THE 4-SURFACE OF STATIONARY VOLUME EMBEDDED IN A 5-DIMENSIONAL PSEUDO-EUCLIDEAN SPACE I

By Kay-Kong Wan and G. H. Derrick

Department of Theoretical Physics, St. Andrews University*

(Received December 20, 1972)

The 4-surface of stationary 4-volume embedded in a 5-dimensional pseudo-Euclidean space is studied as a model field theory. This model field shares many essential features with Einstein's General Relativity. In particular it is generally covariant and intrinsically nonlinear, yet it is much simpler and more manageable especially in quantum theory. It is hoped that a study of such a simpler model will help towards a similar study of much more complex theories such as General Relativity.

1. *Introduction*

Consider a 5-dimensional pseudo-Euclidean space with coordinates ξ^A [1] and metric $\eta_{AB} = (1, -1, -1, -1, -1)$. Any 4-dimensional surface may be fixed by specifying the five coordinates ξ^A as functions of 4 parameters x^μ. In general the 4-surface is a 4-dimensional Riemannian space with a metric

$$g_{\mu\nu}(x) = \eta_{AB}\xi^A_{,\mu}\xi^B_{,\nu}.$$

Let $\underline{\xi}$ denote a column vector with components ξ^A and $\overline{\underline{\xi}}$ denotes a row vector with components $\xi_A = \eta_{AB}\xi^B$. Then we have

$$g_{\tau\kappa} = \overline{\underline{\xi}}_{,\tau}\underline{\xi}_{,\kappa}.$$

The Christoffel symbols of the first kind and second kind are

$$[\tau\kappa, \lambda] = \tfrac{1}{2}(g_{\tau\lambda,\kappa}+g_{\kappa\lambda,\tau}-g_{\tau\kappa,\lambda}) = \overline{\underline{\xi}}_{,\tau\kappa}\underline{\xi}_{,\lambda} = \overline{\underline{\xi}}_{,\lambda}\underline{\xi}_{,\tau\kappa};$$

$$\Gamma^\lambda_{\tau\kappa} = g^{\lambda\mu}[\tau\kappa, \mu] = g^{\lambda\mu}\overline{\underline{\xi}}_{,\mu}\underline{\xi}_{,\tau\kappa} = \overline{\underline{\xi}}^{\lambda}\underline{\xi}_{,\tau\kappa},$$

where $\overline{\underline{\xi}}^\lambda = g^{\lambda\mu}\overline{\underline{\xi}}_{,\mu}$.

---

* Address: Department of Theoretical Physics, St. Andrews University, North Haugh, St. Andrews, Fife, Scotland.

Note that $\underline{\xi}$ is a scalar with respect to arbitrary coordinate transformations in the 4-surface.

Covariant derivatives are defined in the usual way:

$$\underline{\xi}_{;\tau} = \underline{\xi}_{,\tau}, \text{ since } \underline{\xi} \text{ is a scalar.}$$

$$\begin{aligned} \underline{\xi}_{;\tau\kappa} &= \underline{\xi}_{,\tau\kappa} - \underline{\xi}_{,\lambda}\Gamma^{\lambda}_{\tau\kappa} = \\ &= \underline{\xi}_{,\tau\kappa} - \underline{\xi}_{,\lambda}\underline{\xi}^{\lambda}\underline{\xi}_{,\tau\kappa} = \\ &= \underline{\xi}_{;\tau\kappa}. \end{aligned}$$

The curvature tensors are given by

$$R_{\tau\kappa\lambda\mu} = \underline{\xi}_{;\lambda\tau}\underline{\xi}_{;\mu\kappa} - \underline{\xi}_{;\lambda\kappa}\underline{\xi}_{;\mu\tau},$$

$$R_{\kappa\lambda} = g^{\tau\mu}R_{\tau\kappa\lambda\mu}.$$

Now consider those 4-surfaces whose metric satisfies the following conditions. The metric $g_{\mu\nu}$ is nonsingular, *i.e.* $g = \det |g_{\mu\nu}| \neq 0$. Therefore the sign of g is the same throughout the surface and it is an invariant. The contravariant metric tensor $g^{\alpha\beta}$ exists such that $g^{\mu\beta}g_{\beta\nu} = \delta^{\mu}_{\nu}$.

We consider only surfaces with $g < 0$ so that the pseudo-Euclidean Minkowski flat surfaces are included as special cases. The volume of a domain $x^{\mu} \in D$ of the 4-surface is

$$\tau = \int_D \sqrt{-g}\, d^4x,$$

which is an invariant.

A 4-surface of stationary volume is defined to be a surface whose volume τ is at a stationary value with respect to small arbitrary deformation of the surface. The deformation is realized mathematically by variation of the coordinates ξ^A. The variation $\delta\xi^A$ is to be taken as zero at the boundary of the domain D. For a variation $\delta\xi^A$ we have

$$\delta\tau = -\int \delta\xi^B(\sqrt{-g}\, g^{\tau\kappa}\eta_{AB}\xi^A_{,\tau})_{,\kappa}d^4x.$$

The requirement that $\delta\tau$ vanishes for arbitrary $\delta\xi^A$ leads to the set of defining equations for a 4-surface of stationary volume

$$(\sqrt{-g}\, g^{\tau\kappa}\xi^A_{,\tau})_{,\kappa} = 0, \tag{1.1}$$

which may be written in covariant form

$$g^{\mu\nu}\xi^A_{;\mu\nu} = 0. \tag{1.2}$$

The above equations are not all independent and certain identities, the Bianchi identities [2], [3], exist among them. Rewriting (1.2) as

$$(1 - \underline{\xi}_{,\lambda}\underline{\xi}^{\lambda})g^{\tau\kappa}\underline{\xi}_{,\tau\kappa} = 0,$$

we see that the Bianchi identities are

$$\xi_{A,\varrho}(\delta^A_B - \xi^A_{,\mu}g^{\mu\nu}\xi_{B,\nu})g^{\tau\kappa}\xi^B_{,\tau\kappa} = 0$$

$\xi_{A,\varrho}$ being the four linearly independent null eigenvectors of $(1-\xi_{,\lambda}\bar{\xi}^{\lambda})$ Let n^A be the unit vector normal to $\xi_{A,\varrho}$, that is,

$$n^A = n'^A/|n'^A|, \qquad n'^A = \eta^{AB}\varepsilon_{BCDEF}\xi^C_{,0}\xi^D_{,1}\xi^E_{,2}\xi^F_{,3},$$

where ε_{BCDEF} is the 5-dimensional permutation symbol. Then the five field equations (1.2) are equivalent to the single equation

$$g^{\tau\kappa}n_A\xi^A_{,\tau\kappa} = 0.$$

We have now obtained a field theory with field variables ξ^A and field equations (1.1).

2. *Hamiltonian formulation and the constraints* [4], [5], [6]

Take the Lagrangian density for the field to be

$$\mathscr{L} = -Q\sqrt{-g},$$

where Q is a positive number of the dimensions of energy density. The minus sign is inserted to give a positive Hamiltonian. Define canonical momenta conjugate to ξ^A by

$$\pi_A = \frac{\partial\mathscr{L}}{\partial\xi^A_{,0}} = \mathscr{L}\eta_{AB}\xi^B_{,\mu}g^{0\mu}.$$

There are four primary constraints

$$\varphi_\mu \approx 0,$$

where $\varphi_0 = \pi^A\pi_A + Q^2\Delta^{00}$, $\varphi_j = \xi^A_{,j}\pi_A$, $\pi^A = \eta^{AB}\pi_B$, $\quad \Delta^{00} = gg^{00} = \det|g_{ij}|$.

$\mathscr{L}$ being a homogeneous function of $\xi^A_{,0}$ of degree one implies a vanishing Hamiltonian density. The total Hamiltonian density is therefore

$$\mathscr{H}_T = U^\mu\varphi_\mu,$$

where U^μ are arbitrary functions of x^μ. The constraints are all first class and there are no further consistency conditions. The equation of motion of any functional of ξ^A, π_A is derived from the integrated total Hamiltonian in the usual way.

3. *Covariant quantization*

The Dirac covariant quantization [6], [7] scheme may be applied to the present field. The ξ^A, π_A become operators satisfying the standard boson commutation rules and the constraints become subsidiary conditions

$$\varphi_\mu|\Psi\rangle = 0,$$

on physical vectors $|\Psi\rangle$. Somewhat lengthy calculation shows that the first class nature of the constraints as operators is preserved. Therefore we conclude that this quantization scheme can indeed be consistently carried out, a nontrivial result in contrast to the case of General Relativity [8]. Let us examine the subsidiary conditions

$$\xi^A_{,j}\pi_A|\Psi\rangle = 0, \tag{3.1}$$

$$(\pi^A\pi_A + Q^2\Delta^{00})\,|\Psi\rangle = 0, \tag{3.2}$$

in greater detail. From the general formulation of covariant field theories we may expect the constraints to be the generators for infinitesimal canonical transformations which correspond to arbitrary deformation of the constant x^0 surface. Three of them should be the generators for tangential surface deformation while the remaining one should be the generator for deformation normal to the surface. Suppose $\varphi_j = \xi^A_{,j}\pi_A$ are the three generators for tangential surface deformation, then the subsidiary conditions (3.1) express that $|\Psi\rangle$ must be invariant under arbitrary tangential surface deformation. This is equivalent to the requirement that $|\Psi\rangle$ be invariant under 3-dimensional coordinate transformation $x^j \to x'^j =$ function (x^1, x^2, x^3). Using the functional representation

$$\xi^A \to \xi^A,$$

$$\pi_A \to -i\hbar c\delta/\delta\xi^A,$$

we can indeed show explicitly that is the case. Hence (3.1) may be solved without much trouble. The solutions will be invariants in 3-dimensional tensor analysis. An example is

$$|\Psi\rangle = \Phi(\textstyle\int d^3x\zeta_{ABC}(\xi^D)J^{ABC}),$$

where ζ_{ABC} are arbitrary functions of ξ^D,

$$J^{ABC} = \begin{vmatrix} \xi^A_{,1} & \xi^A_{,2} & \xi^A_{,3} \\ \xi^B_{,1} & \xi^B_{,2} & \xi^B_{,3} \\ \xi^C_{,1} & \xi^C_{,2} & \xi^C_{,3} \end{vmatrix}$$

and Φ is an arbitrary function of the integral.

The subsidiary condition (3.2) presents difficulties however. As will be seen later it essentially expresses the requirement that $|\Psi\rangle$ be invariant under arbitrary normal deformation. One needs to compare the values of $|\Psi\rangle$ at different constant x^0 surfaces, so to obtain the solutions of (3.2) is as difficult as solving the equations of motion.

4. *Transformation properties*

On a coordinate transformation $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu$, the scalar field ξ^A transforms according to

$$\bar{\delta}\xi^A = \xi'^A(x) - \xi^A(x) = -\xi^A_{,\mu}\varepsilon^\mu.$$

$\xi^A_{,\mu}$, ε^μ may be decomposed [5] into their respective normal components $\xi^A_\perp$, $\varepsilon_\perp$ and their respective tangential components $\xi^A_{,r}$, $\varepsilon^r_{||}$ resulting in

$$\bar{\delta}\xi^A = -(\varepsilon_\perp\xi^A_\perp + \varepsilon^r_{||}\xi^A_{,r}),$$

where

$$\varepsilon_{\perp} = \varepsilon^0 (g^{00})^{-1/2}, \quad \xi_{\perp}^A = g^{0\mu}\xi_{,\mu}^A (g^{00})^{-1/2}, \quad \varepsilon_{||}^r = \varepsilon^r - \varepsilon^0 g^{0r}/g^{00}.$$

Then one can readily verify that the classical infinitesimal canonical transformation which corresponds to the above coordinate transformation is effected by the generating functional

$$G = -\int d^3x(\varepsilon_{\perp}\varphi_{\perp} + \varepsilon_{||}^r \Phi_r),$$

where

$$\varphi_{\perp} = -(\eta_{AB}\pi^A\pi^B + Q^2\Delta^{00})/2Q(-\Delta^{00})^{1/2} \approx 0,$$

$$\varphi_r = \xi_{,r}^A \pi_A \approx 0.$$

This justifies our previous interpretation of the significance of the constraints. With the help of the canonical equations of motion we can carry out the transformation by the generating functional

$$\mathscr{G} = -\int d^3x(\varepsilon^0 \mathscr{H}_T + \varepsilon^j \varphi_j).$$

In quantum theory the corresponding infinitesimal unitary transformation is effected by

$$U = \exp(i\mathscr{G}/\hbar) = (1 + i\mathscr{G}/\hbar),$$

where $\mathscr{G}$ is Hermitian [7].

Hence a covariant quantum theory for our model field is established. The general features of generally covariant quantum theory manifest themselves explicitly. A detailed physical analysis of these general characteristics is given elsewhere [9].

5. A special coordinate system

The fundamental reason for working in a specific coordinate frame is a subject of great controversy. We shall not go into such a controversial subject here. The choice of a specific frame is somewhat arbitrary in general. For the rest of this paper we shall work in a particular coordinate system uniquely defined by the set of coordinate conditions

$$\xi^\mu = x^\mu, \tag{5.1}$$

that is, we just choose the first four of the original 5-dimensional pseudo-Euclidean coordinates as our coordinates in the four-surface. Note that we take as our coordinates four scalars. In this special coordinate system we have

$$g_{\mu\nu} = \eta_{\mu\nu} - \xi_{,\mu}\xi_{,\nu}, \text{ where } \xi \text{ denotes } \xi^A \text{ with } A = 4;$$

$$g^{\mu\nu} = \eta^{\mu\nu} + (\eta^{\mu\alpha}\eta^{\nu\beta}\xi_{,\alpha}\xi_{,\beta})/(1 - \eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma});$$

$$g = \det(g_{\mu\nu}) = -(1 - \eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma}).$$

The field equation is

$$g^{\mu\nu}\xi_{,\mu\nu} = 0,$$

which may be written as

$$[\eta^{\mu\nu}\xi_{,\nu}(1-\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma})^{-1/2}]_{,\mu} = 0. \tag{5.2}$$

The appropriate Lagrangian and Hamiltonian densities are

$$\mathscr{L} = -Q(1-\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma})^{1/2},$$
$$\mathscr{H} = Q[(1+(\nabla\xi)^2]\,[1+(c\pi/Q)^2])^{1/2}, \tag{5.3}$$

where

$$\pi = \partial\mathscr{L}/\partial\dot{\xi} = (Q\dot{\xi}/c^2)\,(1-\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma})^{-1/2}$$

is the canonical momentum density. There is no longer any constraint. A general solution to the highly nonlinear equation (5.2) is not available. However many exact particular solutions may be found. The most interesting ones in this latter class are those of the form

$$\xi = \varphi_K(x) = A\exp(iK_\mu x^\mu)+A^*\exp(-iK_\mu x^\mu), \tag{5.4}$$

where A, K_μ are constants and K_μ satisfies

$$\eta^{\varrho\sigma}K_\varrho K_\sigma = 0.$$

The above solution defines a space-time and one may easily verify that the "straight" line generated by $x^\mu = K^\mu U$, where U is some parameter, is a null geodesic in the above space-time. Hence solution (5.4) has the form of a wave propagating along this geodesic. Slightly more general solutions may be obtained by superimposing all waves travelling in the same direction. However linear superposition of waves travelling in exactly opposite directions does not lead to another solution, nor does linear superposition of waves moving in different directions. The nonlinear interference effects show up for waves not propagating in the same direction.

6. *Perturbation approach*

Working in the special reference frame defined by (5.1) we are no longer troubled by constraints. Quantization of the field may be effected in the usual way by imposing the standard boson commutation relations on ξ, π. All this is encouraging but the real problem is how one can extract physical information from the intrinsically nonlinear field. The easiest attempt would be a perturbation approach of some kind.

6.1. Weak-field theory

In analogy to the linearized treatment of General Relativity let us consider the case where the field is everywhere-weak, that is, $\xi_{,j}$ and $c\pi/Q$ are of the order od $\lambda^{\frac{1}{2}} \ll 1$ for all x^μ. The exact numerical value of λ is a physical assumption. We may now carry out a binomial expansion of the Hamiltonian density (5.3) regarding 1 to be of the order λ^0, $(\nabla\xi)^2$, $(c\pi/Q)^2$ to be of the order λ and so on. Therefore to the second order we have $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_I$, where

$$\mathscr{H}_0 = Q + \tfrac{1}{2} Q[(\nabla\xi)^2 + (c\pi/Q)^2], \tag{6.1}$$

$$\mathscr{H}_I = -(Q/8)\,[(\nabla\xi)^4 + (c\pi/Q)^4 - 2(\nabla\xi)^2(c\pi/Q)^2]. \tag{6.2}$$

The additive Q in $\mathscr{H}_0$ may be ignored. In the usual perturbation approach $\mathscr{H}_0$ is regarded as representing the free field while $\mathscr{H}_I$ is considered as a small perturbation. The free field is seen to resemble the Klein-Gordon field for scalar mesons except that there is no mass term in our present case. The name real massless scalar meson field would be appropriate for it. The free field equation is

$$\eta^{\varrho\sigma}\xi_{,\varrho\sigma} = 0.$$

The perturbed equations of motion are

$$\dot{\xi} = (c^2\pi/Q)\,(1 + \tfrac{1}{2}\,[(\nabla\xi)^2 - (c\pi/Q)^2]),$$

$$\dot{\pi} = Q\nabla^2\xi + \tfrac{1}{2}\,Q\nabla\cdot(\nabla\xi[(c\pi/Q)^2 - (\nabla\xi)^2]).$$

Eliminating π, we obtain the perturbed field equation

$$\eta^{\alpha\beta}\xi_{,\alpha\beta} + \tfrac{1}{2}\,(\eta^{\alpha\beta}\xi_{,\alpha\beta}\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma} + \eta^{\alpha\beta}\xi_{,\alpha}[\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma}]_{,\beta}) = 0, \tag{6.3}$$

which may be derived from the Lagrangian density

$$\mathscr{L} = -Q + \tfrac{1}{2}\,Q\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma} + \tfrac{1}{8}\,Q(\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma})^2.$$

Many exact solutions to the field equation (6.3) may be found. Observe that solution (5.4) to the exact field equation (5.2) is also a solution of (6.3). Furthermore solutions which resemble interference effects between these plane wave solutions may be obtained by perturbation.

6.2. Quantization in the Schrödinger picture

The standard method may be applied here. Decompose ξ, π by Fourier integrals:

$$\xi(\boldsymbol{x}) = (2\pi)^{-3/2}\int d^3K q_K \exp(i\boldsymbol{K}\cdot\boldsymbol{x}); \qquad \pi(\boldsymbol{x}) = (2\pi)^{-3/2}\int d^3K p_K \exp(i\boldsymbol{K}\cdot\boldsymbol{x}).$$

Define a_K, $a_K^\dagger$ by

$$q_K = (\hbar c^2/2w_K Q)^{1/2}(a_K + a_{-K}^\dagger); \qquad p_K^\dagger = i(Q\hbar w_K/2c^2)^{1/2}(a_K^\dagger - a_{-K}),$$

where $w_K = c|\boldsymbol{K}|$. Then the commutation relations for ξ, π imply that a_K, $a_K^\dagger$ will obey the standard commutation relations for boson creation and annihilation operators. All this is independent of the Hamiltonian. In perturbation theory the unperturbed Hamiltonian is

$$H_0 = \int d^3x\mathscr{H}_0 = \int d^3K\hbar w_K a_K^\dagger a_K.$$

Hence we can construct and give particle interpretation to the various eigenstates of H_0 in the usual way. The problem now is to calculate the perturbation energy ΔE_K of a one-particle state $|K\rangle$ due to the interaction Hamiltonian

$$H_I = \int d^3x \mathscr{H}_I. \tag{6.4}$$

This is achieved using the time-independent perturbation theory (see Appendix A for details). H_I has to be expressed in terms of a_K, $a_K^\dagger$. As will be seen presently, only terms with factors

$$a_K^\dagger a_{K'} a_{\bar{K}} a_{\bar{K}'}, \qquad a_K a_{K'} a_{\bar{K}} a_{\bar{K}'},$$

have nonvanishing contribution to the relevant matrix elements. Therefore only these terms need to be worked out explicitly. The result is

$$H_I = \int d^3K d^3K' d^3\bar{K} d^3\bar{K}' \Omega_{KK'\bar{K}\bar{K}'} [\delta(K+K'+\bar{K}+\bar{K}') a_K a_{K'} a_{\bar{K}} a_{\bar{K}'} -$$
$$-4\delta(-K+K'+\bar{K}+\bar{K}') a_K^\dagger a_{K'} a_{\bar{K}} a_{\bar{K}'}] + \text{irrelevant terms}, \tag{6.5}$$

where

$$\Omega_{KK'\bar{K}\bar{K}'} = -\frac{h^2}{32(2\pi)^3 Q} (w_K w_{K'} w_{\bar{K}} w_{\bar{K}'})^{1/2} [(1-K^0 \cdot K'^0)(1-\bar{K}^0 \cdot \bar{K}'^0)] \leqslant 0,$$

$$K^0 = K/|K|.$$

The perturbation calculation on the nondegenerate vacuum state $|0\rangle$ gives

$$\Delta E_0 = \Delta E_0^{(2)} = \int \frac{d^3K_1 d^3K_2 d^3K_3 d^3K_4}{4!} \frac{[\sum_{p1234} \delta(K_1+K_2+K_3+K_4)\Omega_{K_1K_2K_3K_4}]^2}{-hc(K_1+K_2+K_3+K_4)}, \tag{6.6}$$

where $\sum_{p1234}$ means summation over all the perturbations of K_1, K_2, K_3, K_4. This expression contains the square of a δ-function which appears originally in H_I. Hence it is not very meaningful as it stands. However, as remarked in Appendix A, this difficulty may be bypassed. The usefulness of ΔE_0 as given by (6.6) will be seen in connection with the expression for ΔE_K later.

Let us now consider the perturbation on a one-particle state $|k\rangle$. The energy eigenvalue associated with $|k\rangle$ is degenerate. However this presents no difficulty (see Appendix A). Using the appropriate formulae (A1), (A2), we obtain after some lengthy calculation

$$\Delta E_k = \Delta E_0^{(2)} = A+B+C,$$

where

$$A = \int \frac{d^3K_1 d^3K_2 d^3K_3}{3!} \frac{(4\sum_{p123} \Omega_{kK_1K_2K_3})^2 \delta(K_1+K_2+K_3-k)}{hc(k-K_1-K_2-K_3)},$$

$$B = \int \frac{d^3K_1 d^3K_2 d^3K_3 d^3K_4}{4!} \frac{[\sum_{p1234} \Omega_{K_1K_2K_3K_4}\delta(K_1+K_2+K_3+K_4)]^2}{-\hbar c(K_1+K_2+K_3+K_4)},$$

$$C = \int \frac{d^3K_1 d^3K_2 d^3K_3}{3!} \frac{(\sum_{pk123} \Omega_{kK_1K_2K_3})^2\delta(k+K_1+K_2+K_3)}{-\hbar c(k+K_1+K_2+K_3)}.$$

As pointed in Appendix A we should regard

$$\Delta E_k'^{(2)} = \Delta E_k^{(2)} - \Delta E_0^{(2)} = A + C$$

as the true perturbed energy of the original one-particle state $|k\rangle$. Hence terms involving the square of the δ-function arising from H_I are cancelled out and cause no trouble. The real trouble comes from the actual structure of H_I. The integral A may be shown to diverge towards negative infinity at least like

$$-\int d^3K_1 d^3K_2 K_1 K_2.$$

B diverges similarly.

All this is expected from the form of H_I which involves the derivatives of the field variable. As one goes to a higher order in the perturbation calculation, one gets higher order products of the derivatives. Hence more divergent factors w_K turn up in the numerators leading to a higher order divergence. Therefore the perturbation treatment of our present model field theory is again plagued by infinite quantities.

6.3. The problem of scattering in the interaction picture

The usual S-matrix approach will be used and the Interaction picture will be employed throughout this section.

6.3.1. The first order S-matrix processes

The first order S-matrix is given by

$$S^{(1)} = \frac{1}{i\hbar} \int_{-\infty}^{\infty} H_I dt.$$

A number of processes can happen, notably what we may call the "shower process", that is, an incoming particle is annihilated and three outgoing particles are created. However we shall consider a more conventional two-particle scattering process. The initial state will be $|i\rangle = |k,k'\rangle$ with $k \neq k'$. The transition to final states of the form $|f\rangle = |K, K'\rangle$ will now be studied. Let us consider the simplest case in which $k+k' = 0$. In other words we have a head-on collision of two quanta. This is not such a restricted case as it may appear because any two-particle collision would appear to be head-on one in the centre-of-momentum frame of reference. Using the usual formula we obtain the total scattering cross--section

$$\sigma = \frac{(2\pi)^4}{2\hbar c} \int |T_{fi}|^2 \delta(K+K')\delta(E_f - E_i) d^3K d^3K',$$

where

$$T_{fi} = 8[\Omega_{kk'KK'}+\Omega_{kKk'K'}+\Omega_{kK'k'K}] \quad \text{and} \quad \boldsymbol{k}' = -\boldsymbol{k}.$$

The integration may be explicitly worked out giving

$$\sigma = 7E_k^6/10\pi Q^2(\hbar c)^4, \quad E_k = \hbar ck.$$

To estimate σ, we take Q to be of the order of the energy density of the universe which lies in the range 10^{-8} to 10^{-6} ergs cm^{-3}, say, take

$$Q = 10^{-7} \text{ ergs cm}^{-3} \Rightarrow \sigma \approx 10^{79} E_k^6 \text{ cm}^2,$$

where E_k is to be expressed in the CGS units. One can put in some typical energies for "gravitons" to work out σ. A typical graviton associated with the gravitational wave which may conceivably be generated in a laboratory as envisaged by Weber [10] would have an energy of the order of 10^{-19} ergs. The total scattering cross-section is then $\sigma = 10^{-35}$ cm^2. This is very small. But it might just be measurable since in some experiments on neutrinos a cross-section well below 10^{-40} cm^2 is not unknown [11].

Let us now consider a typical graviton associated with interstellar gravitational radiation. According to J. Wheeler the density of such gravitational radiation could be as high as 10^3 ergs cm^{-3} and that its wavelength λ would be of the order of 10^{24} cm [12]. The energy associated with a graviton may then be taken as $\hbar c/\lambda \approx 10^{-40}$ ergs.

The corresponding scattering cross-section is $\sigma \approx 10^{-161}$ cm^2, which is far too small to be measurable.

6.3.2. The second order S-matrix

The first order S-matrix elements are all finite. We may go on to examine the second order S-matrix elements. However the results are not rewarding. The calculation is very tedious leading to divergences. Therefore we shall not pursue it any further.

7. *Some remarks*

The treatment of curved space-time using such an embedding technique may be extended to other Riemannian spaces. It is known [13] that various Riemannian spaces commonly occurring in general relativity are immersible in pseudo-Euclidean spaces of appropriate dimension. As an example consider those vacuum solutions in general relativity, that is, solutions which give a vanishing Ricci tensor $R_{\mu\nu}$. We know that all vacuum solutions are immersible in a ten-dimensional flat space, while on the other hand, many vacuum solutions of physical significance are immersible in a six-dimensional flat space. All vacuum solutions embedded in a five-dimensional flat space are trivial, leading to Minkowskian space-times only [14]. Therefore one can formulate theories for the above non-trivial space-times in a way similar to the treatment of our present model theory. The field equations will be

$$R_{\mu\nu} = 0, \tag{7.1}$$

where the Ricci tensor $R_{\mu\nu}$ is expressed in terms of the coordinates $\xi^A (A = 0, 1, 2, ..., N; 5 \leqslant N \leqslant 9)$ and their derivatives of the particular embedding flat space concerned. One may then proceed to set up a Lagrangian formulation for the theory, bearing in mind that in the variational principle the variation is effected by $\delta\xi^A$. After this the Hamiltonian formulation may be established and various quantization schemes can then be attempted. In view of the complexity of the field equations (7.1) various technical difficulties may arise in actually carrying out the above programme. There is obviously room for much further work on this.

APPENDIX A

A perturbation theory in quantum field theory

If we do not confine the field in a finite box, H_0 will have a continuous eigenvalue spectrum, apart from the zero eigenvalue of $|0\rangle$ which may be considered as a discrete eigenvalue especially for a massive field. Then it is not difficult to show that the appropriate expressions for the calculation of perturbation energy are:

(1) for the vacuum state

$$\Delta E_0 = \Delta E_0^{(1)} + \Delta E_0^{(2)} + \ldots, \text{ where}$$

$$\Delta E_0^{(1)} = \langle 0|H_I|0\rangle; \qquad \Delta E_0^{(2)} = \sum_{n \neq 0} \frac{\langle 0|H_I|n\rangle \langle n|H_I|0\rangle}{-E_n} \quad \text{and}$$

$\sum_{n\neq 0}$ denotes a summation (and an integration) over all the unperturbed eigenstates except $|0\rangle$;

(2) for the one-particle state $|\boldsymbol{k}\rangle$.

The unperturbed energy E_k is degenerate. But in many practical cases, including our present case, we have $E_n = E_k \Rightarrow \langle n|H_I|\boldsymbol{k}\rangle = 0$. We shall confine ourselves to such cases. Then the degeneracy of E_k causes no trouble. We have $\Delta E_k = \Delta E_k^{(1)} + \Delta E_k^{(2)} + \ldots$, where

$$\Delta E_k^{(1)} \delta(\boldsymbol{k} - \boldsymbol{K}) = \langle \boldsymbol{K}|H_I|\boldsymbol{k}\rangle; \tag{A1}$$

$$\Delta E^{(2)} \delta(\boldsymbol{k} - \boldsymbol{K}) = {\sum_n}' \frac{\langle \boldsymbol{K}|H_I|n\rangle \langle n|H_I|\boldsymbol{k}\rangle}{E_k - E_n} \tag{A2}$$

and $\sum_n'$ is a sum over all the unperturbed eigenstates except the one-particle states and those eigenstates with $E_n = E_k$.

Some remarks

(1) The essential difference between these formulae and those for the case of a discrete spectrum is the appearance of the δ-function. The necessity of the δ-function is most clearly seen by considering a somewhat trivial example in which $H_I = H_0^2$. Our first order expression then leads then leads to the exact result.

(2) In many practical cases, H_I contains a δ-function. Then $\Delta E_0^{(2)}$ may involve the square of the δ-function as in our present ξ-field theory. This means that the expression for $\Delta E_0^{(2)}$ itself is not a meaningful quantity as it stands. However this difficulty may be bypassed. Let us consider the present ξ-field theory. Firstly one can confine the field in a finite yet large box of volume V perform the perturbation calculation. For a large V we obtain

$$\Delta E_0^{(2)} = \frac{V}{8\pi^3} \int \frac{d^3K_1 d^3K_2 d^3K_3 d^3K_4}{4!} \frac{|\sum_{p1234} \Omega_{K_1K_2K_3K_4}|^2 \delta(\boldsymbol{K}_1+\boldsymbol{K}_2+\boldsymbol{K}_3+\boldsymbol{K}_4)}{-\hbar c(K_1+K_2+K_3+K_4)}.$$

Now if we regard the result given in equation (6.6) as the limiting case of large V, we may interpret $\delta^2(\boldsymbol{K}_1+\boldsymbol{K}_2+\boldsymbol{K}_3+\boldsymbol{K}_4)$ appearing in (6.6) as the limiting case of $(V/8\pi^3)$ $\delta(\boldsymbol{K}_1+\boldsymbol{K}_2+\boldsymbol{K}_3+\boldsymbol{K}_4)$. This procedure therefore gives some meaning to (6.6). Secondly, one observes that

$$\Delta E_K^{(2)} = \Delta E_0^{(2)} + \Delta E_K'^{(2)},$$

where $\Delta E_K'^{(2)}$ does not involve the square of a δ-function. Hence we may consider $\Delta E_K'^{(2)}$ as the genuine perturbed energy [15], of the original one-particle state. The difficulty caused by a (δ-function)^2 therefore disappears at least as far as our ξ-field is concerned.

REFERENCES

[1] Notation: Capital Latin indices A, B take the values 0, 1, 2, 3, 4, lower case Greek indices the values 0, 1, 2, 3, and lower case Latin indices the values 1, 2, 3. $x^0 = ct$, $\boldsymbol{x} = (x^1, x^2, x^3)$. The summation convention for repeated indices is adopted.

[2] P. G. Bergmann, *Phys. Rev.*, **75**, 680 (1949).

[3] J. L. Anderson, *Principles of Relativity Physics*, Academic Press, New York 1967.

[4] P. A. M. Dirac, *Canad. J. Math.*, **2**, 129 (1950).

[5] P. A. M. Dirac, *Canad. J. Math.*, **3**, 1 (1951).

[6] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Series, Yeshiva University, New York 1964.

[7] P. A. M. Dirac, *Lectures on Quantum Field Theory*, Belfer Graduate School of Science Series, Yeshiva University, New York 1966, § 3.

[8] J. L. Anderson, in *Relativistic Theories of Gravitation*, Editor L. Infeld, Pergamon Press, Oxford 1962, p. 289.

[9] K-K. Wan, G. H. Derrick, to be published.

[10] J. Weber, *General Relativity and Gravitational Waves*, Interscience Publishers, 1961, p. 149.

[11] H. Muirhead, *The Physics of Elementary Particles*, Pergamon Press, Oxford 1965, p. 589.

[12] J. Weber, Ref. [10], p. 137.

[13] J. Rosen, *Rev. Mod. Phys.*, **37**, 204 (1965).

[14] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton 1964, p. 200.

[15] N. M. Hugenholtz, *Physica*, **23**, 481 (1957).