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ON THE PHYSICAL INTERPRETATION OF THE BEL-ROBINSON TENSOR

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Expansion of the Einstein canonical pseudotensor, ${}_{E}t_{\mu}{}^{\nu}$, for the gravitational field in the normal coordinate system in vacuum shows that the first generalized normal tensor, ${}_{E}t_{\mu}{}^{\nu}{}_{\varrho\sigma}$, belonging to the ${}_{E}t_{\mu}{}^{\nu}$ contains the Bel-Robinson tensor. It is therefore possible to connect some components of that tensor with the variations of the energy and momentum of the free gravitational field.

1. The Bel-Robinson tensor

Bel [1 — 3] has discovered the tensor

$$T^{\beta}{}_{\alpha\gamma\delta} = R^{\beta\mu\nu}{}_{\gamma}R_{\alpha\mu\nu\delta}+R^{\beta\mu\nu}{}_{\delta}R_{\alpha\mu\nu\gamma}-\tfrac{1}{2}\,\delta^{\beta}_{\alpha}R^{\mu\nu\varrho}{}_{\gamma}R_{\mu\nu\varrho\delta} \tag{1.1}$$

$$\alpha, \beta, \gamma, \delta, \mu, \nu, \varrho = 1, 2, 3, 4,$$

which possesses very interesting properties [1 — 6]. Usually, this tensor is called the Bel-Robinson tensor. The Bel-Robinson tensor is connected [7] with the Bianchi identities

$$\nabla_{\alpha}R_{\beta\gamma\mu\nu}+\nabla_{\beta}R_{\gamma\alpha\mu\nu}+\nabla_{\gamma}R_{\alpha\beta\mu\nu} \equiv 0 \tag{1.2}$$

and with their consequences

$$\nabla_{\alpha}R^{\alpha\beta}{}_{\mu\nu} \equiv 2\nabla_{[\mu}R_{\nu]}{}^{\beta} \tag{1.3}$$

in the same manner as the symmetric energy-momentum tensor, $T^{\mu\nu}$, of the electromagnetic field is connected with the system of the Maxwell equations:

$$\nabla_{\alpha}F_{\beta\gamma}+\nabla_{\beta}F_{\gamma\alpha}+\nabla_{\gamma}F_{\alpha\beta} = 0,$$

$$\nabla_{\alpha}F^{\alpha\beta} = 4\pi J^{\beta}. \tag{1.4}$$

Here $F_{\mu\nu} = -F_{\nu\mu}$ denotes the tensor of the electromagnetic field and J^{β} the current;

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∇ denotes the covariant derivative. The physical meaning of the Bel-Robinson tensor is rather obscure.

Bonazzola [8] has looked for a link between the component $T^4{}_{444}$ of the Bel-Robinson tensor and the differences of the gravitational energy.

Our attempt to interpret the Bel-Robinson tensor is different from that of Bonazzola: it is based on the standard formalism of the general relativity. We start from the fact that the first generalized normal tensor [9], [10], ${}_Et_\mu{}^\nu{}_{\varrho\sigma}$, belonging to the Einstein canonical pseudotensor, ${}_Et_\mu{}^\nu$, contains the Bel-Robinson tensor. It is interesting that our construction can be performed most easily for the canonical pseudotensor ${}_Et_\mu{}^\nu$.

2. *The physical interpretation of the Bel-Robinson tensor*

Einstein's canonical pseudotensor for the gravitational field is defined by

$$(-g)^{\frac{1}{2}}{}_Et_\mu{}^\nu = (-g)^{\frac{1}{2}}L\delta^\nu_\mu - g^{\varrho\sigma}{}_{,\mu}\frac{\partial[(-g)^{\frac{1}{2}}L]}{\partial g^{\varrho\sigma}{}_{,\nu}}. \tag{2.1}$$

Here

$$L = \frac{1}{16\pi G}g^{\mu\nu}[\Gamma^\varrho_{\mu\sigma}\Gamma^\sigma_{\nu\varrho} - \Gamma^\varrho_{\mu\nu}\Gamma^\sigma_{\varrho\sigma}],$$

G is the Newtonian gravitational constant; a comma denotes the ordinary derivative.

In the following we limit ourselves to the normal coordinate systems[1] [9], [10], [12], [13]. In these coordinate systems a local, covariant analysis of the gravitational field is possible [14]. It is known [9 — 13] that starting from the general coordinate system, U, we can always geometrically construct the normal coordinate system, $NCS(U; P)$, which is geodesic in P. The point P is the origin of this coordinate system. This coordinate system belongs to the general system U and the point P. If the coordinates U are transformed into U', we get a new normal coordinate system, $NCS(U'; P)$, belonging to the same point P. The transformation $NCS(U; P) \to NCS(U'; P)$ is a linear homogeneous transformation with constant coefficients. Consequently, in the class of normal coordinate systems, $[NCS(U; P)]$, the formal integrals

$$\int_\Sigma {}_Et_a{}^b d\sigma_b = \overline{P}_a(\Sigma) \tag{2.2}$$

form a free vector. ${}_Et_a{}^b$ is the Einstein canonical pseudotensor. $a, b, c, d, e, f, g, = 1, 2, 3, 4$ denote the tensor indices in a normal coordinate system. Σ is a sufficiently small[2] and fixed space-like hypersurface in the neighbourhood of P.

We shall refer the free vector, $\overline{P}_a(\Sigma)$, to the point P and obtain the normal vector $P_a(\Sigma; P)$. $P_a(\Sigma; P)$ can be considered as a vector $P_\mu(\Sigma; P)$ in a general coordinate system U [10]. Let us construct a normal coordinate system in every point P of the neighbourhood

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[1] Our normal coordinates are often called the "Riemann coordinates", see Eisenhart [13]. The normal coordinates of Eisenhart are our "orthogonal normal coordinates".

[2] The hypersurface Σ must lie entirely in the four-dimensional region in which normal coordinates are defined.

of Σ and refer the integrals (2.2) calculated in every normal coordinate system to the suitable P. In this way we obtain the vector field $P_a(\Sigma)$. This is also the vector field $P_\mu(\Sigma)$ in the general coordinate system U [10]. Thus, in general relativity, we can connect locally with the gravitational field a vector field $P_\mu(\Sigma)$. Obviously, the same can be done for the integrals

$$\int_\Sigma T_\nu^\mu d\sigma_\mu$$

where $T^{\mu\nu} = T^{\nu\mu}$ is the symmetric energy-momentum tensor of matter.

In the fixed normal coordinate system $NCS(U; P)$ let us consider a small four-dimensional region Ω defined by two 3-dimensional regions V_0 and V_1 which are located on two space-like hypersurfaces Σ_0 and Σ_1 respectively and by 3-dimensional space-like walls Γ. Using Gauss' theorem we have:

$$\Delta P_a(V_1, V_0; P) = \int_{V_1} {}_E t_a{}^b d\sigma_b - \int_{V_0} {}_E t_a{}^b d\sigma_b = -\int_\Gamma {}_E t_a{}^b d\sigma_b, \tag{2.3}$$

because $[(-g)^{\frac{1}{2}} {}_E t_a{}^b]_{,b} = 0$.

Let Σ_0 and Σ_1 be given by the equations $y^4 = 0$ and $y^4 = \tau$ respectively and let V_0, V_1 be the spheres of the same radius r (see Fig. 1), where $r = [(y^1)^2 + (y^2)^2 + (y^3)^2]^{\frac{1}{2}}$.

We then have

$$\Delta P_a(V_1, V_0; P) = \int_{V_1} {}_E t_a{}^4 d^3v - \int_{V_0} {}_E t_a{}^4 d^3v, \tag{2.4}$$

where $d^3v = (-g)^{\frac{1}{2}} dy^1 dy^2 dy^3$.

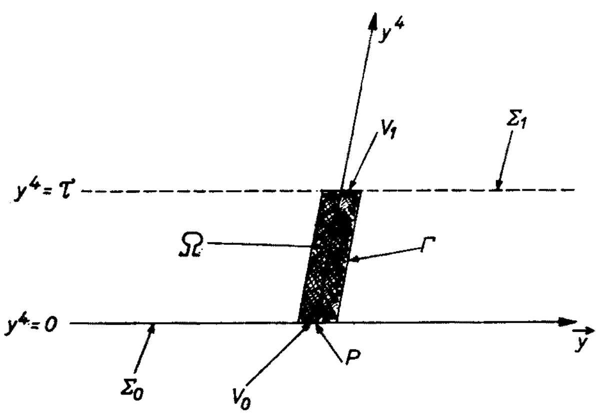

Fig. 1

Let us consider the case when r is infinitesimal and τ is very small. Then, with good accuracy

$$\Delta P_a(V_1, V_0; P) \to \delta P_a(V_1, V_0; P) \cong \tfrac{1}{2}\big(\overset{0}{{}_E t_a{}^4}{}_{,44} (\Delta\tau)^2 +$$

$$+ \overset{0}{{}_E t_a{}^4}{}_{,4} \Delta\tau + \overset{0}{{}_E t_a{}^4} - \overset{0}{{}_E t_a{}^4} \big) V = \tfrac{1}{2} \overset{0}{{}_E t_a{}^4}{}_{,44} (\Delta\tau)^2 V, \tag{2.5}$$

because $\overset{0}{{}_Et_a{}^4} = 0$, $\overset{0}{{}_Et_a{}^4{}_{,4}} = 0$. $\delta P_a(V_1, V_0; P)$ means the difference $P_a(V_1; P) - P_a(V_0; P)$ of the energy-momentum of the free gravitational field between instants $y^4 = \tau$ and $y^4 = 0$ for the infinitesimal 3-region V which equals V_0 for $y^4 = 0$ and equals V_1 for $y^4 = \tau$. The integral $\int_{V_1} d^3v$ differs from the integral $\int_{V_0} d^3v$ by terms of fourth order in the normal coordinates y^a. Consequently, in our approximation in which only quadratic terms are preserved, we may write $\int_{V_1} d^3v = \int_{V_0} d^3v \overset{\text{df}}{=} V$. We have incorporated this in (2.5). $\Delta\tau$ is a small interval defined by the equation $y^4 = \tau$. The index "0" above a quantity denotes its value at the point P. Obviously, the differences δP_a form a vector with respect to the group of transformations of the normal coordinate system $NCS(U; P)$. It is easy to obtain the first, nonvanishing in P, term of the expansion of the canonical pseudotensor ${}_Et_a{}^b$ in the neighbourhood of P. In vacuum it has the form

$$\frac{1}{2}\,\overset{0}{{}_Et_a{}^b{}_{,cd}}\,y^c y^d = \frac{1}{16\pi G}\,\frac{1}{2}\,\frac{2}{9}\Big[\overset{0}{T^b{}_{acd}} + \overset{0}{\overline{T}^b{}_{acd}} -$$

$$- \tfrac{1}{2}\,\delta^b_a\big(\overset{0}{R^{efg}{}_c}\overset{0}{R_{egfd}} + \overset{0}{R^{efg}{}_d}\overset{0}{R_{egfc}}\big)\Big]y^c y^d. \tag{2.6}$$

Here $T^b{}_{acd}$ denotes the Bel-Robinson tensor and $\overline{T}^b{}_{acd}$ the tensor

$$R^{bef}{}_c R_{defa} + R^{bef}{}_d R_{cefa} - \tfrac{1}{2}\,\delta^b_a R^{efg}{}_c R_{efgd}. \tag{2.7}$$

The tensor (2.7) differs from the Bel-Robinson tensor by the position of the indices a and d in the first term and by the position of the indices a and c in the second term. $\overset{0}{{}_Et_a{}^b{}_{,cd}}(P)$ is the generalized normal tensor belonging to the canonical pseudotensor and to the point P. By means of the above-mentioned procedure [10] we can obtain the field of the generalized normal tensor ${}_Et_a{}^b{}_{,cd}$. In the general coordinate system U this field has the form

$${}_Et_\mu{}^\nu{}_{\varrho\sigma} = \frac{1}{16\pi G}\,\frac{2}{9}\Big[T^\nu{}_{\mu\varrho\sigma} + \overline{T}^\nu{}_{\mu\varrho\sigma} -$$

$$- \tfrac{1}{2}\,\delta^\nu_\mu\big(R^{\alpha\beta\gamma}{}_\varrho R_{\alpha\gamma\beta\sigma} + R^{\alpha\beta\gamma}{}_\sigma R_{\alpha\gamma\beta\varrho}\big)\Big]. \tag{2.8}$$

This tensor field describes, in a covariant manner, the variations energy and momentum density in the free gravitation from point to point.

Let us calculate the differences (2.5) in the orthogonal normal coordinate system, $ONCS(U; P)$, *i.e.*, in $NCS(U; P)$ for which

$$\overset{0}{g_{ab}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.9}$$

In this particular normal coordinate system

$$\overset{0}{\underset{-}{T}}{}^{4}{}_{a44} = \overset{0}{T}{}^{4}{}_{a44} \tag{2.10}$$

and, consequently

$$\overset{0}{{}_{E}t}{}_{a}{}^{4}{}_{,44} = \frac{1}{16\pi G}\cdot\frac{4}{9}\,[\overset{0}{T}{}^{4}{}_{a44} - \tfrac{1}{2}\,\delta_a{}^4 \overset{0}{R}{}^{efg}{}_4 \overset{0}{R}{}_{egf4}]. \tag{2.11}$$

On introducing this into (2.5) we get:

$$\delta P_1 = \frac{1}{16\pi G}\cdot\frac{2}{9}\,\overset{0}{T}{}^{4}{}_{144}(\Delta\tau)^2 V,$$

$$\delta P_2 = \frac{1}{16\pi G}\cdot\frac{2}{9}\,\overset{0}{T}{}^{4}{}_{244}(\Delta\tau)^2 V,$$

$$\delta P_3 = \frac{1}{16\pi G}\cdot\frac{2}{9}\,\overset{0}{T}{}^{4}{}_{344}(\Delta\tau)^2 V,$$

$$\delta P_4 = \frac{1}{16\pi G}\cdot\frac{2}{9}\left(\overset{0}{T}{}^{4}{}_{444} - \frac{1}{2}\,\overset{0}{R}{}^{efg}{}_4 \overset{0}{R}{}_{egf4}\right)(\Delta\tau)^2 V. \tag{2.12}$$

We see that the differences $\delta P_a(V_1, V_0; P)$ of the normal vector of the energy-momentum of the free gravitation are to a good approximation proportional to the components $\overset{0}{T}{}^{4}{}_{a44}$ of the Bel-Robinson tensor. This fact affords a possible interpretation of these components.

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REFERENCES

[1] L. Bel, *CR Acad. Sci.* (France), **247**, 1094 (1958).
[2] L. Bel, *CR Acad. Sci.* (France), **249**, 1297 (1959).
[3] L. Bel, *Cahiers Phys.*, **16**, 59 (1962).
[4] R. K. Sachs, *Z. Phys.*, **157**, 462 (1960).
[5] A. Komar, *Phys. Rev.*, **164**, 1595 (1967).
[6] J. D. Zund, *Tensor NS*, **21**, 354 (1970).
[7] F. Öktem, *Nuovo Cimento*, **58B**, 167 (1968).
[8] S. Bonazzola, *Nuovo Cimento*, **52B**, 187 (1967).
[9] O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge University Press 1933.
[10] J. A. Schouten, *Ricci-Calculus*, 2nd edition, Springer, Berlin 1954.
[11] S. Gołąb, *Tensor Calculus* (in Polish), 2nd edition, PWN, Warsaw 1966.
[12] A. Z. Petrov, *New Methods in General Relativity* (in Russian), Moscow 1966.
[13] L. P. Eisenhart, *Riemannian Geometry*, Sixth Printing, Princeton 1966.
[14] F. A. E. Pirani, *Phys. Rev.*, **105**, 1089 (1957).