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EXPONENTIAL INTERACTIONS AT HIGH ENERGY

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It is shown that the exponential interaction exp z^2 leads to Regge-type high-energy elastic scattering amplitudes when summations are carried out in the major coupling constant. A more complicated entire interaction of the same order yields similar results.

1. Introduction

At present, there are many indications [1]–[4] that the non-polynomial interactions can solve the divergence problems in particle physics. Likewise, it is essential to provide asymptotically decreasing or constant amplitudes and this is the task of the paper. Although there exist general estimates [5] showing that rather complicated non-polynomial amplitudes possess reasonable high-energy bounds, it is necessary to discuss the details of the asymptotic behaviour in explicitly solvable models.

In this paper we calculate the high-energy sum of the ladder amplitudes for two entire interactions of the order equal to two. In accordance with the expectation, we get Regge-pole structures for the leading asymptotic behaviour.

Section 2. contains some details of calculating the ladders and the summation over the major coupling constant for a general non-polynomial interaction. These ideas are applied to two models in Section 3, where we also discuss contribution of the cut.

2. Ladders at high-energy

To prepare the summation of the ladders coming from exponential interactions, let us consider a general non-polynomial interaction

$$\mathscr{L}_1(\hat{\kappa}\varphi) = \hat{g} \sum_{k=0}^{\infty} f(\hat{\kappa}) : (\hat{\kappa}\varphi)^k : \tag{1}$$

and show the method of calculating the ladders with superlines as rungs at high-energy. $\hat{g}(\hat{\kappa})$ is called the major (minor) coupling constant, φ is a real scalar field.

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First, it is easier to calculate the ladders with multiple loops coming from the interaction $g : \varphi^{k+2}(x)$. Thus, consider the diagram drawn in Fig. 1. This case gives the essential information necessary for (1).

To make the contribution of Fig. 1 finite, we introduce a Pauli-Villars regularization

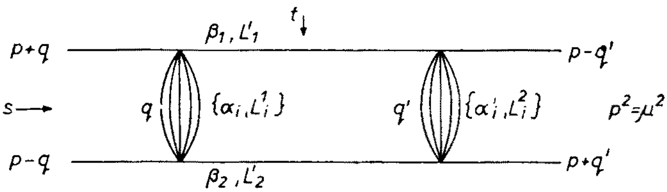

Fig. 1. Ladder with two rungs

into the internal meson lines and write

$$\Delta_F(p) = -i \int_0^{\lambda^2} dL(p^2-\mu^2-L+i\varepsilon)^{-2}, \quad \lambda^2 = M^2-\mu^2. \tag{2}$$

$M > \mu$ is a large regulator mass. By making use of the identity

$$(\prod_{i=1}^{k} a_i^2)^{-1} = (2k-1)! \int_0^1 \prod_{i=1}^{k} (d\alpha_i \alpha_i)\delta(1-\sum_{i=1}^{k}\alpha_i)(\sum_{i=1}^{k} a_i\alpha_i)^{-2k}, \tag{3}$$

we can integrate over the internal momenta [6] and get for the contribution of Fig. 1,

$$\mathcal{M}_4 = 2(-g^2\varrho)^2 5![(2\pi)^4(4\pi)^{4k}]^{-1} \int_0^{\lambda^2} \{dLdL'\} \int_0^1 \{\alpha d\alpha\}\{\beta d\beta\}\{\alpha' d\alpha'\} \times$$
$$\times \delta(1-\sum_i \alpha_i - \sum_i \alpha'_i - \beta_1 - \beta_2) C(\alpha, \alpha', \beta)^4 \times$$
$$\times [t \prod_i \alpha_i \prod_i \alpha'_i + d(\alpha, \alpha', \beta, L, L', s)]^{-6} \tag{4}$$

with

$$\varrho = (k+2)!^2 k!^{-1}, \tag{5}$$

$$C(\alpha, \alpha', \beta) = C_k(\alpha') \prod_i \alpha_i + C_k(\alpha) \prod_i \alpha'_i + (\beta_1+\beta_2) C_k(\alpha) C_k(\alpha')$$

and

$$d(\alpha, \alpha', \beta, L, L', s) = \beta_1\beta_2 C_k(\alpha)C_k(\alpha') + \mu^2(\beta_1+\beta_2)(C_k(\alpha')\prod_i \alpha_i + C_k(\alpha)\prod_i \alpha'_i) -$$
$$-(\mu^2 + \sum_i \alpha_i L^1_i + \sum_i \alpha'_i L^2_i + \beta_1 L'_1 + \beta_2 L'_2 - i\varepsilon) C(\alpha, \alpha', \beta), \tag{6}$$

where

$$C_k(\alpha) = \sum_{l=1}^{k} \alpha_l^{-1} \prod_{i=1}^{k} \alpha_i. \tag{7}$$

(4) has not yet a convenient from to discuss the high-energy behaviour. Therefore we introduce new variables into (4)

$$\gamma_1 = \sum_{i=1}^{k} \alpha_i, \quad \gamma_2 = \sum_{i=1}^{k} \alpha'_i, \quad \alpha_i = \gamma_1 \xi_i, \quad \alpha'_i = \gamma_2 \xi'_i. \tag{8}$$

In the second step we get rid of the δ-functions by introducing the scaling variables

$$\xi_1 = 1-\varrho_1, \quad \xi_j = (1-\varrho_j)\varrho_1 \dots \varrho_{j-1}, \quad \xi_k = \varrho_1 \dots \varrho_{k-1}; \quad j = 2, \dots, k-1 \tag{9}$$

and similarly $\xi'_i \to \varrho'_i$. In such a way, it is easy to find

$$\mathcal{M}_4 = 2(-g^2\varrho)^2 5![(2\pi)^4(4\pi)^{4k}]^{-1} \int_0^{\lambda^2} \{dLdL'\} \int_0^1 \{\gamma d\gamma\} \times$$

$$\times \{\varrho(1-\varrho)d\varrho\} \{\varrho'(1-\varrho')d\varrho'\} \{\beta d\beta\} \delta(1-\gamma_1-\gamma_2-\beta_1-\beta_2) \times$$

$$\tilde{C}(\gamma, \varrho, \varrho', \beta)^4 [t\gamma_1\gamma_2 \prod_{i=1}^{k-1} \varrho_i(1-\varrho_i)\varrho'_i(1-\varrho'_i) + \tilde{d}(\gamma, \beta, \varrho, \varrho', L, L', s)]^{-6}, \tag{10}$$

where

$$\frac{C(\alpha, \alpha', \beta)}{\tilde{C}(\gamma, \varrho, \varrho', \beta)} = (\gamma_1\gamma_2)^{k-1} \prod_{i=1}^{k-1} (\varrho_i\varrho'_i)^{k-i-1} \frac{d(\alpha, \alpha', \beta, L, L', s)}{\tilde{d}(\gamma, \beta, \varrho, \varrho', L, L', s)}. \tag{11}$$

Clearly the leading term of $\mathcal{M}_4$ for $t \to \infty$ is determined by the end-point contributions of (10). Discussing the mixtures of the various end-point contributions by well-known methods [6], one is led to the high-energy amplitude

$$\mathcal{M}_4 = (g^2\varrho)^2 2^{2k-1}(2\pi)^{-4} \left(\frac{M^2-\mu^2}{(4\pi)^2}\right)^{2k} t^{-2} \frac{\left(\ln \frac{t}{\mu^2}\right)^{2k-1}}{(2k-1)!} K(s, M^2) \tag{12}$$

and

$$K(s, M^2) = 3! \int_0^{\lambda^2} dL'_1 dL'_2 \int_0^1 \frac{d\beta_1 d\beta_2 \beta_1 \beta_2 \delta(1-\beta_1-\beta_2)}{[\beta_1\beta_2 s - \mu^2 - \beta_1 L'_1 - \beta_3 L'_2 + i\varepsilon]^4}. \tag{12'}$$

$K(s, M^2)$ represents the contribution of the contracted diagram belonging to Fig. 1; it describes the slope of the Regge-trajectory [6] in summing ladders. $K(s, M^2)$ is bounded for $M \to \infty$ and $s \to \infty$, respectively.

From (12) one can obtain the contribution of the g^2-order ladder by neglecting the factor $g^2\varrho \cdot 2^{k-1} K(s, M^2) \cdot \left(\frac{M^2-\mu^2}{(4\pi)^2}\right)^k$, because by short — circuiting in Fig. 1 the two lines running through, we get one rung with $2k-1$ single lines. Furthermore, from the same reasoning and from the factorization of the reduced graph (Fig. 2) it follows for the high-

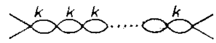

Fig. 2. Contracted graph

-energy contribution of the ladder with m rungs

$$\mathcal{M}_{2m} = \left(\frac{g^2\varrho}{2}\right)^m \frac{2}{(2\pi)^4}\left(\frac{2(M^2-\mu^2)}{(4\pi)^2}\right)^{mk} K(s, M^2)^{m-1} t^{-2} \frac{\left(\ln \frac{t}{\mu^2}\right)^{mk-1}}{(mk-1)!} \,. \tag{13}$$

The summation of (13) in m is described in [7].

Returning to our original task, we first remark that the high-energy behaviour of an m-runged ladder having k_1 single lines in the i'th rung ($i = 1, \ldots, m$) is quite similar to (13). Consequently, the summed ladders with superlines as rungs give the following contribution at $t \to \infty$

$$\mathcal{M} = \sum_m \mathcal{M}_{2m} = 2\left[(2\pi)^4 t^2 \ln \frac{t}{\mu^2} K(s, M^2)\right]^{-1} \sum_{m=1}^{\infty} y^m \times$$

$$\sum_{k_1 \ldots k_m = 1}^{\infty} \left[\left(\sum_{i=1}^{m} k_i - 1\right)!\right]^{-1} \prod_{i=1}^{m} \left(f(k_i+2)^2 (k_i+2)!^2 (k_i!)^{-1} x^{k_i}\right) \equiv$$

$$\equiv (\kappa\mu)^2 [(2\pi)^6 t^2 K(s, M^2)]^{-1} \Psi(x, y). \tag{14}$$

Here

$$x = 2\left(\frac{\kappa\mu}{4\pi}\right)^2 \ln \frac{t}{\mu^2}, \qquad y = \tfrac{1}{2} g^2 \kappa^4 K(s, M^2)$$

$$\kappa^2 = \left(\frac{M^2}{\mu^2} - 1\right)\hat{\kappa}^2, \qquad g^2\hat{\kappa}^4 = \hat{g}^2\hat{\kappa}^4. \tag{15}$$

κ and g are kept finite. As it is seen from (14) the amplitude $\Psi(x, y)$ satisfies a Volterra-type integral equation,

$$\Psi(x, y) = \frac{y}{x}\, \Phi(x) + y \int_0^x \frac{\Phi(x-x')}{x-x'}\, \Psi(x', y)\, dx'. \tag{16}$$

Its kernel generates the g^2-order ladder

$$\mathcal{M}_2 = g^2 \kappa^4 \left[(2\pi)^4 t^2 \ln \frac{t}{\mu^2}\right]^{-1} \Phi(x),$$

$$\Phi(x) = \sum_{k=1}^{\infty} \frac{f(k+2)^2 (k+2)!^2}{k!(k-1)!}\, x^k. \tag{17}$$

In [5], starting from (14) several estimates leading to Regge-type behaviours were described.

3. *Exponential-type interactions*

From the results (14)-(17) it is apparent that for entire interactions there exists the amplitude $\mathcal{M}$. Let us, therefore, consider interactions represented by simple exponential functions. As it is known [5] exponentials of order $\geqslant 2$ generate non-local (local) interactions.

In what follows we consider a less trivial class of Lagrangians [8, 9] than the linear exponentials,

$$\mathcal{L}_1(z) = \hat{g}(e^{\pm z^2} - 1). \tag{18}$$

The relevance of (18) lies in the fact [8] that equal-time current commutators require second-order exponentials in a vector field.

To solve (16), we take its Laplace transform, then

$$\omega(p, y) = y\varphi(p)(1 - y\varphi(p))^{-1}, \tag{19}$$

where $\varphi(p)$ denotes the Laplace transform of $\dfrac{\Phi(x)}{x}$ and $\omega(p, y)$ determines $\Psi(x, y)$ according to the inversion formula

$$\Psi(x, y) = (2\pi i)^{-1} \int_{-i\infty+\sigma}^{i\infty+\sigma} \omega(p, y) e^{px} dp, \qquad \sigma > 0. \tag{20}$$

From (17) and (18)

$$\frac{\Phi(x)}{x} = 8 \sum_{k=0}^{\infty} \frac{(2k+3)^2}{k!(k+1)!} x^{2k+1} \tag{21}$$

and

$$\varphi(p) = 8[p^5 + 8p^3 - (p^2 - 4)^{5/2}](p^2 - 4)^{-5/2}. \tag{22}$$

Consequently $\omega(\mathrm{p}, y)$ has poles of finite number, whence $\Psi(x, y)$ gets contributions of the type exp. $(p_k x) \sim t^{\alpha(s)}$, that is a Regge-pole behaviour arises. There is, however, a cut--contribution, too. This is due to the square roots of (22) and it amounts to the integral

$$\int_{-2}^{2} \beta(p, y) \left(\frac{t}{\mu^2}\right)^{2(4\pi)^{-2}\kappa^2\mu^2 p} dp \tag{23}$$

where $\beta(p, y)$ denotes the discontinuity on the cut

$$\beta(p, y) = 8yp^3(p^2+8)(p^2-4)^2[(1+8y)^2(p^2-4)^5 - (8yp^3(p^2+8))^2]^{-1}. \tag{24}$$

It follows that the interaction (18) produces a fixed cut in the complex angular momentum plane between $-\left(\dfrac{\kappa\mu}{4\pi}\right)^2$ and $\left(\dfrac{\kappa\mu}{4\pi}\right)^2$.

Summarizing, from the point of view of asymptotic properties the non-polynomial interaction (18) is perfectly sensible.

Now, we want to show that there exist entire interactions of the same order without cuts. Namely, consider the Lagrangian

$$\mathscr{L}_1(z) = \hat{g} \sum_{k=1}^{\infty} (k \cdot k!)^{-\frac{1}{2}} (\pm z)^k, \qquad z = \hat{\kappa}\varphi. \tag{25}$$

One can easily find

$$x^{-1}\Phi(x) = x^2(x+2) \exp x \tag{26}$$

and

$$\varphi(p) = (1-2p)(p-1)^{-2}. \tag{27}$$

Substituting into (19) and (20) we have for the complete ladder amplitude $\mathscr{M}$ at $t \to \infty$

$$\mathscr{M} = \frac{g^2\kappa^6}{2(2\pi)^6\mu^2}\left[\beta_1(s)\left(\frac{t}{\mu^2}\right)^{2\alpha_1(s)} - \beta_2(s)\left(\frac{t}{\mu^2}\right)^{2\alpha_2(s)}\right] \tag{28}$$

with

$$\beta_{1,2}(s) = [1+2(y-1) \mp 2\sqrt{y(y-1)}]\,[4y(y-1)]^{-\frac{1}{2}},$$

$$\alpha_{1,2}(s) = \left(\frac{\kappa\mu}{4\pi}\right)^2 [1-y \pm (y(y-1))^{\frac{1}{2}}] - 1. \tag{29}$$

4. Discussion

In the present paper we have presented a high-energy summation of regularized ladders defined by non-polynomial interactions. Non-polynomial interactions have been considered as limits of power-type interactions. A similar construction of rational Lagrangians was investigated in defining the local commutativity for nonlocalizable fields [10] with the result that many essential properties can be preserved from those od the localizable fields.

For entire interactions there exists the complete amplitude (14). We have investigated the class (18), (25) where the nonlocalizable feature starts to come in and found a completely reasonable power-behaviour at high-energy. We know from [5] that this is no longer true for higher-order entire Lagrangians.

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