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A PHYSICAL ANALYSIS ON THE COVARIANCE OF QUANTUM FIELD THEORY

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In this paper a model field theory is proposed especially for the study of general covariance. A detailed analysis is given on the determination of physical states, the role of coordinate conditions and the fundamental differences between a generally covariant theory and a Lorentz covariant one.

1. Introduction

Consider a general four-dimensional space which is not assumed to possess a metric but in which coordinate frames may be set up. Let A_μ $(\mu = 0, 1, 2, 3)$ be a covariant vector and $\varepsilon^{\tau\kappa\mu\lambda}$ be the Levi-Civita symbol which is a tensor density of weight 1 [1]. Then

$$\varepsilon^{\tau\kappa\mu\lambda}(A_{\kappa,\tau}-A_{\tau,\kappa})(A_{\mu,\lambda}-A_{\lambda,\mu}),$$

is a scalar density of weight 1. We may therefore use this as a Lagrangian density to obtain a variational principle

$$\delta\int \mathscr{L} d^4x = 0, \text{ where } \mathscr{L} = \tfrac{1}{4}\varepsilon^{\tau\kappa\mu\lambda}(A_{\kappa,\tau}-A_{\tau,\kappa})(A_{\mu,\lambda}-A_{\lambda,\mu}),$$

which leads to the covariant field equations

$$\varepsilon^{\alpha\beta\tau\kappa}A_{\kappa,\tau\beta} = 0. \tag{1.1}$$

The general solutions are just four arbitrary functions since the left-hand side of (1.1) is identically zero, a fact which could have been anticipated because $\mathscr{L}$ may be written as the divergence

$$(\varepsilon^{\tau\kappa\lambda\mu}A_{\kappa,\tau}A_\mu)_{,\lambda}.$$

Hence we have a field with no genuine field equations in the usual sense. Still we shall formally carry on with the Hamiltonian formulation [2] to see what sort of quantum theory turns out at the end.

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2. *Hamiltonian formulation*

Define canonical momenta conjugate to A_μ by

$$P^\mu = \frac{\partial \mathscr{L}}{\partial A_{\mu,0}} = 2\varepsilon^{0\mu\varrho\sigma}A_{\varrho,\sigma}.$$

One immediately obtains four primary constraints $\varphi^\mu \approx 0$ with

$$\varphi^0 \equiv P^0,$$

$$\varphi^j \equiv P^j - 2\varepsilon^{0jmn}A_{m,n}.$$

The total Hamiltonian density is

$$\mathscr{H}_T = \varepsilon^{j\varrho i\sigma}A_{\varrho,j}A_{\sigma,i} + U_\mu\varphi^\mu,$$

where U_μ are arbitrary functions of x^τ. The constraints are all first class and there are no further consistency conditions. The canonical equations of motion for A_μ and P^μ give $A_{\mu,0} = U_\mu$ which are left to be arbitrary, and $P^\mu_{,0} = 2\varepsilon^{0\mu\varrho\sigma}A_{\varrho,\sigma 0}$.

Again there are no genuine equations of motion, as expected. The integrated total Hamiltonian is

$$H_T = \int d^3x\mathscr{H}_T = \int d^3xU_\mu\varphi^\mu \approx 0.$$

We have dropped the first term in $\mathscr{H}_T$ which does not contribute to the equations of motion and is a perfect divergence. We may go on formally to discuss physical states of the field. Consider an infinite constant x^0 surface S in the four-dimensional space. A specification of the set of values of A_μ and P^μ on S should define a physical state of the field on that surface. However as we move away from this initial surface, the values of A_μ become completely arbitrary. We must conclude that all the different sets of A_μ, P^μ on another constant x^0 surface S' correspond to the same physical state on that surface. We now have the situation that completely arbitrary A_μ correspond to the same physical state on S'. By reversing the procedure, one has the same situation on S. Therefore we consider that only one physical state is possible for our model field theory.

3. *Quantization* [3]

We assume the existence of a linear vector space such that A_μ, P^μ are operators in this space with commutation relations

$$[A_\mu(x), P^\nu(x')] = i\hbar c\delta^\nu_\mu\delta(x-x'); \quad [A_\mu(x), A_\nu(x')] = 0; \quad [P^\mu(x), P^\nu(x')] = 0.$$

State vectors $|\Psi\rangle$ are those vectors which satisfy the subsidiary conditions

$$\Phi^\mu(x)\,|\psi\rangle = 0. \tag{3.1}$$

This procedure works since the first class nature of the constraints as operators is preserved in our case. (3.1) may be solved in the Schrödinger picture using the functional representation in which $P^{\mu}(x)$ is represented by $-i\hbar c\delta/\delta A_{\mu}(x)$ and $|\Psi\rangle$ becomes a functional of $A_{\mu}(x)$. Then (3.1) becomes

$$(i\hbar c\delta/\delta A_{\mu}+2\varepsilon^{0\mu\varrho\sigma}A_{\varrho,\sigma})\,|\psi\rangle = 0,$$

with the solution

$$|\Psi\rangle = \text{constant}\,\exp\left(\frac{i}{c\hbar}\int\varepsilon^{0jmn}A_jA_{m,n}d^3x\right).$$

This is rather a surprising result showing that there is only one physical state although it agrees with the previous analysis in classical theory. We note that

$$H_T|\Psi\rangle = 0, \tag{3.2}$$

which implies that $|\Psi\rangle$ is a time-independent state vector corresponding to a zero eigenvalue of H_T. All this is most reasonable since we actually started with an "empty" field.

4. *Transformation properties*

Consider an infinitesimal coordinate transformation from x^{μ} to $x'^{\mu} = x^{\mu}+\varepsilon^{\mu}$. As a result $A_{\mu}(x)$, $P^{\mu}(x)$ go to $A'_{\mu}(x)$, $P'^{\mu}(x)$. This transformation in classical theory is effected by the generating functional

$$G = \int d^3x\overline{\delta A}_{\mu}(P^{\mu}-2\varepsilon^{0\mu mn}A_{m,n}), \tag{4.1}$$

where

$$\overline{\delta A}_{\mu} = A'_{\mu}(x)-A_{\mu}(x) = \delta A_{\mu}-A_{\mu,\nu}\varepsilon^{\nu},$$

$$\delta A_{\mu} = A'_{\mu}(x')-A_{\mu}(x) = -A_{\nu}(x)\varepsilon^{\nu}{}_{,\mu}.$$

The involvement of velocity variables $A_{\mu,0}$ is unavoidable because the transformation law for A_{μ} contains such quantities explicitly. However there are in fact no real difficulties in calculations as the $\overline{\delta A}_{\mu}$ are multiplied by the weakly vanishing constraints. The preservation of the constraint equations may be readily verified.

In quantum theory the corresponding infinitesimal unitary transformation is effected by

$$U = (1+iG/\hbar),$$

where G is formally the same as in (4.1) except that the variables occurring are now operators. This generator G is Hermitian in the sense defined by Dirac [4]. We see that the present theory confirms Dirac's statement that physical vectors and operators are invariants with respect to coordinate transformations. We observe that this is the case even for a Lorentz transformation which is surprising since this is not true for the usual Lorentz invariant field theory. For example the generators of time and space translations are the components of the energy-momentum vector, not scalars. The above observation is not restricted to our present model only. A thorough examination of this point will be given presently.

5. *Coordinate conditions*

Since the solutions of the field equations consist of four arbitrary functions we are free to impose four conditions on the field variables. One possibility would be the requirements

$$\eta^{\varrho\sigma} A_{\mu,\varrho\sigma} = 0, \tag{5.1}$$

where $\eta^{\varrho\sigma} = (1, -1, -1, -1)$ is the Minkowskian metric. The common practice is to interpret such A_μ as the field viewed by an observer in a particular type of coordinate system. We can actually manufacture all sorts of arbitrary conditions leading to totally different effective field equations. A paradoxical situation appears. An observer with conditions (5.1) may treat the field as a zero mass vector meson field. It will then appear to him that different physical states exist defined by the various plane wave solutions according to the usual way of counting states. All this contradicts our previous conclusion that only a single physical state is possible. However imposition of conditions (5.1) merely reduces the completely arbitrary nature of the field variables. The fact that all the different field variables compatible with (5.1) still represent the same physical state cannot be changed. Let us go a step further to see why this must be the case and indeed to see exactly how the paradox arises in the first place. In a generally covariant theory it is usually assumed that all coordinate systems are equivalent for the description of the physical system concerned and that field equations must be generally covariant. What is sometimes forgotten is that the coordinate variables x^μ appearing in the covariant field equations cannot be identified with a particular observer's coordinates until we impose further restrictions on the field variables. In order to relate the x^μ appearing in the covariant equations to the actual coordinates of a particular observer it is necessary to impose non-covariant coordinate conditions.

Furthermore the number of such conditions must be such that they exclude all but one coordinate system [5], [6], [7]. Only after all this has been done can we identify x^μ as the actual coordinates of an observer. The observer may now proceed in the usual way to define physical states for the system concerned. Two distinct solutions satisfying all those conditions and the field equations will now mean two distinct physical states. The conditions (5.1), though four in number, obviously do not satisfy the above requirements. These conditions only restrict coordinate systems to a special set related by Lorentz transformations. Even at this stage it is still not permissible to identify the x^μ in (5.1) with the actual coordinates of a member of the set. Hence two distinct solutions do not in general represent two different states. In order to achieve a one-one relation between solutions and states we have to impose further conditions (which may take the form of initial data on the $x^0 = 0$ surface) in such a way as to exclude any Lorentz transformation. This initial data will then lead to a unique solution to (5.1). Therefore we see that for a particular observer there is indeed only one unique set of values for the field variables leading to only a single physical state. Different solutions to (5.1) are now seen to correspond to the same state as viewed by different observers. One must be very careful in handling coordinate conditions and in the subsequent interpretation. It is not sufficient just to count the number of coordinate conditions.

A similar situation exists in quantum theory. One should not take (5.1) as field equations to carry out quantization and interpret the results in the usual manner. There is nothing to prevent one from blindly quantizing the theory with arbitrary coordinate conditions such as (5.1), but having done so one may have to identify physical states wich subspaces rather than single vectors in the linear vector space. The vectors in the subspaces are related by unitary transformations as allowed by the coordinate conditions.

6. *General covariance versus Lorentz covariance*

6.1. Classical theory

From the discussion in the previous section, a fundamental difference between a generally covariant theory and a Lorentz covariant one emerges. In the latter case the equations of motion are Lorentz covariant. The important point now is that the coordinate variables x^μ appearing in the field equations in a Lorentz covariant theory are meant to be the actual coordinates of an inertial observer (measured on standard metre sticks and clocks at rest relative to himself). As a result, two distinct solutions to the field equations imply two distinct physical states. In a Lorentz covariant theory this is not an arbitrary assumption and the above conclusion may be tested by physical measurements made by the observer.

A Lorentz covariant theory may be extended to become formally generally covariant by the introduction of more variables such as the non-Minkowskian metric $g_{\mu\nu}$. The best know example is the extension of Maxwell's theory of electromagnetic field to a formally generally covariant theory [8], [9]. Let us examine the situation step by step in detail.
(1) The original Lorentz covariant theory of electromagnetic field:

The space-time is assumed flat. We have a group of inertial reference frames related by Lorentz transformations and in which the metric tensor is $g_{\mu\nu} = \eta_{\mu\nu}$. The electromagnetic field is described by the antisymmetric electromagnetic field tensor $F_{\mu\nu}$ with the field equations

$$F^{\mu\nu}(x)_{,\nu} = 0,$$
$$F_{\mu\nu}(x)_{,\lambda}+F_{\lambda\mu}(x)_{,\nu}+F_{\nu\lambda}(x)_{,\mu} = 0. \tag{6.1}$$

The x^μ are the actual coordinates of a particular inertial observer who can enumerate and determine different physical states of the field from a knowledge of the distinct solutions to (6.1). A Lorentz transformation from x^μ to x'^μ will lead to another set of field equations with x'^μ as coordinate variables. The new equations will be identical with (6.1) in form. The new variables x'^μ are to be interpreted as the actual coordinates of another inertial observer who may count physical states in exactly the same way as the first observer does. It is in this sense we say that all inertial observers are physically equivalent in this particular context.
(2) Extension to a "formally" covariant theory:

One can establish a new set of field equations which is generally covariant and which reduces to (6.1) in an inertial frame. To do this we introduce arbitrary coordinate variables x^μ

in terms of which the metric tensor $g_{\mu\nu} \neq \eta_{\mu\nu}$ in general even though the flat nature of the space-time has not been changed. The electromagnetic field is described by an antisymmetric tensor $F^{\mu\nu}(x)$ which satisfies the generally covariant equations

$$F^{\mu\nu}(x)_{;\nu} = 0,$$

$$F_{\mu\nu}(x)_{;\lambda} + F_{\lambda\mu}(x)_{;\nu} + F_{\nu\lambda}(x)_{;\mu} = 0. \tag{6.2}$$

In this extended theory, we have brought in 10 new variables $g_{\mu\nu}$ which satisfy 20 equations expressing the flatness of the space-time

$$R_{\alpha\beta\mu\nu} = 0, \tag{6.3}$$

where $R_{\alpha\beta\mu\nu}$ is the Riemann curvature tensor. Since we are given that the original Lorentz covariant theory is the correct one, the extension leads to nothing physically new at all. There can be no ambiguity in fixing various physical states as we can refer things back to an inertial observer. It should be clear from our previous analysis that the arbitrary coordinate variables x^μ in (6.2) cannot be identified with the actual coordinates of a particular (non-inertial) observer for the purpose of state determination. In this extended theory a whole set of different solutions to (6.2) may correspond to the same physical state.

(3) Reduction of the extended theory to the original Lorentz covariant theory:

The reduction process is trivial. All we need to do is to impose the coordinate conditions $g_{\mu\nu} = \eta_{\mu\nu}$.

(4) A fundamental question:

Consider a completely new situation. Suppose we do not know the original Lorentz covariant theory of electromagnetic field, and suppose we are given a theory formulated in terms of arbitrary coordinate variables x^μ in a generally covariant manner with the field described by an antisymmetric tensor $F^{\mu\nu}(x)$ satisfying equations (6.2) while the $g_{\mu\nu}$ satisfy equations (6.3). Now how does one count and determine physical states? The analysis given in the previous section tells us that there is no unique answer to this question as it stands. One can attempt to answer the question in one of the following two ways.

(a) Although we are not given any preferred coordinate systems, we may still make the assumption that there are preferred reference frames defined by certain coordinate conditions, say, $g_{\mu\nu} = \eta_{\mu\nu}$.

Furthermore one assumes that after the imposition of the coordinate conditions the coordinate variables x^μ in the field equations may be identified with the actual coordinates of a particular observer who can then count physical states in the usual way. It is in this sense these reference frames are termed "preferred". It is not because that the field equations become simpler using them. An important observation must be emphatically pointed out here, that is, the above are new **physical** assumptions not contained in the original theory. These assumptions imply that the theory given is only formally generally covariant and is extended from a Lorentz covariant theory.

(b) One considers the given theory as a "genuinely" generally covariant theory despite the fact that field equations become simpler in a certain set of coordinate frames. There are therefore no preferred set of reference frames in the above sense. One has to impose sufficient number of coordinate conditions (*e.g.* more than $g_{\mu\nu} = \eta_{\mu\nu}$) to single out a unique coordinate frame and only in such a unique frame can we start to distinguish physical states in the conventional way. Note that one is again making new physical assumptions here.

We see clearly now that additional physical assumptions are necessary to answer the fundamental question raised. There is nothing in the given theory which tells us definitely what is the correct answer. The final test must lie in the actual physical measurements for the determination of states. By comparing experimental results with the predictions of (a) and (b) respectively we can find out which of them is the correct one.

More examples may be given. An extreme one would be to consider the relationship between our model field and the Lorentz covariant vector meson field theory. The above arguments may be repeated step by step leading to the same conclusion.

6.2. Quantum theory

In quantum theory a similar situation exists. In a "genuinely" generally covariant theory, physical states are described only by physical vectors which are invariant with respect to arbitrary transformations of the coordinate variables x^μ, in particular Lorentz transformations. Vectors which are not Lorentz invariant cannot be used to describe states. It is therefore clear that two such vectors related by a Lorentz transformation cannot be regarded as representing two physical states. All this is fundamentally different from a Lorentz covariant quantum theory (or a "formally" generally covariant quantum theory). Another striking feature of a "genuinely" generally covariant quantum theory is that the physical system concerned seems to be "dead" [7] in that the state vectors in Schrödinger picture are "time"-independent on account of the weakly vanishing nature of the Hamiltonian. We can resolve this paradox with the help of our present analysis. To illustrate the situation more vividly, let us consider the parametric formulation of classical mechanics [10]. In such theory the time t is promoted to the status of an additional canonical variable q_0 while another variable τ is introduced to act as the independent variable. The original variational principle is

$$\delta \int L(q_1, \ldots, q_n, t; \dot{q}_1, \ldots, \dot{q}_n) dt = 0, \quad \text{where} \quad \dot{q}_i = \frac{dq_i}{dt}.$$

In the parametric formulation the variational principle becomes

$$\delta \int \mathscr{L}_1 d\tau = 0,$$

where

$$\mathscr{L}_1 = L\left(q_1, \ldots, q_n, q_0; \frac{q_1'}{q_0'}, \ldots, \frac{q_n'}{q_0'}\right) q_0'; \quad q_\mu' = \frac{dq_\mu}{d\tau} \quad \text{and} \quad \mu = 0, 1, \ldots, n.$$

The canonical momentum conjugate to q_0 may be defined and the Hamiltonian formulation set up in the usual way. $\mathscr{L}_1$ being homogeneous in the velocity variables q_μ' of degreee one implies a vanishing Hamiltonian for the system. There will certainly be one primary constraint due to the introduction of an additional variable. Let it be

$$\varphi(q_1, \ldots, q_n, q_0, p_1, \ldots, p_n, p_0) \approx 0.$$

Assuming there are no more consistency equations nor constraints, the total Hamiltonian is

$$H_T = U\varphi \approx 0,$$

U being an arbitrary coefficient. With this Hamiltonian, equations of motion of various canonical variables may be obtained. Among other things this parametric formulation is highly valued for its consistency with the spirit of special relativity in that spatial coordinates and the time are treated on an equal footing. Instead of the variable t which is the actual time coordinate of an observer (measured by his clock), an unspecified variable τ is introduced to act as the independent variable of the theory. The τ, being unspecified, can no longer be identified with the actual time coordinate of an observer. Now one may go one step further to quantize the parametric theory. The constraint will become the subsidiary condition imposed on physical vectors. As a result, physical vectors in the Schrödinger picture are "time"-independent, that is,

$$\frac{\partial}{\partial \tau}|\Psi\rangle = 0, \qquad H_T|\Psi\rangle = 0. \tag{6.4}$$

Let us illustrate the situation with a concrete one-dimensional example of a particle in an external potential V. The Lagrangian is

$$L = \tfrac{1}{2} m\dot{q}_1^2 - V(q_1).$$

In the parametric formulation we introduce two new variables $q_0 = t$ and τ;

$$\mathscr{L}_1 = \left[\frac{1}{2} m \left(\frac{q_1'}{q_0'}\right)^2 - V(q_1)\right] q_0'.$$

The primary constraint is

$$\varphi = P_0 + \frac{P_1^2}{2m} + V(q_1) \approx 0.$$

The equations of motion are obtained from the total Hamiltonian

$$H_T = U\varphi \approx 0,$$

U being an arbitrary function of q_0 and q_1. To quantize we have

$$P_0 \to -i\hbar\partial/\partial q_0; \qquad P_1 \to -i\hbar\partial/\partial q_1.$$

The constraint equation now takes the form

$$\left(P_0 + \frac{P_1^2}{2m} + V(q_1)\right)|\Psi\rangle = 0,$$

or

$$i\hbar \frac{\partial}{\partial q_0} \Psi(q_0, q_1, \tau) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q_1^2} + V(q_1) \right) \Psi(q_0, q_1, \tau).$$

The Schrödinger equation of motion is

$$i\hbar \frac{\partial}{\partial \tau} \Psi = H_T \Psi = 0.$$

Thus the wave function Ψ is actually independent of the parameter τ while its q_0-dependence is given by the standard Schrödinger equation. We have the apparent paradox showing up clearly now. However in the present case we know with certainty that the physical system as viewed by an individual observer is not changeless — the wave function does indeed depend on his actual time coordinate q_0. This example shows that expressions like (3.2), (6.4) cannot be interpreted literally and that they do not imply a changeless situation. In general x^0 and τ in these expressions cannot be identified with clock readings of physical observers, nor can H or H_T be interpreted as a physical energy.

7. Harmonic coordinates in general relativity

Fock [9] has suggested that the harmonic coordinate conditions

$$\{\sqrt{-g}\, g^{\mu\nu}\}_{,\mu} = 0,$$

together with certain conditions at infinity lead to a preferred set of coordinate systems. In particular, he has shown, though not with complete mathematical rigour, that in the case of an isolated system of masses the harmonic conditions together with some proper supplementary conditions determine the coordinate system uniquely apart from Lorentz transformations. We shall not go into the philosophical argument as to what preferred coordinate systems mean in this context. Instead we set ourselves the following definite question. Is it possible to tell by physical means whether General Relativity is "genuinely" generally covariant or whether it is just "formally" generally covariant and is extended from a theory covariant to a more restricted set of coordinate transformations such as the set of harmonic coordinates? According to our previous analysis the answer should be affirmative. In the case of harmonic coordinates we can do one of the two things. We can identify the coordinate variables with the actual coordinates of an "harmonic" observer for the purposes of determining physical states as Fock apparently did. The results can then be put to a physical test. An experimental confirmation on one way or the other will solve our problem which may not be conclusively answered by theoretical argument alone.

8. Concluding remarks

One can express the ideas of the previous sections in the language of group representation theory. In ordinary Lorentz invariant field theory, if we have two inertial observers S' and S whose coordinates are related by $x'^\lambda = L^\lambda_\mu x^\mu + a^\lambda$, then the relation between their quantum state vectors is of the form

$$|\Psi\rangle' = D(L, a)\,|\Psi\rangle. \tag{8.1}$$

Here $D(L, a)$ is a unitary representation of the Poincaré group in the Hilbert space of state vectors. The generators of infinitesimal translations and infinitesimal Lorentz transformations are the energy-momentum vector and the angular momentum tensor respectively, which satisfy the standard commutation relations. The observables of the theory are tensor operators belonging to representations of the Poincaré group. In contrast, if we have a genuinely fully covariant quantum field theory, a change of coordinates $x^\lambda \to x'^\lambda$ must be regarded as a mere "gauge transformation" which leaves the physical vector unchanged, *i.e.* $|\Psi\rangle' = |\Psi\rangle$. Thus in this case the only admissible representation of the Einstein group is the identity representation and all the generators are represented by zero. All observables must be scalars. One wonders whether this severe requirement might not be too restrictive for the quantization of the gravitational field. Perhaps one should instead have a relation of the form (8.1) with the Poincaré group replaced by some "physical" group which describes the relation between a preferred set of observers. Fock has in fact conjected that his "harmonic" coordinate systems transform among themselves according to a group isomorphic to the Poincaré group [11].

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[1] Notation: $x^0 = ct$, $\boldsymbol{x} = (x^1, x^2, x^3)$. Unless otherwise stated lower case Greek indices take the values 0, 1, 2, 3 and lower case Latin indices the values 1, 2, 3.

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