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NEW CLASS OF THE QUANTUM MARKOVIAN PROCESSES AND PARTIAL WAVE EXPANSION OF TRANSITION AMPLITUDE

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New infinite family of the quantum Markovian processes of diffusional type is found. These processes appear naturally when passing from the Cartesian coordinate system to the polar one in the path integral. They permit us to write closed formulae for partial transition amplitudes and for various physical characteristics of the central potential scattering.

1. Introduction

It is essential, for an extension of the range of applications of the Feynman formulation of the quantum mechanics [1], to develop the techniques of handling the path integrals. In particular, for various concrete physical problems it is convenient to use coordinate systems other than Cartesian and the polar system on the first place. Therefore, it became interesting to know how to translate a given path integral written in the Cartesian coordinate system into a path integral expressed in the polar coordinates. This problem has been discussed in literature [2], [3] on the level of the Feynman prescription of evaluation of path integrals which involves the known limiting procedure.

We are going to present here a corresponding discussion of the problem in terms of so-called quantum stochastic processes which provide a closed form of path integral independent of the limiting procedure [4].

Our main task here will consist of revealing some new infinite family of the quantum stochastic processes which are present in the polar formulation of a path integral and which will permit us to write the appearing path integrals in closed form, independent of the Feynman limiting procedure. We obtain a sort of partial wave expansion for path integrals and corresponding quantum stochastic processes. Possible applications of path integrals in the potential scattering theory was demonstrated by Gelman and Spruch [5].

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2. *New family of quantum stochastic processes*

It was shown by Peak and Inomata [3] that the fundamental solution the Schrödinger equation

$$\left[-\partial_t+\frac{i\hbar}{2m}\Delta_y+c(y)\right](t;y,x)=0 \qquad x,y\in\boldsymbol{R}^3 \tag{2.1}$$

$$\lim_{t\downarrow 0}(t;y,x)=\delta(y-x)$$

with a central potential $V(y) = V(|y|) = i\hbar c(y)$ may be written in the following limiting form

$$(t;y,x)=\frac{1}{4\pi|y|\cdot|x|}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\widehat{yx})\times$$

$$\times\lim_{n\to\infty}\int_0^\infty dr_1\ldots\int_0^\infty dr_{n-1}\prod_{k=0}^{n-1}(\Delta\tau_k;r_k,r_{k+1})_0^{(l)}\exp c(r_k)\Delta\tau_k, \tag{2.2}$$

where $r_0 = |y|$, $P_l(\cos\widehat{xy})$ is the Legendre function and

$$(\tau;\eta,\zeta)_0^{(l)}=\frac{\sqrt{\eta\xi}}{\lambda\tau}I_{l+\frac{1}{2}}\left(\frac{\eta\zeta}{\lambda\tau}\right)\exp\left(-\frac{\eta^2+\zeta^2}{2\lambda\tau}\right) \tag{2.3}$$

$\lambda=\dfrac{i\hbar}{m}$ and $I_\nu(z)$, the modified Bessel function. This result follows when passing from Cartesian to polar coordinates and performing integrations over angles in the Feynman path integral representing the fundamental solution of the equation (2.1).

Using our definition of the Feynman path integral in terms of the quantum stochastic processes we may write

$$(t;y,x)=Q_y\{\delta[z(t)-x]\exp\int_0^t c[z(\tau)]d\tau\} \tag{2.4}$$

where $z(t)$ is the quantum Brownian motion in the Euclidean space $\boldsymbol{R}^3$ and Q_y is a complex-valued quantum average operation defined previously [4]. Notice that this definition of the Feynman path integral does not depend on limiting procedure like that shown in the formula (2.2). In fact we shall see that it is possible to rewrite the formula (2.2) in a closed form using some quantum Markovian processes. Namely, we shall verify it explicitly, that the functions $(\tau;\eta,\zeta)_0^{(l)}$ yield the transition probability amplitudes for such processes.

To this end one should check the following axioms [6]

(*i*) $\qquad (\tau;\eta,\zeta)^*=(-\tau;\zeta,\eta)$

(*ii*) $\qquad \lim_{\tau\downarrow 0}(\tau;\eta,\zeta)=\delta(\eta-\zeta)$

(*iii*) $$\int_0^\infty d\xi(\tau;\eta,\xi)\,(\tau;\zeta,\xi)^* = \delta(\eta-\zeta)$$

(*iv*) $$\int_0^\infty d\xi(\sigma;\eta,\xi)\,(\tau;\xi,\zeta) = (\sigma+\tau;\eta,\zeta)$$

(*v*) $(\tau;\eta,\zeta)$ is continuous function in η,ζ.

Moreover, we shall prove the existence of the following limits

A $$a(\eta) = \lim_{\tau\downarrow 0}\tau^{-1}\int_0^\infty d\zeta(\tau;\eta,\zeta)\,(\zeta-\eta)$$

B $$b(\eta) = \lim_{\tau\downarrow 0}\tau^{-1}\int_0^\infty d\zeta(\tau;\eta,\zeta)\,(\zeta-\eta)^2$$

C $$c(\eta) = \lim_{\tau\downarrow 0}\tau^{-1}\Big[\int_0^\infty d\zeta(\tau;\eta,\zeta)-1\Big]$$

D $$\lim_{\tau\downarrow 0}\tau^{-1}\int_0^\infty d\zeta(\tau;\eta,\zeta)\,(\zeta-\eta)^n = 0, \qquad n\geqslant 3.$$

We omit hereafter the indices of the amplitude for simplicity of the notations.

The first axiom is satisfied due to the reality of the modified Bessel function and imaginarity of the parameter λ.

The second axiom may be easily verified if one takes into account an asymptotic expansion for the modified Bessel function [7]

$$I_\nu\left(\frac{z}{\varepsilon}\right) \simeq \left(\frac{\varepsilon}{2\pi z}\right)^{\frac{1}{2}} \exp\left[\frac{z}{\varepsilon} - \tfrac{1}{2}(\nu^2-\tfrac{1}{4})\frac{\varepsilon}{z} + O(\varepsilon^2)\right] \tag{2.5}$$

for $\varepsilon\downarrow 0$.

Then, according to the known property of the free motion transition amplitude we have

$$\lim_{\tau\downarrow 0}(\tau;\eta,\zeta) = \lim_{\tau\downarrow 0}(2\pi\lambda\tau)^{-\frac{1}{2}}\exp\left[-\frac{(\eta-\zeta)^2}{2\lambda\tau}\right] = \delta(\eta-\zeta). \tag{2.6}$$

In order to verify the unitarity condition (*iii*) we notice first that

$$\int_0^\infty d\xi(\tau;\eta,\xi)\,(\tau;\zeta,\xi)^* = \frac{\sqrt{\eta\zeta}}{|\lambda|^2\tau^2}\exp\frac{\zeta^2-\eta^2}{2\lambda\tau}\int_0^\infty d\xi\xi J_{l+\frac{1}{2}}\left(\frac{\eta\xi}{|\lambda|\tau}\right)J_{l+\frac{1}{2}}\left(\frac{\xi\zeta}{|\lambda|\tau}\right) \tag{2.7}$$

because of the following relations being satisfied by the modified Bessel functions

$$I_\nu^*(z) = I_\nu(z^*), \qquad I_\nu(-z) = (-1)^\nu I_\nu(z), \qquad I_\nu(ia) = (-1)^{-\frac{\nu}{2}}J_\nu(-a). \tag{2.8}$$

The last integral may be expressed as a limit of some known integral [8]

$$\int_0^\infty J_\nu(\alpha x)J_\nu(\beta x)x dx = \lim_{\varrho\downarrow 0}\int_0^\infty e^{-\varrho^2x^2}J_\nu(\alpha x)J_\nu(\beta x)x dx =$$

$$= \lim_{\varrho\downarrow 0}\frac{1}{2\varrho^2}I_\nu\left(\frac{\alpha\cdot\beta}{2\varrho^2}\right)\exp\left(-\frac{\alpha^2+\beta^2}{4\varrho^2}\right) \quad \text{for } \alpha, \beta > 0. \tag{2.9}$$

Taking into account the asymptotic expansion (2.5) and the formula (2.6) we obtain at $\alpha = \frac{\eta}{|\lambda|\tau}$, $\beta = \frac{\zeta}{|\lambda|\tau}$ the unitarity relation (*iii*).

The causality condition (*iv*) may be verified using an analogous formula which follows from the previous one by analytic continuation [7]

$$\int_0^\infty d\xi\xi e^{i\alpha\xi^2}I_\nu\left(\frac{a\xi}{i}\right)I_\nu\left(\frac{b\xi}{i}\right) = \frac{i}{2\alpha}I_\nu\left(\frac{a\cdot b}{2i\alpha}\right)\exp\left(\frac{a^2+b^2}{4i\alpha}\right) \tag{2.10}$$

valid for $\operatorname{Re}\nu > -1$, $\operatorname{Re}\alpha > 0$.

Indeed, using it one gets the equalities

$$\int_0^\infty d\xi(\sigma;\eta,\xi)(\tau;\xi,\zeta) = \frac{\sqrt{\eta\zeta}}{\lambda^2\sigma\tau}\exp\left(-\frac{\eta^2}{2\lambda\sigma}-\frac{\zeta^2}{2\lambda\tau}\right)$$

$$\int_0^\infty d\xi\xi I_{l+\frac{1}{2}}\left(\frac{\eta\xi}{\lambda\sigma}\right)I_{l+\frac{1}{2}}\left(\frac{\xi\zeta}{\lambda\tau}\right)\exp\left[-\frac{\xi^2}{m\lambda}\left(\frac{1}{\sigma}+\frac{1}{\tau}\right)\right] =$$

$$= \frac{\sqrt{\eta\zeta}}{\lambda(\sigma+\tau)}I_{l+\frac{1}{2}}\left[\frac{\eta\zeta}{\lambda(\sigma+\tau)}\right]\exp\left[-\frac{\eta^2+\zeta^2}{2\lambda(\sigma+\tau)}\right] = (\sigma+\tau;\eta,\zeta). \tag{2.11}$$

The validity of the fifth postulate is evident, hence all the axioms are fulfilled. The remaining task is to calculate the coefficient functions $a(\eta)$, $b(\eta)$, $c(\eta)$ and to check the condition *D*. In all these cases we use the asymptotic expansion (2.5) and introduce a proper regularization of the integrals involved. For instance, for the coefficient $a(\eta)$ defined in A we find

$$a(\eta) = (2\pi\lambda)^{-\frac{1}{2}}\lim_{\tau\downarrow 0}\tau^{-\frac{3}{2}}\int_{-\eta}^\infty d\zeta\zeta\exp\left(-\frac{\zeta^2}{2\lambda\tau}\right) =$$

$$= (2\pi\lambda)^{-\frac{1}{2}}\lim_{\tau\downarrow 0}\tau^{\frac{1}{2}}\lambda\left[\exp\left(-\frac{\eta^2}{2\lambda\tau}\right)-\exp\left(-\frac{\infty}{2\lambda\tau}\right)\right]. \tag{2.12}$$

We omit the last term on the basis of regularization consisting of introducing a cutting term $-\varepsilon\zeta^2$ under the exponent and taking the limit $\varepsilon\downarrow 0$ after the limit $\tau\downarrow 0$. Similarly for the coefficient function $b(\eta)$ we get after using the asymptotic expansion (2.5), shifting the variable of integration and integrating by parts

$$b(\eta) = (2\pi\lambda)^{-\frac{1}{2}} \lim_{\tau\downarrow 0} \tau^{-\frac{1}{2}}\lambda\left[\int_{\eta}^{\infty} d\zeta \exp\left(-\frac{\zeta^2}{2\lambda\tau}\right) - \eta \exp\left(-\frac{\eta^2}{2\lambda\tau}\right)\right]. \tag{2.13}$$

The second term vanishes as the result of regularization which is tacitly assumed. The first term yields the Fresnel integral in the limit and we obtain finally

$$b(\eta) = \lambda. \tag{2.14}$$

In order to calculate the coefficient $c(\eta)$ it is worth finding first an asymptotic expansion of an integral of the transition amplitude. Using once again the asymptotic expansion (2.5) we get at small τ

$$\int_0^{\infty} d\zeta(\tau;\eta,\zeta) \simeq (2\pi\lambda\tau)^{-\frac{1}{2}} \int_0^{\infty} d\zeta \exp\left[-\frac{\lambda l(l+1)\tau}{2\eta\zeta} - \frac{(\zeta-\eta)^2}{2\lambda\tau}\right]. \tag{2.15}$$

Furthermore, using the stationary phase method, we find

$$\int_0^{\infty} d\zeta(\tau;\eta,\zeta) \simeq (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{\lambda l(l+1)\tau}{2\eta^2}\right] \int_{-\eta(2|\lambda|\tau)^{-1/2}}^{\infty} d\zeta \exp i\zeta^2 \simeq$$

$$\simeq 1 - \frac{\lambda l(l+1)\tau}{2\eta^2} + O(\tau^2). \tag{2.16}$$

From this it follows immediately that the coefficient function $c(\eta)$ is equal to

$$c(\eta) = -\frac{\lambda l(l+1)}{2\eta^2}. \tag{2.17}$$

The higher order moments of the transition amplitude may be estimated without difficulty if we introduce a proper regularization for the integrals

$$d_n(\tau) = \int_{-\eta}^{\infty} d\zeta\zeta^n \exp\left(-\frac{\zeta^2}{2\lambda\tau}\right). \tag{2.18}$$

Replacing ζ^2 by $(1+i\varepsilon)\zeta^2$ under the exponential and considering separately the even and odd moments we may convince ourselves that

$$d_n(\tau) = O(\tau^2) \quad \text{for } n \geqslant 3 \tag{2.19}$$

and that the condition D is satisfied.

Concluding one sees that each of the transition amplitudes $(\tau;\eta,\zeta)_0^{(l)}$ defines some quantum Markovian process of diffusional type on the positive axis [9]. Let us denote these processes by $\{r_0^{(l)}(\tau); \tau \geqslant 0\}$ $l = 0, 1, 2, \ldots$ then we will have

$$Q_\eta\{\chi_{A_0} r_0^{(l)}(\tau)\} = (\tau; \eta, A)_0^{(l)} \tag{2.20}$$

where A is a Borel subset on the positive axis. Moreover we obtain the following closed expression for the limit which appears in the formula (2.2)

$$\lim_{n\to\infty} \int_0^\infty d\zeta_1 \ldots \int_0^\infty d\zeta_{n-1} \prod_{k=0}^{n-1} (\Delta\tau_k; \zeta_k, \zeta_{k+1})_0^{(l)} \exp[c(\zeta_k)\Delta\tau_k] =$$

$$= Q_{|y|}\{\delta[r_0^{(l)}(t) - |x|] \exp \int_0^t c[r_0^{(l)}(\tau)]d\tau\} = (t; |y|, |x|)^{(l)}. \tag{2.21}$$

Thus finally we get the following partial wave expansion for the transition amplitude

$$Q_y\{\delta[z(t) - x] \exp \int_0^t c[z(\tau)]d\tau\} =$$

$$= \frac{1}{4\pi|y|\cdot|x|} \sum_{l=0}^{\infty} (2l+1)P_l(\cos \widehat{yx}) Q_{|y|} \left\{\delta[r_0^{(l)}(t) - |x|] \exp \int_0^t c[r_0^{(l)}(\tau)]d\tau\right\} \tag{2.22}$$

or in another notation

$$(t; y, x) = \frac{1}{4\pi|y|\cdot|x|} \sum_{l=0}^{\infty} (2l+1)P_l(\cos \widehat{yx})\,(t; |y|, |x|)^{(l)} = \tag{2.23}$$

$$= \frac{1}{|y|\cdot|x|} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta_y, \varphi_y) Y_l^{*m}(\theta_x, \varphi_x)\,(t; |y|, |x|)^{(l)}. \tag{2.24}$$

The transition amplitudes $(\tau; \eta, \zeta)^{(l)}$ satisfy the following Schrödinger equation

$$\left[-\partial_\tau + \frac{\lambda}{2}\frac{d^2}{d\eta^2} - \frac{\lambda l(l+1)}{2\eta^2} + c(\eta)\right](\tau; \eta, \zeta)^{(l)} = 0. \tag{2.25}$$

A corresponding equation for the free amplitudes $(\tau; \eta, \zeta)_0^{(l)}$ follows from these when the coefficient function $c(\eta)$ vanishes.

3. *Concluding remarks*

As it is well known the use of the polar coordinates in a discussion of the central potential scattering problem is natural and fruitful. Therefore, our formulae may be applied to the scattering problem in order to obtain the closed expressions for various physical quantities appearing in the discussion. For instance, for the scattering lengths of different order we may write [5]

$$A_l = \lim_{r\to\infty} \frac{2l+1}{2r} \cdot \frac{r^{2l+2}}{[(2l+2)!!]^2} -$$

$$- \lim_{\tau\to\infty} \frac{\tau^{l+\frac{3}{2}}}{a(l)} Q_r\{\delta[r_0^{(l)}(\tau)-r] \exp \int_0^t c[r_0^{(l)}(s)]ds\} \tag{3.1}$$

where

$$a(l) = \pi^{-\frac{1}{2}} 2^{-(l+1)} (2l+1)!! \left(\frac{2m}{\hbar^2}\right)^{l+\frac{3}{2}} \tag{3.2}$$

$$l = 0, 1, 2, \ldots$$

The scattering lengths A_l appear in an asymptotic expansion of the real solution of the zero energy radial equation

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right] + V(r)\right\} u_l(r) = 0 \tag{3.3}$$

satisfying the conditions

$$u_l(0) = 0$$

$$u_l(r) \simeq \frac{r^{l+1}}{(2l+1)!!} - A_l \frac{(2l-1)!!}{r^l}, \qquad r \to \infty. \tag{3.4}$$

In the same way the other interesting quantities like the partial cross-sections, density matrices and their variational bounds may be expressed in closed form using our formulae [10].

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