PURELY INTERNAL GEODESIC MOTION AND RESULTING NECESSARY CONDITIONS FOR RELATIVISTIC SPHERES

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The study of geodesic motion in internal gravitational fields is of relevance for neutrino emission processes in advanced evolutionary stages of superdense stars. From a general formula for neutrino trajectory in a static field of spherical symmetry certain conditions for the metric are derived in the case when trapped neutrino trajectories are allowed.

Several years ago a study has been performed of the geodesic motion of massless particles in internal gravitational fields. This problem is of interest because of this applications to astrophysics, where neutrino emission plays an important role in final evolutionary stages of celestial objects. In the geometric radii of such objects are of the order of the Schwarzschild radius (but, of course, always larger than it), then the ultrastrong gravitational field inside constitutes an essential hindrance for the escape of neutrinos. Results of a series of papers have been summarized (Kuchowicz 1968, 1969) and point out to the important astrophysical conclusion: Neutrino cooling rates of celestial bodies have to be reduced (from their standard values obtained for the Universal Fermi Interaction in flat space) when the effects of space curvature are taken into account. This has been verified by Vilhu's computations for some stellar models (Vilhu 1968).

Though the problem is to be regarded as essentially solved, a brief addendum to the previous paper (Kuchowicz 1968) seems to be advisable. The type of neutrino trajectories which are of the highest importance for our aims are the so-called trapped trajectories, which are confined to a bounded region of space. Three types of such trajectories are possible for neutrinos:

A. purely internal trajectories, *i. e.* those contained completely inside the relativistic matter distribution we are studying;

B. partially external trajectories, with the neutrino returning periodically to the interior of the massive sphere;

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C. even external stationary trajectories (of planetary type), provided the neutrino has a non-zero rest mass.

The first two types are of the highest relevance to the problem of gravity-induced opacity for neutrinos. Now, if we are trying to regard exact solutions of the Einstein equations for fluid spheres as some simplified models for relativistic objects, it is necessary to apply them also to the neutrino escape problem. If the model has to allow for some kind of trapped trajectories of the types A or B, some conditions have to be fulfilled. Such conditions for the case of canonical Schwarzschild coordinates have been presented earlier (see *e. g.* Kuchowicz 1968).

Recently, with available exact solutionss in other coordinate systems, like *e. g.* in isotropic (Kuchowicz 1972) and in polar Gaussian coordinates (Kuchowicz 1973a), it would be good to have a general form of these conditions. This will be given below.

With a general metric of spherical symmetry

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - e^{\sigma}r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{1}$$

we obtain from the Hamilton-Jacobi equation for the trajectory of a particle with rest mass m:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} + m^2c^2 = 0 \tag{2}$$

the following equation:

$$\varphi = \int \frac{e^{\lambda/2-\sigma}dr}{r\sqrt{Ar^2e^{-\nu} - (e^{-\sigma} + Br^2)}}, \tag{3}$$

where

$$A = \frac{\mathscr{E}_0^2}{M^2c^2}, \qquad B = \frac{m^2c^2}{M^2}. \tag{4}$$

It represents the motion (in a plane) of a particle of energy $\mathscr{E}_0$, and angular momentum M. This is a generalization of Eq. (2.3) from the previous paper (Kuchowicz 1968). The former equation is obtained for canonical coordinates: $\sigma = 0$.

The quantity under the root has to be positively-definite for some internal region: $r < r_b$, where r_b denotes the boundary of the fluid sphere in a given coordinate system. If it is, in addition, negative for $r > r_b$, then we know that purely internal trajectories (type A) are admitted.

As the previous investigations have shown that neutrinos with small mass do not differ essentially in their behaviour from massless neutrinos, we shall restrict ourselves to massless neutrinos only. The necessary and sufficient condition for purely internal (type A) trajectories has the form

$$F(r_b) > A > F(r) \text{ for } r \in (r_1, r_2) \text{ and } r_2 < r_b, \tag{5}$$

where

$$F(r) = \frac{e^{\nu-\sigma}}{r^2}. \tag{6}$$

The left part of the inequality (5) guarantees that the trajectory defined by Eq. (3) cannot reach the surface of the fluid sphere.

The necessary and sufficient condition for partially external trajectories (Type B) can be formulated in the following way: if an internal solution for a perfect fluid sphere can be joined in a continuous way at a boundary r_b to the external Schwarzschild solution which has the standard form in canonical coordinates:

$$e^{\gamma} = e^{-\lambda} = 1 - \frac{r_s}{r}, \qquad \sigma = 0, \tag{7}$$

then partially external trajectories are allowed for mass concentrations $a > 2/3$ and for $A(r_{bc})^2 \leqslant \frac{4}{27a^2}$. The symbol r_{bc} denotes here the boundary radius in canonical coordinates, and the mass concentration parameter is defined as the ratio of the Schwarzschild radius r_S to the geometrical radius r_{bc}.

From this formulation of the condition for type B trajectories it is evident that such trajectories exist always for model spheres permitting mass concentrations higher than 2/3 (the quantity a cannot exceed unity!). In the case of isotropic coordinates (with $\lambda = \sigma$) the parameter α (which also cannot exceed unity) replaces the quantity a. Partially external trajectories occur now for $\alpha > 2 - \sqrt{3}$, and for $Ar_f^2 \leqslant \frac{1}{108\,\alpha^2}$, where r_f is the geometrical radius in isotropic coordinates.

The necessary condition for purely internal trajectories has the following form in canonical coordinate (Kuchowicz 1968):

$$A(r_{bc})^2 \leqslant 1 - a. \tag{8}$$

It reads in isotropic coordinates:

$$A(r_f)^2 \leqslant \frac{(1-\alpha)^2}{(1+\alpha)^6}. \tag{9}$$

Another interesting condition for type A trajectories is obtained in isotropic coordinates if we use the general formal solution (Kuchowicz 1972). With the new variable $\xi = r^2$ the function $F(r)$ is now

$$F(r) = \frac{C_1}{C_2}\,\frac{1}{\xi}\,e\int 2[v(\xi) - u(\xi)]d\xi. \tag{10}$$

We use here the notation of the previous paper (Kuchowicz 1972). Since the general condition (5) demands that the value of $F(r)$ in some internal region of the fluid sphere is smaller than the value at its boundary, $F(r)$ has to increase somewhere between r_1 and r_f. Its derivative has to be there positively-definite which gives with the help of Eq. (10):

$$2v - u > \frac{1}{\xi}. \tag{11}$$

This inequality may be considered together with the differential inequalities for physically meaningful fluid spheres which are given in another paper (Kuchowicz 1973b). It has to be fulfilled by the functions u and v, if we wish that in our fluid sphere neutrinos should accumulate in its central regions.

The two inequalities (9) and (11) are only necessary conditions. Even if they are fulfilled, we have to investigate whether the left-hand side of the inequality (5) is fulfilled in some layer of the fluid sphere. The investigation of the possibility of partially external trajectories is much simpler as we need only to know the admissible range of the values of α. Thus we find that the solutions 3.6 and 4.1.2 from the previous paper (Kuchowicz 1972) allow for type B trajectories.

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