\vec{p}_2, \vec{p}_3)\} + \pi_n^{(3)}(\vec{p}_1, \vec{p}_2, \vec{p}_3), \tag{C.11}$$

in general

$$g^{(n,j)}(\vec{p}_1, ..., \vec{p}_j) = \frac{n(n-1)...(n-j+1)}{\langle n(n-1)...(n-j+1)\rangle} \{f^{(j)}(\vec{p}_1, ..., \vec{p}_j) - \varrho^{(j)}(\vec{p}_1, ..., \vec{p}_j)\} + \\ + \pi_n^{(j)}(\vec{p}_1, ..., \vec{p}_j). \tag{C.12}$$

It follows from (C.7), (C.12) that the condition (C.1) is fulfilled,

$$\sum_n \frac{\sigma_n}{\sigma} \pi_n^{(j)}(\vec{p}_1, ..., \vec{p}_j) = \varrho^{(j)}(\vec{p}_1, ..., \vec{p}_j). \tag{C.13}$$

Notice that in the definition (C.12), the factor $\{f^{(j)} - \varrho^{(j)}\}$ can be expressed in terms of $\varrho^{(1)}, \varrho^{(2)}, ..., \varrho^{(j-1)}$. Consequently the set of definitions (C.9), (C.10), (C.11) ... can be used with the help of (C.13) to define $\pi_n^{(1)}, \pi_n^{(2)}, \pi_n^{(3)}, ...$ in succession.

The condition 2 is also satisfied. This can be seen from the fact that

$$g^{(n,j)}(\vec{p}_1, ..., \vec{p}_j) = \frac{n(n-1)...(n-j+1)}{\langle n(n-1)...(n-j+1)\rangle} f^{(j)}(\vec{p}_1, ..., \vec{p}_j) \tag{C.14}$$

if $\varrho^{(j)} = 0$ for all $j \geqslant 2$.

---

[43] The inverse of this statement follows trivially from the condition 1.

The conditions 1 and 2, however, do not determine the semi-inclusive correlations uniquely. If we define, for instance,

$$\tau_n^{(j)}(\vec{p}_1, \dots, \vec{p}_j) = \pi_n^{(j)}(\vec{p}_1, \dots, \vec{p}_j) - \frac{n(n-1)\dots(n-j+1)}{\langle n(n-1)\dots(n-j+1)\rangle} \varrho^{(j)}(\vec{p}_1, \dots, \vec{p}_j), \tag{C.15}$$

which in general do not vanish, one can show immediately that

$$\sum \sigma_n \tau_n^{(j)}(\vec{p}_1, \dots, \vec{p}_j) = 0. \tag{C.16}$$

Thus for a given $\pi_n^{(j)}$, another set

$$\pi_n^{(j)} + a\tau_n^{(j)}$$

also meets the two conditions.

APPENDIX D

Mathematical model of Quigg, Wang and Yang

In the high energy limit one can consider right-going particles ($x > 0$) and left-going particles ($x < 0$) separately, because in the LFH they come from fragmentation of the projectile and the target hadrons, respectively, and are expected not to affect each other. The requirement of overall energy-momentum conservation is also expressed in a factorized form

$$\delta(1 - \sum_{x_j > 0} x_j)\delta(1 + \sum_{x_k < 0} x_k). \tag{D.1}$$

For the right-going particles, for example, this model assumes the following form of exclusive differential cross-sections[44]

$$\frac{d\sigma_{\text{excl}}^{(R)}}{dx} = 0, \tag{D.2}$$

$$\frac{d^n \sigma_{\text{excl}}^{(R)}}{dx_1 \dots dx_n} = K(n-1)!(n-2)!\delta(1 - \sum_{j=1}^{n} x_j), \tag{D.3}$$

$$\text{for } n \geqslant 2,$$

where K is a constant, $0 < x_j < 1$.

---

[44] The suffix (R) is to remind that the quantity concerns only the right-going particles.

This leads to

$$\sigma_n^{(R)} = \frac{1}{n!}\int_0^1 \ldots \int_0^1 dx_1 \ldots dx_n \frac{d^n\sigma_{\text{excl}}^{(R)}}{dx_1\ldots dx_n} = \frac{K}{n(n-1)}\,, \tag{D.4}$$

$$\sigma_{\text{tot}}^{(R)} = \sum_{n=2}^{N^{(R)}} \sigma_n^{(R)} = K\left(1-\frac{1}{N^{(R)}}\right) \approx K, \tag{D.5}$$

with

$$N^{(R)} = \frac{\sqrt{s}}{2m}\,, \tag{D.6}$$

$$\langle n^{(R)}\rangle = \sum_{n=2}^{N^{(R)}} n\sigma_n^{(R)} / \sum_{n=2}^{N^{(R)}} \sigma_n \sim \ln s, \tag{D.7}$$

$$\langle (n^{(R)})^2\rangle = \sum_{n=2}^{N^{(R)}} n^2\sigma_n^{(R)} / \sum_{n=2}^{N^{(R)}} \sigma_n \sim s^{1/2}. \tag{D.8}$$

The asymptotic behaviour (D.5), (D.7), (D.8) is known to characterize the DEM.

From (D.2), (D.3) one can easily calculate the inclusive cross-sections

$$\frac{d\sigma_{\text{incl}}}{dx_1} = \sum_{l=1}^{\infty} \frac{1}{l!}\int \ldots \int \frac{d^{l+1}\sigma_{\text{excl}}}{dx_1 dx_2 \ldots dx_{l+1}}\, dx_2 \ldots dx_{l+1} =$$

$$= K^2 \sum_{l=1}^{\infty} (1-|x_1|)^{l-1} \approx \frac{K^2}{|x_1|}\,. \tag{D.9}$$

$$\frac{d^2\sigma_{\text{incl}}}{dx_1 dx_2} = \sum_{l=0}^{\infty} \frac{1}{l!}\int \ldots \int \frac{d^{l+2}\sigma_{\text{excl}}}{dx_1 dx_2 \ldots dx_{l+2}}\, dx_3 \ldots dx_{l+2} =$$

$$= K^2\delta(1-|x_1|-|x_2|) + K^2 \sum_{l=1}^{\infty} l(l+1)\,(1-|x_1|-|x_2|)^{l-1} =$$

$$= K^2\delta(1-|x_1|-|x_2|) + 2K^2/|x_1+x_2|^3 \quad \text{if} \quad x_1x_2 > 0,\ |x_1+x_2| \leqslant 1. \tag{D.10}$$

$$\frac{d^2\sigma_{\text{incl}}}{dx_1 dx_2} = \frac{K^2}{|x_1 x_2|} \quad \text{if} \quad x_1 x_2 < 0. \tag{D.11}$$

In our definition of distribution and correlation functions (normalization being taken as $\sigma_{tot} = K^2$), these give

$$f^{(1)}(x) = \varrho^{(1)}(x) \approx 1,$$

$$f^{(2)}(x_1, x_2) = \begin{cases} 0 \text{ if } x_1x_2 > 0,\ |x_1+x_2| > 1, \\ \bar{x}_1\bar{x}_2\delta(1-|x_1|-|x_2|)+\dfrac{2\bar{x}_1\bar{x}_2}{|x_1+x_2|^3} \\ \quad \text{if } x_1x_2 > 0,\ |x_1+x_2| \leqslant 1, \\ 1 \text{ if } x_1x_2 < 0, \end{cases} \tag{D.12}$$

and consequently

$$\varrho^{(2)}(x_1, x_2) = \begin{cases} -1 \text{ if } x_1x_2 > 0,\ |x_1+x_2| > 1, \\ \bar{x}_1\bar{x}_2\delta(1-|x_1|-|x_2|)+\dfrac{2\bar{x}_1\bar{x}_2}{|x_1+x_2|^3}-1 \\ \quad \text{if } x_1x_2 > 0,\ |x_1+x_2| \leqslant 1, \\ 0 \text{ if } x_1x_2 < 0. \end{cases} \tag{D.13}$$

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