NONLOCAL QUANTUM ELECTRODYNAMICS

By Tran hu'u Phat*

Institute of Theoretical Physics, Warsaw University**

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In this paper nonlocal quantum electrodynamics is constructed satisfactorily. The theory is free of ultraviolet infinities, its S-matrix is macrocausal, unitary on the mass shell and fulfils gauge invariance.

It is known that the construction of quantum electrodynamics free of ultraviolet divergences encounters serious difficulties. For instance, the nonlocal quantum electrodynamics suggested by [1] does not fulfil the gauge invariance requirement. The gravity-modified quantum electrodynamics of Salam *et al.* [2] is gauge-invariant, but also encounters a great difficulty connected with the so-called equivalence theorem.

Recently, Efimov [3] has outlined a variant of nonlocal electrodynamics. However, some defects of this theory appear, for example, the regularization procedure of Pauli-Villars is still used to suppress the divergence of the self-energy photon graph. Basically this procedure is equivalent to renormalization. In this paper a new approach at building nonlocal quantum electrodynamics is outlined. It will be shown that the S-matrix fulfils all the physical requirements: convergence, macrocausality, unitarity on the mass shell and gauge inavariance.

In paragraph 1 the way of introducting nonlocality is presented. Some basic problems, such as convergence, macrocausality unitarity on the mass shell and gauge invariance are considered in paragraph 2. In paragraph 3 the self-energy graphs of photon and electron of second order are considered in detail. The general problems of quantum electrodynamics and related topics are considered in paragraphs 4 and 5.

1

In papers [4–11] nonlocality was introduced by supposing that in the case of interaction the field operator $\varphi(x)$ is replaced as follows

$$\varphi(x) \to \Phi(x) = \int d^4y V(x-y)\varphi(y) = V(\Box_x)\varphi(x)$$

* On leave of absence from Hanoi University, Viet-Nam.

** Present address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

here $V(z)$ is defined by

$$V(z) = e^{\omega(z)}\psi(z) \tag{1.1}$$

$e^{\omega(z)}$ is a generalized analytic function and $\psi(z)$ has one of the following forms

$$\text{a)} \quad \psi(\Box) = \int\limits_{\varrho^2 < l^2} d^4\varrho a(\varrho^2) \exp\left(i\varrho_0\partial_0 + \vec{\varrho}\vec{\partial}\right) \tag{1.2}$$

$$\text{b)} \quad \psi(\Box) = \int\limits_{\varrho^2 < l^2} d^4\varrho a(\varrho^2) \exp\left(\varrho_0\partial_0 + i\vec{\varrho}\vec{\partial}\right). \tag{1.3}$$

However, in the case when this approach is applied to construct nonlocal electrodynamics, we encounter either a divergence connected with the self-energy graph of the photon or a violation of gauge invariance. Therefore, we need a new manner of building satisfactorily nonlocal quantum electrodynamics.

Let us recall that the nonlocality introduced by [3] bases on the Lagrangian formalism. In the case when there is no interaction the field operators participate locally in the Lagrangian. In an opposite case, that is when there is interaction, they participate nonlocally in the Lagrangian by means of the above-mentioned procedure. From here one obtains the interaction Lagrangian

$$\mathscr{L}_1 = gU(V(\Box)\varphi(x))$$

and then the S-matrix is formally obtained as follows

$$S = T\exp\{ig \int U(\Phi)dx\}.$$

In our opinion, this procedure certainly leads to nonlocal quantum electrodynamics, in which we should encounter either divergences for some graphs or violation of gauge invariance. Another procedure for introducing nonlocality is now discussed.

We also assume that if there is no interaction, then the field operators participate locally in the Lagrangian. In the case when interaction exists, it is nonlocal and the nonlocality is introduced by starting from the S-matrix.

It is known that the S-matrix can be defined as a power series of the coupling constant g

$$S = \sum_{n=0}^{\infty} \int g^n S_n(x_1, x_2, \ldots, x_n)dx_1 \ldots dx_n$$

and the main problem now is to define all the members $S_n(x_1, x_2, \ldots, x_n)$. In the case when the interaction Lagrangian is given, we can define easily $S_n(x_1, x_2, \ldots, x_n)$ by using the Wick theorem. However, in our view, the more interesting problem is the following.

Let us indicate an algorithm for defining the S-matrix in each order of perturbation theory. This can be realized by formulating the Feynman rule in momentum space for an arbitrary graph of nth order, and the summation of all the matrix elements corresponding to nth order graphs gives us the Fourier transform of $S_n(x_1, x_2, \ldots, x_n)$ in momentum space. The next problem is to prove that by means of such a procedure the S-matrix is free of divergences, unitary on the mass, shell, macrocausal and also fulfils gauge invariance.

We assume that the Lagrangian of free fields is as usual

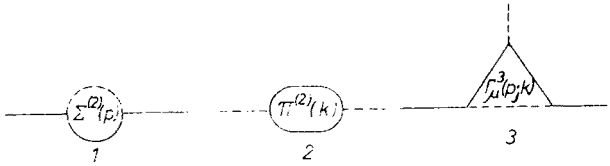
$$\mathscr{L} = \frac{i}{2}\bar{\psi}(\gamma^\mu\vec{\partial}_\mu - \overleftarrow{\partial}_\mu\gamma^\mu)\psi - m\bar{\psi}\psi - \tfrac{1}{2}\,\partial_\nu A_\mu \partial_\nu A_\mu$$

and the interaction of these fields is defined by an S-matrix of trilinear type in the sense that at every vertex there are two electron lines and one photon line.

Now the new approach is formulated in terms of the Feynman rule in momentum space for an arbitrary graph of n orders.

1. Factor $\dfrac{1}{\hat{p}-m}$ due to every interior line of electron with 4-momentum p.
2. Factor $\dfrac{1}{k^2-i\varepsilon}$ due to every interior photon line with 4-momentum k.
3. Factor $ie\gamma_\mu$ for every vertex.
4. Factor $(2\pi)^{-\frac{3}{2}}\dfrac{e_\lambda}{2\sqrt{\omega_{\vec{k}}}}$ due to every exterior photon line with energy $\omega_{\vec{k}}$ and polarization λ.
5. Factor $(2\pi)^{-\frac{3}{2}}\,\bar{w}^s(p)$ due to every exterior electron line with 4-momentum p and spinor index s, leaving graph.
6. Factor $(2\pi)^{-\frac{3}{2}}\,w^s(p)$ for every exterior electron line with 4-momentum p and spinor index s, entering graph.
7. Factor $(2\pi)^4\,\delta^4(p-p'\pm k)$ for every vertex corresponding to energy-momentum conservation.
8. Factor (-1) for every closed loop of electron.
9. The expression so obtained is to be integrated over all the interior momenta.
10. Each integration with respect to the independent momentum contains the weight function $V(l^2p^2)$.

As an illustrative example, let us consider the well known graphs: self-energy and vertex graphs.



We obtain easily

$$\Sigma^{(2)}(p) = \frac{e^2}{(2\pi)^4 i}\sum_\sigma g^{\sigma\sigma}\int \frac{V(l^2k^2)dk}{k^2+i\varepsilon}\,\gamma^\sigma\,\frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma^\sigma,$$

$$\Pi^{(2)}_{\mu\nu}(k) = \frac{e^2}{(2\pi)^4 i}\int V(l^2p^2)dp\,Sp\left(\gamma^\mu\,\frac{\hat{p}+m}{p^2-m^2+i\varepsilon}\,\gamma^\nu\,\frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon}\right)$$

$$\Gamma^{(3)}_\nu(p;k) = \frac{e^2}{(2\pi)^4 i}\sum_\sigma g^{\sigma\sigma}\int \frac{V(l^2q^2)dq}{(p-q)^2+i\varepsilon}\,\gamma^\sigma\,\frac{\hat{q}+\hat{k}+m}{(q+k)^2-m^2}\,\gamma^\nu\,\frac{\hat{q}+m}{q^2-m^2}\,\gamma^\sigma$$

and the approach given by [3-11] leads to the following expressions for $\Sigma(p)$, $\Pi_{\mu\nu}(k)$ and $\Gamma_\nu(p; k)$

$$\Sigma^{(2)}(p) = \frac{e^2}{(2\pi)^4 i} \sum_\sigma g^{\sigma\sigma} \int \frac{V(l^2k^2)dk}{k^2+i\varepsilon} \gamma^\sigma \frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon} \gamma^\sigma$$

$$\Pi^{(2)}_{\mu\nu}(k) = \frac{e^2}{(2\pi)^4 i} \int dp Sp \left(\gamma^\mu \frac{\hat{p}+m}{p^2-m^2+i\varepsilon} \gamma^\nu \frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon} \right)$$

$$\Gamma^{(3)}_\nu(p; k) = \frac{e^2}{(2\pi)^4 i} \sum_\sigma g^{\sigma\sigma} \int \frac{V[l^2(p-q)^2]}{(p-q)^2+i\varepsilon} \gamma^\sigma \frac{\hat{q}+\hat{k}-m}{(q+k)^2-m^2} \gamma^\nu \frac{\hat{q}+m}{q^2-m^2} \gamma^\sigma dq.$$

These expressions for $\Sigma(p)$, $\Pi_{\mu\nu}(k)$ and $\Gamma(p; k)$ allow us to see clearly the difference between two approaches.

2

In this paragraph some basic problems are considered. Namely, we shall prove that S-matrix is convergent, macrocausal, unitary on the mass shell and fulfils gauge invariance.

Let us firstly consider the convergence of members of perturbation theory series. In momentum space the matrix elements of a certain process in nth approximation of perturbation theory has the following form

$$F = \int \dots \int \prod_i V(l^2 r_i^2) dr_i \prod_j \frac{1}{k_j^2+i\varepsilon} \prod_l \gamma^{\sigma_l} \prod_m \frac{1}{\hat{q}_m - m} \tag{2.1}$$

here k_j and q_m are respectively 4-momenta corresponding to interior photon and electron lines, r_i are the 4-momenta, with respect to which one integrates.

It is easily seen that the above integral would be convergent in the case when the weight function $V(l^2k^2)$ were chosen to be the form factors given above.

Next, the macrocausality is proved to be valid. To do this, let us consider the following expression

$$C(x, y) = \frac{\delta}{\delta A(x)} \left(\frac{\delta S}{\delta A(y)} S^{-1} \right). \tag{2.2}$$

Expanding the S-matrix with respect to the coupling constant $e^2/4\pi$ one obtains a series of the form

$$S = \sum_n \frac{1}{n!} \int dx_1 \dots dx_n S_n(x_1, \dots, x_n) \times$$

$$\times : A(x_1) \dots A(x_i) \bar{\psi}(x_{i+1}) \dots \bar{\psi}(x_k) \psi(x_{k+1}) \dots \psi(x_n) : \tag{2.3}$$

The Fourier transform $\tilde{S}_n(p_1, \ldots, p_n)$ of $S_n(x_1, \ldots, x_n)$ contains the expressions of the form (2.1). Therefore, it is easily seen that in fact $S_n(x_1, \ldots, x_n)$ is built from the causal Green functions $D^c(x)$, $S^c(x)$ and the local generalized functions $V(\Box_x)\, D^c(x-y)$, $V(\Box_x)\, S^c(x-y)$.

Substituting (2.3) into (2.2) and passing to N-product, the commutation function $D^{\pm}$ and the anticommutation functions $D^{\pm}$ appear. Thereby, in the case when $S_n(x_1, \ldots, x_n)$ does not contain the function of the form $V(\Box)D^c$, $V(\Box)S^c$, we obtain

$$\frac{\delta}{\delta A(x)}\left(\frac{\delta S}{\delta A(y)}\, S^{-1}\right) = 0 \quad \text{for } x \leqslant y.$$

But with the aid of $V(z)$-function given by (1.1), we have the macrocausality principle:

$$\frac{\delta}{\delta A(x)}\left(\frac{\delta S}{\delta A(y)}\, S^{-1}\right) = 0$$

beside the regions

$$G: x^0 \geqslant y^0,\ (x-y)^2 > 0$$

$$G_l: -l^2 \leqslant (x-y)^2 \leqslant l^2.$$

Now the unitarity on the mass shell is proved. For simplicity, let us restrict ourselves to the case of self-interaction of a certain scalar field $\varphi(x)$ and the weight function $V(l^2k^2)$ is defined by (1.2) or (1.3).

Suppose we have a certain Feynman graph with n exterior lines.

Then the amplitude corresponding to this graph is described by the following integral

$$F = \int \ldots \int \prod_i V(l^2 k_i'^2) dk_i' \prod_j \frac{1}{k_j^2 + m^2} \tag{2.4}$$

here k_j-Euclidean 4-momentum, corresponding to the given line in the graph. The integration (2.4) is taken over the Euclidean momentum space of four dimensions and k_j'-4-momentum of integration. We shall prove the following property of amplitude (2.4). Suppose the graph, corresponding to the amplitude F, can be divided into two blocks F_{I} and F_{II} to be connected with each other by n interior lines. We have then

$$F = \int \ldots \int \prod_{v=1}^{r-1} \frac{V(l^2 k_v^2)}{k_v^2 + m^2}\, dk_v F_{\mathrm{I}}(q_j, k_i) \frac{1}{k_r^2 + m^2} F_{\mathrm{II}}(q_j', k_i)$$

here $q_j\,(j = 1, 2, \ldots, n_1)$ and $q_j'\,(j = 1, 2, \ldots, n_2)$ are the exterior momenta corresponding to the blocks I and II respectively. The following equalities are realized: $q = q_1' + \ldots + q_{n_2}' = -(q_1 + \ldots + q_{n_1})$ $(n = n_1 + n_2)$ and $k_r = q - k_1 - \ldots - k_{r-1}$. The above-mentioned expression for F can be rewritten in terms of F_{I} and F_{II} that describe the blocks I and II as follows

$$F = \int \ldots \int dk_1 \ldots dk_r F_{\mathrm{I}}(q_j, k_i) \prod_{v=1}^{r} \frac{V_v(l^2 k_v^2)}{k_v^2 + m^2} \times$$

$$\times F_{\mathrm{II}}(q_j', k_i) \delta^4(q - k_1 - \ldots - k_r) \tag{2.5}$$

Here

$$V_1(l^2k^2) = V_2(l^2k^2) = \ldots = V_{r-1}(l^2k^2) \equiv V(l^2k^2)$$

and

$$V_r(l^2k^2) \equiv 1.$$

Then we have the following equality

$$\Delta F(z) = i(2\pi)^r \prod_{\nu=1}^{r} V_\nu(-m^2) \int \tilde{d}k_1 \ldots \int \tilde{d}k_r \times$$

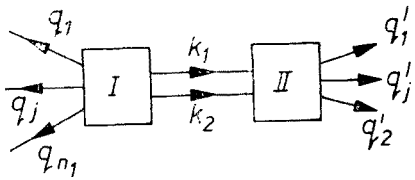
$$\times \prod_{\nu=1}^{r} \theta(\tilde{k}_{\nu 0})\delta(k_\nu^2 + m^2)\delta^4(\tilde{q} - \tilde{k}_1 - \ldots - \tilde{k}_r)F_{\mathrm{I}}(q_j, \tilde{k}_i)F_{\mathrm{II}}(q'_j, \tilde{k}_i) \tag{2.6}$$

here $\Delta F(z)$ is the discontinuity of $F(z)$ across the cut in the complex z-plane, $z = -q^2$ and $\tilde{k}_j$ is the vector $(\vec{\tilde{k}}_j, \tilde{k}_{j_0})$ so that $\tilde{k}_j^2 = \vec{\tilde{k}}_j^2 - \tilde{k}_{j_0}^2$. It is necessary to understand that $F_{\mathrm{I}}(q_j, \tilde{k}_i)$ and $F_{\mathrm{II}}(q'_j, \tilde{k}_i)$ are analytic continuations of the starting functions with respect to corresponding values of scalar arguments $(q_j\tilde{k}_i)$ and $(q'_j\tilde{k}_i)$.

As it is known, the above-mentioned property is called the Cutkosky rule [12].

Equality (2.6) shall be proved in two stages. Firstly, it is proved for the case of two intermediate interior lines, where we have

$$F = \int dk_1 \frac{V(l^2k_1^2)}{k_1^2 + m^2} F_{\mathrm{I}}(q_j; k_1) \frac{1}{(q-k_1)^2 + m^2} F_{\mathrm{II}}(q'_j; k_1)$$



or

$$F = \int dk_1 \int dk_2 \frac{V_1(l^2k_1^2)}{k_1^2 + m^2} \cdot \frac{V_2(l^2k_2^2)}{k_2^2 + m^2} \delta^4(q - k_1 - k_2) \times$$

$$\times F_{\mathrm{I}}(q_j; k_1, k_2)F_{\mathrm{II}}(q'_j; k_1, k_2),$$

here $V_1 \equiv V$ and $V_2 \equiv 1$. We see that this form of F is totally identical to that of [13], therefore the equality is proved for this case.

Now we consider the case when I and II are connected to each other by $r+1$ intermediate interior lines. It is easily seen that the situation is similar to the first case, that is the amplitude F can be also written in the following form

$$F = \int \frac{dk_{r+1} V_{r+1}(l^2k_{r+1}^2)}{k_{r+1}^2 + m^2} \int dk P(q_j, q'_j; k_{r+1}, k)\delta^4(q - k_{r+1} - k)$$

here

$$P(q_j, q'_j; k_{r+1}, k) = \int dk_1 \ldots dk_r \prod_{\nu=1}^{r} \frac{V_\nu(l^2 k_\nu^2)}{k_\nu^2 + m^2} \times$$

$$\times \delta^4(k - k_1 - \ldots - k_r) F_{\mathrm{I}}(q_j; k_i) F_{\mathrm{II}}(q'_j; k_i)$$

in which

$$V_1 = \ldots = V_r = V$$

and

$$V_{r+1} \equiv 1.$$

The mentioned expression for F is also totally identical to that of [13], therefore the equality (2.6) is proved for the second case.

Hence, in our theory the Cutkosky rule is proved entirely. By using an approach similar to that of [14] the unitarity property of S-matrix on the mass shell is proved in high approximation of perturbation theory.

Now let us return to the case when $V(l^2k^2)$ is defined by (1.1). It is easily seen that for this case the unitarity property of S-matrix on the mass shell is still valid for the reason that the forms of $V(l^2k^2)$ given respectively by (1.1) and (1.2) or (1.3) are only different from each other by a real factor $\exp \omega$.

Finally the Ward identity is proved. The basis for proving is the following relation

$$\frac{\partial}{\partial p_\mu} \frac{1}{\hat{p} - m} = \frac{1}{\hat{p} - m} \gamma_\mu \frac{1}{\hat{p} - m}.$$

It is easily seen that in the second order of perturbation theory, the Ward identity is valid. Indeed, by differentiating $\Sigma^{(2)}(p)$ we have

$$\frac{\partial}{\partial p_\mu} \Sigma^{(2)}(p) = \frac{e^2}{(2\pi)^4 i} \sum_\sigma g^{\sigma\sigma} \frac{\partial}{\partial p_\mu} \int \frac{V(l^2 q^2) dq}{q^2 + i\varepsilon} \gamma^\sigma \frac{1}{\hat{p} - \hat{q} - m + i\varepsilon} \gamma^\sigma =$$

$$= \frac{e^2}{(2\pi)^4 i} \sum_\sigma g^{\sigma\sigma} \int \frac{V(l^2 q^2) dq}{q^2 + i\varepsilon} \gamma^\sigma \frac{1}{\hat{p} - \hat{q} - m + i\varepsilon} \gamma^\mu \frac{1}{\hat{p} - \hat{q} - m + i\varepsilon} \gamma^\sigma =$$

$$= \Gamma_\mu^{(3)}(p; k)_{|k=0}.$$

Now this identity is proved for a compact self-energy graph of electron of arbitrary order. It is easily seen that we can always choose the interior momentum variable for which the matrix elements corresponding to a certain self-energy graph is of the following form

$$\Sigma_w^{(n)}(p) = \frac{e^2}{(2\pi)^4 i} \sum_\sigma g^{\sigma\sigma} \int V(l^2 q^2) F(p, q) \gamma^\sigma \frac{\hat{p} - \hat{q} + m}{(p - q)^2 - m^2 + i\varepsilon} \gamma^\sigma dq$$

here $F(p, q)$ is the function characterizing the matrix element corresponding to the remaining part of the graph. Form here we have

$$\frac{\partial}{\partial p_\mu}\Sigma_w^{(n)}(p) = \frac{e^2}{(2\pi)^4 i}\sum_\sigma g^{\sigma\sigma}\int V(l^2q^2)F(p, q)dq \times$$

$$\times\gamma^\sigma\frac{1}{\hat{p}-\hat{q}-m+i\varepsilon}\gamma^\mu\frac{1}{\hat{p}-\hat{q}-m+i\varepsilon}\gamma^\sigma = \Gamma_{w\mu}^{(n+1)}(p, k)_{|k=0}. \tag{2.7}$$

If we now differentiate all the self-energy graphs of electron $\Sigma_w^{(n)}(p)$ with respect to p, we should obtain a set of vertex functions $\Gamma_{w\mu}^{(n+1)}(p; k)$ corresponding to all the possibilities of attaching the photon line to the self-energy electron graphs. Therefore, from the fact that the summation of all the $\Sigma_w^{(n)}(p)$ gives us the compact self-energy electron function of nth order,

$$\Sigma^{(n)}(p) = \sum_w \Sigma_w^{(n)}(p)$$

we deduce that

$$\Gamma_\mu^{(n+1)}(p; k) = \sum_w \Gamma_{w\mu}^{(n+1)}(p; k)$$

gives us the compact vertex function of $(n+1)$th order. Using (2.7) we have then the Ward identity for nth approximation of the perturbation theory

$$\frac{\partial}{\partial p_\mu}\Sigma^{(n)}(p) = \Gamma_\mu^{(n+1)}(p; k)_{|k=0}.$$

Hence the Ward identity is proved entirely. Next, gauge invariance is considered.

The S-matrix can be expanded as follows

$$S = \sum_{m,n,l}\frac{1}{m!n!l!}\int dk_1\ldots\int dk_n\int dp_1\ldots\int dp_m\int dq_1\ldots\int dq_l \times$$

$$\times F_{\mu_1\ldots\mu_n}(k_1, \ldots, k_n; p_1, \ldots, p_m; q_1, \ldots, q_l) : A_{\mu_1}(k_1)\ldots\psi(p_1)\ldots\bar{\psi}(q_l) :$$

Gauge invariance means that

$$k_{\mu_i}F_{\mu_1\ldots\mu_i\ldots\mu_n} = 0,$$

$$k_{\mu_i}k_{\mu_j}F_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_n} = 0, \tag{2.8}$$

and so on for every k_{μ_i}. Each of the above conditions is fulfilled in the case when the remaining momenta, on which F depends, belong to mass shell.

The proof of gauge invariance under the form (2.8) is rather simple and is based on the following equality

$$k_\mu \left\{ \frac{1}{\hat{p}_1 - m} \gamma^\mu \frac{1}{\hat{p}_2 - m} \right\} = \frac{1}{\hat{p}_2 - m} - \frac{1}{\hat{p}_1 - m}$$

if $k = p_1 - p_2$.

To summarize, in this paragraph we have proved that our nonlocal quantum electrodynamics fulfils all the imposed conditions: it is free of ultraviolet infinities, macrocausal, unitary on the mass shell and gauge invariant in each order of perturbation theory.

3

The self-energy graphs for electron and photon of second order are considered in this paragraphs.

Firstly, let us study the vacuum polarization described by the diagram 2.

Its matrix element is given as:

$$\Pi_{\mu\nu}(q) = \int V(l^2 p^2) d^4 p Sp \left(\gamma_\mu \frac{\hat{p} + \hat{q} + m}{(p+q)^2 - m^2} \gamma_\nu \frac{\hat{p} + m}{p^2 - m^2} \right) =$$

$$= \int_0^1 dz \int V(l^2 p^2) dp Sp \frac{\gamma_\mu (\hat{p} + \hat{q} + m) \gamma_\nu (\hat{p} + m)}{[(p+qz)^2 + q^2(z - z^2) - m^2]^2} .$$

Let us notice that for $p^2 \gg 1$ we can write

$$V(l^2(p+k)^2) \approx V(l^2 p^2)$$

and for $p^2 \ll 1$

$$V(l^2 p^2) \approx 1.$$

Then by the change of variable $p \to p - qz$ the expression for $\Pi_{\mu\nu}(q)$ can be rewritten as

$$\Pi_{\mu\nu}(q) \approx \int_0^1 dz \int V(l^2 p^2) dp Sp \frac{\gamma_\mu (\hat{p} + \hat{q}(1-z) - m) \gamma_\nu (\hat{p} - \hat{q}z - m)}{[p^2 + q^2(z - z^2) - m^2]^2} =$$

$$= 4 \int_0^1 dz \int V(l^2 p^2) dp \frac{-(2 q_\mu q_\nu - g_{\mu\nu} q^2) z(1-z) - g_{\mu\nu} (\frac{1}{2} q^2 - m^2)}{[q^2(z - z^2) + p^2 - m^2]^2} .$$

From gauge invariance

$$\Pi_{\mu\nu}(q) q^\nu = 0$$

we deduce that

$$\int V(l^2q^2)d^4q\,\frac{-p^2(z-z^2)-\frac{1}{2}q^2+m^2}{[q^2+p^2(z-z^2)-m^2]^2}=0$$

and therefore the expression for $\Pi_{\mu\nu}(q)$ finally takes the form

$$\Pi_{\mu\nu}(q)=(q_\mu q_\nu-g_{\mu\nu}q^2)\Pi(q^2)$$

where

$$\Pi(q^2)=-8\int_0^1 dzz(1-z)\int\frac{V(l^2p^2)dp}{[p^2+q^2(z-z^2)-m^2]^2}.$$

If

$$\Pi(q^2)=\Pi(0)+\frac{\partial\Pi}{\partial q^2}\Big|_{q^2=0}q^2+\ldots$$

then

$$\Pi(0)=-8\int_0^1 dz(z-z^2)\int\frac{V(l^2p^2)dp}{[p^2-m^2]^2}=-\frac{4}{3}\int\frac{V(l^2p^2)dp}{[p^2-m^2]^2},$$

$$\frac{\partial\Pi(q^2)}{\partial q^2}\Big|_{q^2=0}=16\int_0^1 dz(z-z^2)^2\int\frac{V(l^2p^2)dq}{[p^2-m^2]^3}=\frac{8}{15}\int\frac{V(l^2p^2)dp}{[p^2-m^2]^3}.$$

It is clear that for $l\to 0$, $V\to 1$, $\Pi(0)$ diverges logarithmically and $\frac{\partial}{\partial q^2}\Pi(q^2)$ converges. Therefore, for sufficiently small l, $\Pi(0)$ can take the following form

$$\Pi(0)\approx-\frac{4}{3}\log\frac{1}{l^2m^2}+O(l^2m^2)$$

and because $\left.\frac{\partial\Pi}{\partial q^2}\right|_{q^2=0}$ converges for $l\to 0$, we can write

$$\frac{\partial\Pi(q^2)}{\partial q^2}\Big|_{q^2=0}=\frac{8}{15}\frac{i\pi^2}{2m^2}\beta$$

here β characterizes the nonlocal factor taking part in the integral. Finally, we have

$$\Pi_{\mu\nu}(q^2)=4\pi^2 i(q_\mu q_\nu-g_{\mu\nu}q^2)\left[-\frac{1}{3}\ln\frac{1}{l^2m^2}+\frac{1}{15m^2}\beta q^2+\ldots\right]$$

from where we obtain the renormalized current density

$$J_R^\mu(x) = \left[1 + \frac{\alpha}{3\pi} \ln \frac{1}{l^2 m^2}\right] J_\mu(x)$$

and the effective potential for electron as a result of vacuum polarization is equal to

$$a_{\text{eff}}^\mu(q) = \left[1 - \frac{\alpha\beta}{15\pi m^2} q^2\right] a^\mu(x).$$

For a point charge we have the effective potential for the electron

$$-Ze^2\left(\frac{1}{r} - \beta \frac{\alpha}{15\pi m^2} 4\pi\delta(\vec{r})\right).$$

It is possible that this expression allows us to check the nonlocal effect by evaluating the influence of the vacuum polarization on the energy level of the hydrogen atom.

Let us consider the self-energy graph of the electron. As was mentioned earlier, its matrix element is of the following form

$$\Sigma^{(2)}(p) = \frac{e^2}{i(2\pi)^4} \sum_\mu g^{\mu\mu} \int \frac{V(l^2k^2)dk}{k^1 + \lambda^2} \gamma^\mu \frac{\hat{p} - \hat{k} + m}{(p-k)^2 - m^2 + i\varepsilon} \gamma^\mu$$

or

$$\Sigma^{(2)}(p) = -\frac{\alpha}{(2\pi)^3} \int_0^1 dx \int dk V(l^2k^2) \frac{i(\hat{p} + \hat{k}) + 2m}{[k^2 + (2pk + p^2 + m^2 - \lambda^2)x + \lambda^2]^2}.$$

In order to evaluate the correction to the electron mass, let us expand $\Sigma^{(2)}(p)$ in terms of $(\hat{p} + m)$

$$\Sigma^{(2)}(p) = -i\{A^0 + B^0(\hat{p} + m) + \ldots\}.$$

Then using some simple transformations we obtain

$$A^0 = -\frac{ie^2}{(2\pi)^4} \int_0^1 dx \int V(l^2k^2) \frac{2(x+1)x}{(k^2 + x^2m^2 - i\varepsilon)^2} dk.$$

As $V(l^2p^2)$ is regular in the whole p^2-plane we can pass to Euclidean metric by means of the rotation $k_0 \to e^{i\pi/2}k_0$, which gives us

$$A^0 = \frac{e^2\pi^2}{(2\pi)^4} \int_0^1 dx \int_0^{+\infty} V(l^2z) \frac{2x(1+x)zdz}{[z + x^2m^2]^2}$$

where $z = p^2$.

As $l^2 \to 0$, $V \to 1$ this integral is logarithmically divergent. Hence, for a sufficiently small value of l we can write

$$\delta m = A^0 = \frac{e^2}{16\pi^2} m \left(\ln \frac{1}{l^2 m^2} + O(l^2 m^2) \right).$$

4

In this paragraph the equations and properties of propagators and vertex function are considered.

It is usual to define propagators for the electron and photon as follows

$$G^e_{\alpha\beta}(x-y) = \langle 0 | T(\check{\psi}_\alpha(x) \check{\bar{\psi}}_\beta(y)) | 0 \rangle$$

$$G^\gamma_{\mu\nu}(x-y) = \langle 0 | T(\check{A}_\mu(x) \check{A}_\nu(y)) | 0 \rangle$$

here $\check{O}$ denotes the operator in the Heisenberg picture. Passing to the interaction picture, the propagators are written as

$$G^e_{\alpha\beta}(x-y) = \langle 0 | T(\psi_\alpha(x) \bar{\psi}_\beta(y) S) | 0 \rangle \tag{4.1a}$$

$$G^\gamma_{\mu\nu}(x-y) = \langle 0 | T(A_\mu(x) A_\nu(y) S) | 0 \rangle. \tag{4.1b}$$

Let Σ and $\Pi_{\mu\nu}$ be the self-energy functions of electron and photon respectively, we can the easily obtain the following equations for propagators G^e and G^γ by means of (4.1)

$$G^e(x-y) = \tilde{S}^c(x-y) + \int \tilde{S}^c(x-x') \Sigma(x'-y') \tilde{S}^c(y'-y) dx' dy'$$

$$G^\gamma_{\mu\nu}(x-y) = \tilde{D}^c_{\mu\nu}(x-y) + \int \tilde{D}^c_{\mu\lambda}(x-x') \Pi_{\lambda\sigma}(x'-y') \tilde{D}^c_{\sigma\nu}(y'-y) dx' dy'$$

where

$$\tilde{S}^e_{\alpha\beta}(x-y) = \langle 0 | T(\psi_\alpha(x) \bar{\psi}_\beta(y)) | 0 \rangle$$

$$\tilde{D}^c_{\mu\nu}(x-y) = \langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle.$$

In momentum space these equations take the form

$$G^e(p) = \tilde{S}^c(p) + \tilde{S}^c(p) \Sigma(p) \tilde{S}^c(p)$$

$$G^\gamma(k) = \tilde{D}^c(k) + \tilde{D}^c(k) \Pi(k) \tilde{D}^c(k).$$

In analogy with the ordinary quantum electrodynamics the mass and polarization operators $M(p)$ and $P(k)$ are defined by the following equalities

$$G^e(p)^{-1} = \tilde{S}^e(p)^{-1} + iM(p)$$

$$G^\gamma(k)^{-1} = \tilde{D}^c(k)^{-1} + iP(k).$$

We now introduce the compact self-energy parts of electron and photon, Σ^* and Π^*. They are connected with the mass and polarization operators $M(p)$ and $P(k)$ as follows

$$\Sigma^*(p) = -iM(p),$$
$$\Pi^*(k) = -iP(k).$$

Through Λ_μ we express the matrix elements corresponding to the compact vertex graphs and the vertex function is defined as follows

$$\Gamma_\mu = \gamma_\mu + \Lambda_\mu.$$

Now basing on the skeleton graphs of the following forms

we can write down the equations for Π^*, Σ^* and Γ_μ

$$\Sigma^*(p) = \frac{e^2}{(2\pi)^4}\int \gamma_\mu G^e(p-k)\Gamma_\nu(p, p-k; k)G^\gamma_{\mu\nu}(k)V(l^2k^2)dk$$

$$\Pi^*(k) = \frac{e^2}{(2\pi)^4}\frac{1}{3}Sp\int \gamma_\mu G^e(p)\Gamma_\mu(p, p-k; k)G^e(p-k)V(l^2p^2)dp,$$

from where we obtain two integral equations connecting three functions G^e, G^γ and Γ_μ

$$G^e(p) = \tilde{S}^c(p) + \frac{e^2}{(2\pi)^4}S^c(p)\int \gamma_\mu G^e(p-k)\Gamma_\nu(p, p-k; k)\times$$
$$\times G^e(p)G^\gamma_{\mu\nu}(k)V(l^2k^2)dk, \tag{4.2a}$$

$$G^\gamma(k) = \tilde{D}^c(k) + \frac{e^2}{3(2\pi)^4}D^c(k)G^\gamma(k)Sp\int \gamma_\mu G^e(p)\times$$
$$\times\Gamma_\mu(p, p-k; k)G^e(p-k)V(l^2p^2)dp. \tag{4.2b}$$

In this paragraph we consider also the asymptotic behaviour of propagators and the vertex function. As it is known, for $l \to 0$ the perturbation theory series is divergent from the second order. Therefore, it is clear that the real parameters of decomposition of perturbation theory must depend on e^2 and l^2. It is easily seen that for the vertex function, if, say, we have the following decomposition

$$\Gamma(p_1, p_2; k) = \sum_{n=0}^{\infty} a_n\lambda^n$$

where

$$\lambda = \ln\frac{1}{l^2m^2}$$

and a_n are the functions of p_1^2/m^2, p_2^2/m^2, k^2/m^2 and e^2.

A discussion similar to that of [15] leads to the following expression for Γ:

$$\Gamma(p_1, p_2; k) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} a_{nr}(e^2)^r(e^2\lambda)^n$$

here the a_{nr} are the functions of p_1^2/m^2, p_2^2/m^2 and k^2/m^2. The series for Γ_μ contains two parameters, e^2 and $e^2\lambda$, therefore the condition for applying the perturbation theory is the following

$$e^2 \ll 1 \quad \text{and} \quad e^2 \ln \frac{1}{l^2 m^2} \ll 1.$$

The second condition shows that the value of l cannot be too small, for example, if

$$l^{-1} \approx 100 \div 1000 \text{ GeV}$$

then

$$e^2 \ln \frac{1}{l^2 m^2} \approx \frac{1}{3} \div \frac{1}{2}.$$

The perturbation series for propagators can be obtained similarly

$$G^e(p) = s(p)\tilde{S}^c(p)$$

$$G^\gamma(k) = d(k)\tilde{D}^c(k)$$

where

$$s(p) = \sum_{n,r=0}^{\infty} s_{nr}(e^2)^r(e^2\lambda)^n$$

$$d(k) = \sum_{n,r=0}^{\infty} d_{nr}(e^2)^r(e^2\lambda)^n$$

in which s_{nr} and d_{nr} depend only on k^2/m^2. If we represent $\tilde{S}^c$ and $\tilde{D}^c$ by the expressions

$$\tilde{S}^c(p) = \xi(p)S^c(p)$$

$$\tilde{D}^c(k) = \eta(k)D^c(k)$$

then it is easily seen that $\xi(p)$ and $\eta(k)$ could be represented by the power series of l as follows

$$\xi(p) = \sum_{m=0}^{\infty} \xi_m(l^2p^2)^m$$

$$\eta(k) = \sum_{n=0}^{\infty} \eta_n(l^2k^2)^n$$

here $\xi_0 = \eta_0 = 1$ and ξ_n, η_n are constants. Finally we obtain

$$G^e(p) = \sum_{m,n,r=0}^{\infty} s_{nr}\xi_m(e^2)^r(e^2\lambda)^n(l^2p^2)^m S^c(p),$$

$$G^\gamma(k) = \sum_{m,n,r=0}^{\infty} d_{nr}\eta_m(e^2)^r(e^2\lambda)^n(l^2k^2)^m D^c(k).$$

For $|k^2| \gg m^2$ the functions $s(k)$ and $d(k)$ do not depend asymptotically upon m, they have the forms

$$s(k) = \sum_{n,r} s^0_{nr}(e^2)^r \left(e^2 \ln \frac{1}{l^2|k^2|}\right)^n$$

$$d(k) = \sum_{n,r} d^0_{nr}(e^2)^r \left(e^2 \ln \frac{1}{l^2|k^2|}\right)^n,$$

where s^0_{nr} and d^0_{nr} are constants.

For the vertex function we obtain an analogous asymptotic expression.

5

Now let us try to find the equation, basing on which we can determine the weight function $V(l^2p^2)$. To do this, we propose that the form factor V, similar to the other functions of quantum electrodynamics, is not given *a priori*. It will be defined by the structure of nonlocal quantum electrodynamics. Thus our opinion is different from that of Efimov and other authors who believe that the form factor $V(l^2p^2)$ needs to be introduced *a priori* in nonlocal quantum electrodynamics [1], [3–11], [16–22]. We first consider the following transformations

$$G^e \to G^e_R = Z_1^{-1} G^e, \quad G^\gamma \to G^\gamma_R = Z^{-1} G^\gamma,$$
$$\Gamma \to \Gamma_R = Z_1 \Gamma, \quad e^2 \to e^2_R = Ze^2,$$
$$V \to V_R = V, \tag{5.1}$$

leaving invariant the matrix elements of a certain process. These transformations constitute a group called the renormalization group. Because the divergences do not appear in our theory, the concept on renormalization need not be imposed. However, formally, the preceding transformations are called renormalization ones and it is interesting to study this problem. As it is well known, basing on (5.1) we can obtain the functional equations for propagators and vertex function [23], for instance, for propagators of electron and photon the following equations are known

$$s(x, y, e^2) s\left(1, \frac{y}{t}, e^2 d(t, y, e^2)\right) = s(t, y, e^2) \times$$
$$\times s\left(\frac{x}{t}, \frac{y}{t}, e^2 d(t, y, e^2)\right), \tag{5.2}$$

$$e^2 d(x, y, e^2) = e^2 d(t, y, e^2) d\left(\frac{x}{t}, \frac{y}{t}, e^2 d(t, y, e^2)\right). \tag{5.3}$$

Owing to [24] these equations do not give us uniquely any solutions, namely their general solutions are defined up to arbitrary functions of two arguments.

It is expected that the solutions could be defined uniquely if we had some restricted requirements added to the preceding equations. In that spirit, let us study the possible transformations leaving the matrix elements invariant. It is easily seen that these are the following

$$p \to \alpha^{-1} p, \quad G^e \to \alpha^{-1} G^e,$$
$$V \to \alpha V, \quad G^\gamma \to \alpha^{-1} G^\gamma,$$
$$e^2 \to \alpha e^2, \quad \Gamma \to \alpha \Gamma, \tag{5.4}$$

which have the structure of scale transformation. Let us now seek the functional equation for the form factor V basing on (5.4). It has been mentioned above that this function is defined by the structure itself of nonlocal quantum electrodynamics. Hence, in principle it is a function of the mass and the charge of electron, that is, it has the following form

$$V = V(l^2 p^2, l^2 m^2, e^2).$$

Owing to (5.4) this function contains an arbitrary factor. We can carry this arbitrary factor attached to V into its arguments. Then it is possible to consider that l becomes now a variable and we can impose the normalized condition on V

$$V = 1 \quad \text{for} \quad p^2 = (l^2)^{-1}.$$

For a certain infinitesimal transformation

$$p \to (1+\varepsilon)^{-1} p \approx (1-\varepsilon) p$$

we have

$$V(l_2^2 p^2, l_2^2 m^2, e_2^2) - 2\varepsilon l_1^2 p^2 \frac{\partial}{\partial (l_1^2 p^2)} V(l_1^2 m^2, l_1^2 m^2, e_1^2) =$$
$$= (1+\varepsilon) V(l_1^2 p^2, l_1^2 m^2, e_1^2). \tag{5.5}$$

Putting $p^2 = l_1^{-2}$ and using the normalized condition one obtains

$$1 - 2\varepsilon t \left.\frac{\partial a}{\partial x}\right|_{x=t} = (1+\varepsilon) a$$

where

$$x = l_1^2 p^2, \quad t = l_1^2 / l_2^2$$

and

$$a = V(x, l_1^2 m^2, e_1^2).$$

From here ε is obtained

$$\varepsilon = (1-a)(a+2bt)^{-1} \tag{5.6}$$

where

$$b = \left.\frac{\partial a}{\partial x}\right|_{x=t}.$$

Consequently we obtain the following expression for e_2^2

$$e_2^2 = (1+2bt)\,(a+2bt)^{-1}e_1^2. \tag{5.7}$$

Substituting (5.6) into (5.5) we obtain the functional equation for V

$$V\left[\frac{x}{t}, \frac{y}{t}, (2bt+1)\,(V(t, y, e^2)+2bt)^{-1}e^2\right] -$$
$$-2(1-V(t, y, e^2))\,(V(t, y, e^2)+2bt)^{-1}bx =$$
$$= (1+2bt)\,(V(t, y, e^2)+2bt)^{-1}V(x, y, e^2), \tag{5.8}$$

in which $y = l^2, m^2$ and the value of b may be given *a priori*.

Thus we have obtained the functional equation for the form factor which together with the analytic condition can allow us to define V.

The functional equations for propagators and vertex function can be found in a similar manner. For instance, the equation for photon propagator is the following

$$d\left[\frac{x}{t}, \frac{y}{t}, (1+2ct)\,(d(t, y, e^2)+2ct)^{-1}e^2\right] =$$
$$= 2(1-d(t, y, e^2))\,(d(t, y, e^2)+2ct)^{-1}cx +$$
$$+(1+2ct)\,(d(t, y, e^2)+2ct)\,(d(x, y, e^2)) \tag{5.9}$$

where

$$G^{\gamma}_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)d(k)$$

$$c = \frac{\partial}{\partial\xi}\,d(\xi, y, e^2)\Big|_{\xi=t}.$$

(5.9) together with (5.8) can allow us to find uniquely the propagator $d(k)$.

In resuming, in this paragraph the functional equations for propagators and vertex function were obtained, and with the aid of these equations we were able to evaluate exactly these fundamental functions of nonlocal quantum electrodynamics. In particular, the functional equation for the form factor V is also obtained. Although these equations still cannot give us uniquely the form of V, this reveals a great perspective in defining this unknown function.

6

To end, let us discuss the obtained results. In this paper we have made an effort to build a variant of nonlocal quantum electrodynamics. The main idea is that nonlocality is not attached to field operators, or alternatively, to propagators as usual. The nonlocality is attached to the S-matrix itself. To do this, we utilized the Feynman graph technique to give us the algorithm for finding each term of the S-matrix in the frame of the perturbation theory.

The obtained nonlocal S-matrix fulfils all the necessary physical requirements: it is free of divergence, macrocausal, unitary on the mass shell and gauge invariant.

The corrections to the mass and the charge of electron are calculated. The general equations for propagators and vertex function are established. From the asymptotic decomposition of propagators and vertex function it is shown that the quantity l having the sense of fundamental length is limited by the following condition

$$e^2 \ln \frac{1}{l^2 m^2} \ll 1.$$

Finally, basing on the assumption that the form factor characterizing the nonlocality, in reality, is also defined by the structure itself of quantum electrodynamics and on the group structure of the matrix elements, we found one functional equation for this form factor. Although this equation is rather complicated, it may allow us to study some properties of the form factor and it is very interesting that this equation together with the analytic condition, imposed on the form factor, can give us uniquely the solution. In addition, the equations for propagators and vertex function are also obtained outside the equation of renormalization group.

In our opinion, an interesting question is raised: whether we are able to define uniquely the fundamental functions of quantum electrodynamics and the form factor with the aid of the equations (5.2)–(5.9).

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