THE 4-SURFACE OF STATIONARY VOLUME EMBEDDED IN A 5-DIMENSIONAL PSEUDO-EUCLIDEAN SPACE II

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(*Received December 20, 1972*)

In a previous paper a generally covariant and intrinsically nonlinear model field, the 4-surface of stationary volume embedded in a 5-dimensional pseudo-Euclidean space, was introduced. In this paper two new methods are put forward to study the model. The first one is an everywhere-slowly-varying field approach which is essentially perturbative. The second one is a variational method which is able to lead to some finite results.

1. *Introduction*

In a previous paper [1] (hereafter referred to as I), a model field theory, the 4-surface of stationary volume embedded in a 5-dimensional pseudo-Euclidean space, was introduced. Any 4-dimensional surface immersed in a 5-dimensional flat space with coordinates $\xi^A(A=0, 1, 2, 3, 4)$ and metric $\eta_{AB}=\mathrm{diag}\,(1, -1, -1, -1, -1)$ may be fixed by specifying ξ^A as functions of 4 parameters x^μ [2]. In general the surface is a 4-dimensional Riemannian space with a metric

$$g_{\mu\nu} = \eta_{AB}\xi^A_{,\mu}\xi^B_{,\nu}.$$

We shall consider those surfaces whose metric is non-singular, *i. e.* $g = \det|g_{\mu\nu}| \neq 0$ and in which g is negative. The volume of a domain $x^\mu \in D$ of the 4-surface is

$$\tau = \int_D \sqrt{-g}\, d^4x, \tag{1.1}$$

which is an invariant with respect to arbitrary coordinate transformations in the 4-surface. A 4-surface of stationary volume is defined to be a surface whose volume τ is at a stationary value with respect to small arbitrary deformation of the surface, the deformation being realized mathematically by the variation $\delta\xi^A$ which is assumed to vanish at the boundary of the domain D. The action integral (1.1) then leads directly to the set of defining equa-

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tions for a 4-surface of stationary volume

$$(\sqrt{-g}\, g^{\tau\kappa}\xi^A_{,\tau})_{,\kappa} = 0, \tag{1.2}$$

or in covariant form

$$g^{\mu\nu}\xi^A_{;\mu\nu} = 0, \tag{1.3}$$

where the semicolon denotes a covariant differentiation. In I a generally covariant quantization scheme was carried through. Then a special set of coordinate conditions $x^\mu = \xi^\mu$ were chosen singling out a unique reference frame in which the field equation is

$$[\eta^{\mu\nu}\xi_{,\nu}(1-\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma})^{-1/2}]_{,\mu} = 0, \text{ where } \xi = \xi^A \text{ with } A = 4, \tag{1.4}$$

with the appropriate Lagrangian and Hamiltonian

$$L = \int \mathscr{L} d^3x,$$
$$H = Q\int([1+(\nabla\xi)^2]\,[1+(c\pi/Q)^2])^{1/2}d^3x, \tag{1.5}$$

where $\mathscr{L} = -Q(1-\eta^{\varrho\sigma}\xi_{,\varrho}\xi_{,\sigma})^{1/2}$, $\pi = \partial\mathscr{L}/\partial\dot{\xi}$ which is the canonical momentum density and Q is a positive constant of the dimensions of energy density. A weak field perturbation approach commonly used in general relativity was adopted to study the quantum behaviour of the field. However the usual divergence problem in quantum field theory arises. In this paper two new treatments are introduced for the study of the field in this special coordinate system. The first one is again essentially perturbative. The second one is a variational method which is able to lead to some finite results.

2. *The theory of an everywhere-slowly-varying field*

The weak field approximation common in the treatment of General Relativity is a very restrictive one. It is more realistic to consider a field which is everywhere-slowly-varying. By this we mean that the deviation of the four-surface from the tangent plane at a nearby point on the surface is always small. In terms of canonical variables this is equivalent to the assumption that $\nabla\xi$, $c\pi/Q$ be slowly varying function of x^μ. The problem now is to obtain an approximation expansion of the Hamiltonian (1.5) in accord with this assumption. Mathematically we want an approximate expression for an integral

$$I = \int_{-\infty}^{\infty} dx F(\theta_a(x)),$$

where $\theta_a(x)$ are a set of slowly-varying real functions of a single variable x (we shall later generalize our results to integrals over three spatial variables $\boldsymbol{x}$). Let us introduce a set of functions $U_{Kn}(x)$ defined by

$$U_{Kn}(x) = \frac{1}{\sqrt{L}}\, e^{iKx}\, \frac{\sin\pi\left(\dfrac{x}{L}-n\right)}{\pi\left(\dfrac{x}{L}-n\right)},$$

where L is a numerical constant; $K = \frac{2\pi}{L} \times$(zero or integers); n = zero or integers. $U_{Kn}(x)$ is defined for all x from $-\infty$ to $+\infty$. Some relevant properties of this set of functions are listed in Appendix A. In particular we note that $U_{Kn}(x)$ are a complete orthonormal set and $|U_{Kn}|$ has the absolute maximum at $x_n = nL$. $|U_{Kn}|$ decreases as x moves away from x_n and becomes small compared with $|U_{Kn}(x_n)|$ for $|x-x_n| \gg L$. To make use of this set of functions we firstly adjust the value of L to be such that $\theta_a(x)$ changes only slightly over a dimension L. The slowly varying nature of $\theta_a(x)$ means that L may be large. We can write

$$\theta_a = \sum_{K,n} \theta_{aKn} U_{Kn}(x), \text{ with } \theta_{aKn} = \int_{-\infty}^{\infty} dx U^*_{Kn} \theta_a.$$

Define

$$\langle \theta_a \rangle_n = \int dx U_{0n} \theta_a / \int dx U_{0n} = L^{-1/2} \theta_{a0n},$$

and

$$\theta'_{an}(x) = \theta_a(x) - \langle \theta_a \rangle_n.$$

$\langle \theta_a \rangle_n$ may be regarded as the average value of θ_a in a neighbourhood of $x = nL$ of a dimension of the order L. Then $\theta'_{an}(x)$ is small in the neighbourhood. Now we are in a position to state a Theorem:

If $\theta_a(x)$ is a set of slowly-varying real functions of x, then

$$I = \int_{-\infty}^{\infty} dx F(\theta_a(x)) \approx \sum_n [LF(\langle \theta_a \rangle_n) + \tfrac{1}{2} (\partial^2 F/\partial\theta_a \partial\theta_b)_n \sum_{K \neq 0} \theta^*_{aKn} \theta_{bKn}],$$

where $(\partial^2 F/\partial\theta_a\partial\theta_b)_n = (\partial^2 F/\partial\theta_a\partial\theta_b)$ whith $\theta_a = \langle\theta_a\rangle_n$, $\theta_b = \langle\theta_b\rangle_n$, and summations over a, b are implied.

The proof together with an examination of the approximation involved is given in Appendix B. Extension to the 3-dimensional case is easily obtained by defining the complete orthonormal set of functions

$$U_{Kn}(\boldsymbol{x}) = L^{-3/2} \exp(i\boldsymbol{K} \cdot \boldsymbol{x}) \prod_{j=1,2,3} \frac{\sin \pi \left(\frac{x^j}{L} - n^j \right)}{\pi \left(\frac{x^j}{L} - n^j \right)}.$$

The Theorem takes exactly the same form with L in the first sum replaced by L^3.

We can now use the Theorem to obtain an approximate expression for the Hamiltonian given by (1.5). As shown in Appendix C the result is

$$H = QL^3 \sum_n [(1 + \langle \nabla\xi \rangle_n^2)(1 + \langle c\pi/Q \rangle_n^2)]^{1/2} + \tfrac{1}{2} \sum_{n,K\neq 0} \frac{icK^j g_n^{0j}}{g_n^{00}} (q^*_{Kn} p_{Kn} - p^*_{Kn} q_{Kn}) +$$

$$+ \tfrac{1}{2} \sum_{n,K\neq 0} \frac{c^2 \sqrt{-g_n}}{Q g_n^{00}} p^*_{Kn} p_{Kn} + \tfrac{1}{2} \sum_{n,K\neq 0} \frac{-QK^j K^k}{\sqrt{-g_n}} \left(\frac{g_n^{0j} g_n^{0k}}{g_n^{00}} - g_n^{jk} \right) q^\dagger_{Kn} q_{Kn} +$$

$$+ \text{higher terms},$$

where $g^{\tau\kappa}$ is the metric tensor for the 4-surface;

$$q_{Kn} = \int d^3x U^*_{Kn}\xi, \quad p_{Kn} = \int d^3x U^*_{Kn}\pi.$$

$g^{\tau\kappa}$ may be expressed in terms of $\xi_{,j}$, π (see Appendix C). $g_n^{\tau\kappa}$ is then the value of $g^{\tau\kappa}$ at $\xi_{,j} = \langle\xi_{,j}\rangle_n$, $\pi = \langle\pi\rangle_n$. Note that $\langle\xi_{,j}\rangle_n$, $\langle\pi\rangle_n$, $g_n^{\tau\kappa}$ depend solely on q_{0n} and p_{0n}. Now define

$$a_{Kn} = \left[\frac{(g_n^{0j}g_n^{0k} - g_n^{jk}g_n^{00})K^jK^kQ^2}{4\hbar^2(-g_n)}\right]^{1/4} q_{Kn} + ic\left[\frac{(-g_n)}{4\hbar^2Q^2(g_n^{0j}g_n^{0k} - g_n^{jk}g_n^{00})K^jK^k}\right]^{1/4} p_{Kn},$$

$$w_{Kn} = \frac{[(g_n^{0j}g_n^{0k} - g_n^{jk}g_n^{00})K^jK^k]^{1/2} + g_n^{0j}K^j}{g_n^{00}}, \tag{2.1}$$

where repeated indices other than n imply a summation.

Then

$$H = QL^3 \sum_n [(1+\langle\nabla\xi\rangle_n^2)(1+\langle c\pi/Q\rangle_n^2)]^{1/2} + \\ + \sum_{n,K\neq 0} \hbar w_{Kn} a^*_{Kn} a_{Kn} + \text{higher terms.} \tag{2.2}$$

Note that if we write

$$(k_{Kn})_0 = w_{Kn}/c \text{ and } (k_{Kn})_j = -K^j,$$

then $g_n^{\lambda\mu}(k_{Kn})_\lambda(k_{Kn})_\mu = 0$ (no summation over $\boldsymbol{K}$, $\boldsymbol{n}$).

This means that

$$(w_{Kn}/c, -\boldsymbol{K}),$$

are the covariant components of a null vector with respect to the metric $g_n^{\tau\kappa}$.

3. *Quantization*

The field is quantized by imposing the standard equal-time boson commutation relations on ξ and π which in turn imply the commutation relations for q_{Kn}, p_{Kn} to be

$$[q_{Kn}, q^\dagger_{K'n'}] = i\hbar\delta_{KK'}\delta_{nn'}; \quad [q_{Kn}, q_{K'n'}] = [p_{Kn}, p_{K'n'}] = 0.$$

The everywhere-slowly-varying nature of the field means that $\langle\xi_{,j}\rangle_n$, $\langle c\pi/Q\rangle_n$ may be taken as unquantized c-numbers (see Appendix D for details). As a result, $g_n^{\tau\kappa}$ may also be similarly treated as c-numbers. To this approximation we see that a_{Kn}, $a^\dagger_{Kn}$ defined in (2.1) obey the standard commutation rules for creation and annihilation operators, *i. e.*,

$$[a_{Kn}, a^\dagger_{K'n'}] = \delta_{KK'}\delta_{nn'}; \quad [a_{Kn}, a_{K'n'}] = [a^\dagger_{Kn}, a^\dagger_{K'n'}] = 0.$$

Indeed with the Hamiltonian (2.2) we may regard a_{Kn}, $a^\dagger_{Kn}$ respectively as the annihilation and creation operators for the quantum of energy $\hbar w_{Kn}$. The situation is that we have on the one hand the slowly-varying classical field $g_n^{\tau\kappa}$ and on the other hand the quantized field with the Hamiltonian (2.2). The background metric $g_n^{\tau\kappa}$ depends on the classical variables $\langle\xi_{,j}\rangle_n$ and $\langle c\pi/Q\rangle_n$ so that it will vary slowly with time as these variables develop in time according to the classical canonical equations generated by the Hamiltonian

$$QL^3 \sum_n [(1+\langle\nabla\xi\rangle_n^2)(1+\langle c\pi/Q\rangle_n^2)]^{1/2}.$$

The quanta created by $a_{Kn}^\dagger$ from the vacuum state are mainly in the region $n^j - L/2 < x_j < n^j + L/2$, but there is some overlap with neighbouring regions. The total number operator is $N = \sum_n N_n$, where $N_n = \sum_K a_{Kn}^\dagger a_{Kn}$ and $[N_n, N_{n'}] = 0$.

It is seen that we have essentially a free field theory. To bring in interaction one may proceed to higher order terms in the Taylor series. However the calculation becomes very lengthy and the usual divergence problem remains. We are in a position now to compare the present everywhere-slowly-varying field theory and the weak field approach discussed in the previous paper I. It is clear that both methods are perturbative leading to free field theories in the lowest order approximation. Higher order terms are obtained essentially from a binomial expansion of the square root expression for the Hamiltonian density. As one goes to a higher order, one has a higher order product of $\xi_{,j}$, π leading to a higher order divergence in quantum theory. However, substantial differences also exist between the two cases. While the weak field treatment in the lowest order leads exactly to a massless real scalar meson field, the slowly-varying field theory gives a quantized field which is superimposed on a classical background field. This background field also contributes to the energy eigenvalues of the quanta. The use of U_{Kn} functions leads to quanta which are localized in domains of volume of the order L^3. The appearance of the classical background field is one of the most striking features of our intrinsically nonlinear field. We have also noted that the background field may vary slowly with time. Hence the energy associated with each quantum will depend on time as well as the spatial position of the cell in which it is created. These properties appear to be in accord with the very concept of curved space-time in which not all world points are equivalent as in the case of a flat space-time.

4. *A variation treatment*

4.1. The general idea

In order to check whether the divergences obtained are a spurious result of the perturbation method used it is desirable to devise a non-perturbative approach. In this section a variational procedure is adopted for the calculation of energy eigenvalues. The essential idea lies in the approximation of the continuum field by a system of countable degrees of freedom. Once this is done many of the usual techniques of quantum mechanics, in particular the variational method for the calculation of eigenvalues, may readily be employed to study the system. There are a number of ways to achieve a discrete set of coordinates and momenta. We shall adopt a method of approximation by finite differences [3], [4], [5].

To begin with, confine the field in a large yet finite cubic box of side L with the usual periodic boundary conditions. Divide this spatial box into $M = (2N+1)^3$ small cubic cells, each of side $d = L/(2N+1)$, N being a positive integer. The centre of the $\boldsymbol{n}$-th cell is specified by $\boldsymbol{x} = \boldsymbol{x}_n = (n_1, n_2, n_3)d$, where $-N \leqslant n_i \leqslant N$. Then we may make the following approximations for the quantities in the $\boldsymbol{n}$-th cell:

(1) the field variable $\xi(\boldsymbol{x}) \approx \xi_n(\boldsymbol{x}) \approx \xi(\boldsymbol{x}_n)$, where $\boldsymbol{x}_n = (n_1, n_2, n_3)d$,

(2) $\nabla\xi(x) \approx \Delta\xi_n = \frac{1}{2d}(\xi_{n_1+1,\, n_2,\, n_3} - \xi_{n_1-1,\, n_2,\, n_3},\ \xi_{n_1,\, n_2+1,\, n_3} - \xi_{n_1,\, n_2-1,\, n_3},\ \xi_{n_1,\, n_2,\, n_3+1} - \xi_{n_1,\, n_2,\, n_3-1})$,

(3) the canonical momentum density $\pi(x) \approx p_n/d^3$, where p_n is the momentum conjugate to ξ_n,

(4) the total Hamiltonian $H \approx \sum_n H_n$, where $H_n \approx Qd^3([1+(\Delta\xi_n)^2][1+(cp_n/Qd^3)^2])^{1/2}$,

(5) the total linear momentum $\boldsymbol{p} \approx \sum_n \boldsymbol{p}_n$, where $\boldsymbol{p}_n \approx -p_n\Delta\xi_n$. (4.1)

The quantization may be effected by the usual procedure with the explicit representation

$$\xi_n \to \xi_n; \quad p_n \to -i\hbar\partial/\partial\xi_n.$$

Note that $[1+(\Delta\xi_n)^2]$ commutes with $[1+(cp_n/Qd^3)^2]$. Hence there is no ambiguity in the expression for H_n.

To illustrate the variational method to be used for the study of the Hamiltonian $H = \sum_n Qd^3([1+(\Delta\xi_n)^2][1+(cp_n/Qd^3)^2])^{1/2}$, let us consider the much simpler and well-defined case of the massless real Klein-Gordon field. In the finite difference approximation, the Hamiltonian is

$$H = \frac{Qd^3}{2}\sum_n [(\Delta\xi_n)^2 - (\hbar c/Qd^3)^2\partial^2/\partial\xi_n^2].$$

The eigenfunctionals of H in the functional representation in the continuum case are explicitly known [6], [7]. The exact eigenfunctions of H in our present discrete case may also be similarly established. The vacuum state is

$$\Psi_0 = A\exp[-\sigma^2 \sum_{k,m,m'} w_k \exp(i\boldsymbol{k}\cdot(\boldsymbol{m}-\boldsymbol{m}')d)\xi_m\xi_{m'}],$$

where

$$\sigma^2 = Qd^3/(2M\hbar c^2);$$

$$A = \text{normalization constant};$$

$$\boldsymbol{k} = \frac{2\pi}{L}(n_1, n_2, n_3) \text{ and } -N \leqslant n_i \leqslant N;$$

$$w_k = \frac{c}{d}[(\sin k_x d)^2 + (\sin k_y d)^2 + (\sin k_z d)^2]^{1/2}.$$

One can verify that

$$H\Psi_0 = E_0\Psi_0, \quad \boldsymbol{P}\Psi_0 = 0,$$

where $E_0 = \frac{1}{2}\sum_k \hbar w_k$ which is clearly seen to approach the well-known infinite zero point energy $\frac{1}{2}\sum_k \hbar kc$ in the continuum case as $d \to 0$ and $V = Md^3 \to \infty$. For $d \to 0$ keeping the volume $V = Md^3$ large but finite we have

$$E_0 = \frac{1}{2}\sum_k \hbar w_k \to \frac{\hbar c V}{d^4}\,\frac{1}{16\pi^3}\iiint_{-\pi}^{\pi} dxdydz(\sin^2 x+\sin^2 y+\sin^2 z)^{1/2}+ \\ +O(\hbar c V^{1/3} d^{-2}). \tag{4.2}$$

The one-particle eigenstates are of the form

$$\Psi_k = B\sum_m \exp(i\boldsymbol{k}\cdot \boldsymbol{m}d)\xi_m \Psi_0,$$

with

$$H\Psi_k = E_k\Psi_k; \quad P\Psi_k = \boldsymbol{p}\Psi_k,$$

where

$$E_k = \frac{1}{2}\sum_{k'} \hbar w_{k'}+\hbar w_k; \quad \boldsymbol{p} = \frac{h}{d}(\sin dk_x, \sin dk_y, \sin dk_z).$$

These eigenvalues tend to the original continuum values as d approaches zero. All other many-particle eigenstates may be similarly obtained. Now suppose we did not know the exact eigenfunctions. We can use the variational technique to estimate the eigenvalues now that we have a discrete system. Take the normalized trial wave function for the vacuum to be $\Phi_0 = \prod_n \Phi_{0n}$, where $\Phi_{0n} = (\pi\sigma^2)^{-\frac{1}{4}}\exp(-\xi_n^2/2\sigma^2/2\sigma^2)$ and σ^2 is to be the variational parameter. The vacuum expectation value is then

$$(\Phi_0, H\Phi_0) = (Qd^3M/2)(3\sigma^2/4d^2+d^2/2\varrho^2\sigma^2),$$

where $\varrho = Qd^4/c\hbar$ is dimensionless. Hence the estimated zero point energy is given by the minimum of $(\Phi_0, H\Phi_0)$:

$$(\Phi_0, H\Phi_0)_{\min} = (3/8)^{1/2}(\hbar c V/d^4). \tag{4.3}$$

Thus our variational method gives a factor $(3/8)^{1/2} \approx 0.612$ compared with the true factor

$$\frac{1}{16\pi^3}\iiint_{-\pi}^{\pi} dxdydz\,(\sin^2 x+\sin^2 y+\sin^2 z)^{1/2} \approx 0.593\pm 0.004.$$

The above integration was performed numerically. Φ_0 is seen to be a very good trial wave function as far as the most singular contribution to E_0 is concerned, *i. e.* the term proportional to V/d^4. The natural trial wave function for a one-particle state would be

$$\Phi_k = (2/M\sigma^2)^{1/2}\sum_m \exp(i\boldsymbol{k}\cdot \boldsymbol{m}d)\xi_m\Phi_0,$$

where $(2/M\sigma^2)^{1/2}$ is the normalization constant and σ^2 is the variational parameter. Some properties of Φ_k are

$$(\Phi_k, \Phi_0) = 0; \qquad (\Phi_k, \Phi_{k'}) = \delta_{k,k'};$$
$$(\Phi_0, \xi_n \Phi_k) = \sigma(2M)^{-1/2} \exp(i\boldsymbol{k}\cdot\boldsymbol{x}_n); \qquad \boldsymbol{P}\Phi_k = \boldsymbol{p}\Phi_k.$$

We have

$$(\Phi_k, H\Phi_k) = \frac{d^3QM}{2}\left(\frac{3\sigma^2}{4d^2} + \frac{1}{2}\frac{d^2}{\varrho^2\sigma^2}\right) + \frac{1}{2}d^3Q\left(\frac{d^2}{\varrho^2\sigma^2} + \frac{3}{2}(1-Z_k)\frac{\sigma^2}{d^2}\right),$$

where $Z_k = \frac{1}{3}(\cos 2dk_x + \cos 2dk_y + \cos 2dk_z)$ and $\boldsymbol{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$, $-N \leqslant n_i \leqslant N$.

Observe that

$$(\Phi_k, H\Phi_k) = (\Phi_0, H\Phi_0) + \frac{1}{2}d^3Q\left(\frac{d^2}{\varrho^2\sigma^2} + \frac{3}{2}(1-Z_k)\frac{\sigma^2}{d^2}\right)$$

and that

$$(\Phi_0, H\Phi_0) \gg \frac{1}{2}d^3Q\left(\frac{d^2}{\varrho^2\sigma^2} + \frac{3}{2}(1-Z_k)\frac{\sigma^2}{d^2}\right) \text{ since } M \to \infty.$$

Furthermore one finds that

$$[(\Phi_k, H\Phi_k) - (\Phi_0, H\Phi_0)]_{\min} = \left[\frac{1}{2}d^3Q\left(\frac{d^2}{\varrho^2\sigma^2} + \frac{3}{2}(1-Z_k)\frac{\sigma^2}{d^2}\right)\right]_{\min} =$$
$$= (d^3Q/p)\,[3(1-Z_k)/2]^{1/2} \to \hbar ck \text{ as } d \to 0.$$

Hence an exact eigenvalue spacing is obtained. The conventional variational procedure would be to minimize $(\Phi_k, H\Phi_k)$ giving the eigenvalue spacing as

$$(\Phi_k, H\Phi_k)_{\min} - (\Phi_0, H\Phi_0)_{\min} \to \sqrt{\frac{3}{8}}\frac{\hbar c}{d},$$

which has quite the wrong form, and indeed gives an infinite value in the small d limit instead of the correct spacing $\hbar ck$. The reason is that for a fixed V, $(\Phi_k, H\Phi_k)_{\min}$ contains terms in $1/d^4$, $1/d$, ... while $(\Phi_0, H\Phi_0)_{\min}$ contains $1/d^4$ only. On subtraction the $1/d^4$ terms cancel but we are left with $1/d$ which diverges as $d \to 0$. Since we know that the correct one-particle energy relative to the vacum is finite, *viz.*, $\hbar ck$, we must conclude that more sophisticated trial wave functions are needed to effect cancellation of the singular terms. Such functions may readily be constructed for the massless real Klein-Gordon field, but unfortunately these functions prove impracticable in the nonlinear case. We shall therefore retain the same simple forms for Φ_0 and Φ_k when we turn to the nonlinear field and shall adopt the following rather questionable procedure. Instead of estimating the eigenvalue spacing from $(\Phi_k, H\Phi_k)_{\min} - (\Phi_0, H\Phi_0)_{\min}$ we shall take $[(\Phi_k, H\Phi_k) - (\Phi_0, H\Phi_0)]_{\min}$. Since we are trying to estimate the finite difference between two indefinitely large energies it is not unreasonable to use the same parameter σ in both Φ_k and Φ_0. On minimizing this difference with respect to σ we find complete cancellation of all the singular terms, both for the linear massless Klein-Gordon field above and also for the nonlinear case. As we have already shown that this rather dubious procedure yields the correct answer $\hbar ck$ for the energy of a massless Klein-Gordon particle of momentum $\hbar \boldsymbol{k}$, we hope it will also give reasonable results when applied to our nonlinear field.

Returning now to the nonlinear field, we have to ascertain the meaning of the square root in (4.1) first. We define the operator

$$(1-\gamma\partial^2/\partial y^2)^{1/2},\ \gamma \text{ being a numerical constant,}$$

by

$$(1-\gamma\partial^2/\partial y^2)^{1/2}\psi(y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1+\gamma k^2)^{1/2}\hat{\psi}(k,t)\exp(iky),$$

where $\hat{\psi}(k,t)$ is the Fourier transform of $\psi(y,t)$.

Au anambiguous meaning for the nonlinear Hamiltonian (4.1) is therefore established for the subset of wave functions for which all relevant integrals of the above type exist. We are now in a position to attempt a variation treatment.

4.2. The vacuum state

In view of the similarity of our model field and the massless Klein-Gordon field as seen in the weak field treatment we again take

$$\Phi_0 = \prod_n \Phi_{0n}, \text{ where } \Phi_{0n} = (\pi\sigma^2)^{-1/2}\exp(-\xi_n^2/2\sigma^2),$$

as the trial wave function for the vacuum state. Since

$$\Phi_{0n} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\eta_n\sigma\exp\left(-\frac{1}{2}\sigma^2\eta_n^2\right)\frac{\exp(i\eta_n\xi_n)}{(\pi\sigma^2)^{1/4}},$$

we have

$$H_n\Phi_{0n} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\eta_n\left\{\sigma\exp\left(-\frac{1}{2}\sigma^2\eta_n^2\right)Qd^3[1+(\Delta\xi_n)^2]^{1/2}[1+ +(ch\eta_n/Qd^3)^2]^{1/2}\frac{\exp(i\eta_n\xi_n)}{(\pi\sigma^2)^{1/4}}\right\}.$$

The vacuum expectation value may be calculated exactly in terms of modified Bessel functions of the second kind $K_\nu(z)$ [8].

The result is

$$(\Phi_0, H\Phi_0) = [Q^2d^7M/(\sqrt{2}\,\pi ch)]\exp(\lambda+\beta)K_1(\beta)[K_0(\lambda)+K_1(\lambda)], \tag{4.4}$$

where $\beta = d^2/\sigma^2$; $\varrho = Qd^4/ch$; $\lambda = \varrho/2\beta$.

The minimization of (4.4) gives (Appendix E)

$$(\Phi_0, H\Phi_0)_{\min} \approx \sqrt{2}\,\hbar cV/\pi d^4. \tag{4.5}$$

This is a significant result. Comparing this with (4.2), (4.3) we see that the present zero point energy and the vacuum energy of the Klein-Gordon field diverges in exactly the same manner. In contrast if we apply this variational method to estimate the vacuum energy of the Hamiltonian obtained in the weak field approach in the previous paper I a much higher order of divergence is obtained, showing that some of the divergences of the perturbation approach are spurious. One might then hope for a similar behaviour of the one-particle states.

4.3. One-particle states

In general we talk about a one-particle state in a nonliner theory only in the context of a perturbation approach. The situation is however different in the present case. Our nonlinear field equation admits individual plane wave solutions [1]. Moreover the Hamiltonian and the linear momentum take on a linear form if we confine ourselves to a plane wave solution. Hence we may be able to formulate one-particle states in an exact manner. Indeed this can be done as will be seen in the next section. Therefore in our variational treatment it is reasonable to use the trial wave function

$$\Phi_k = (2/M\sigma^2)^{1/2} \sum_m \exp(i\boldsymbol{k}\cdot\boldsymbol{m}d)\xi_m\Phi_0,$$

hoping that it would at least give a qualitatively correct result.

After some calculation, whose details are available in Appendix F, the energy expectation value is found to be

$$(\Phi_k, H\Phi_k) = (\Phi_0, H\Phi_0)+(Q^2d^7/\sqrt{2}\,\pi c\hbar)\exp(\lambda+\beta)\,[A+B],$$

where

$$A = K_1(\beta)\,[2K_1(\lambda)-(K_0(\lambda)+K_1(\lambda))Z_k] > 0,$$

$$B = 2\beta(1-Z_k)\,[K_0(\beta)-K_1(\beta)]\,[K_0(\lambda)+K_1(\lambda)] < 0,$$

$$Z_k = \tfrac{1}{3}\,[\cos 2k_xd+\cos 2k_yd+\cos 2k_zd].$$

Let

$$\langle H\rangle^k = (\Phi_k, H\Phi_k)-(\Phi_0, H\Phi_0).$$

For reasons specified previously we shall take $\langle H\rangle^k_{\min}$ as our estimate for the energy of the one-particle state concerned. Appendix F gives

$$\langle H\rangle^k_{\min} \approx (8/3\pi)^{1/2}\hbar ck.$$

Observe that $(8/3\pi)^{1/2} \approx 0.9$ which, under the circumstances, may be regarded as a good approximation to unity. This means that $\langle H\rangle^k_{\min}$ is approximately the same as the corresponding value of the one-particle state in the massless Klein-Gordon field case. This result appears very reasonable. Since the theory ought to be Lorentz invariant one would expect that the energy of something which resembles a free and massless particle should be $\hbar ck$ in order to give a Lorentz energy-momentum 4-vector with momentum $\hbar \boldsymbol{k}$.

In conclusion we note that if we attempt to estimate the one-particle energy by

$$(\Phi_k, H\Phi_k)_{\min} - (\Phi_0, H\Phi_0)_{\min},$$

we again are faced with a divergent result. The same applies to the expression

$$[(\Phi_k, H\Phi_k) - (\Phi_0, H\Phi_0)]_{\sigma_0}$$

where both expectation values are evaluated at the parameter value $\sigma = \sigma_0$ which optimizes $(\Phi_0, H\Phi_0)$.

4.4. Two-particle states, I

Considerable difficulties begin to emerge as we try to construct states resembling those two-particle states of linear theories. The inevitably approximate or even precarious nature of the idea of two-particle states in a nonlinear theory which allows strong interaction is obvious. In our present case however we expect, from the knowledge of plane wave solutions (I 5.4) that well-defined two-particle states may be formulated at least for two particles moving in the same direction. There should be no interaction between these particles. The idea can be formulated in an exact fashion. Let us confine the field in a box V with the usual periodic boundary conditions. Let

$$\varphi_{nk} = V^{-1/2} \exp(ink_\varrho x^\varrho),$$

where $k_j = (2\pi/L) \times$ integer; $k_0 = |\boldsymbol{k}| = k$; $n =$ integer; $k_\varrho k^\varrho = n^{\mu\nu} k_\mu k_\nu = 0$. Then (no summation over n)

$$\xi_{nk} = A_n \overset{*}{\varphi}_{nk} + \overset{*}{A}_n \varphi_{nk} = A_n \varphi_{-nk} + \overset{*}{A}_n \overset{*}{\varphi}_{-nk}, \tag{4.6}$$

is a real solution to our nonlinear equation (1.4). The general solution representing a wave travelling in the direction specified by a $\boldsymbol{k}$ is (we assume that the three integers $\boldsymbol{L}\boldsymbol{k}/2\pi$ have no common factors so that we need not consider submultiples of $\boldsymbol{k}$)

$$\xi_k = \sum_{n>0} \xi_{nk}. \tag{4.7}$$

Hence

$$\pi_k = (Q\dot{\xi}_k/c)(1 - \eta^{\varrho\sigma}\xi_{k,\varrho}\xi_{k,\sigma})^{-1/2} = Q\dot{\xi}_k/c^2,$$

$$(c\pi_k/Q)^2 = (\nabla\xi_k)^2.$$

Note that no summation over the subscript $\boldsymbol{k}$ is meant for all expressions in the Section. One finds easily that

$$\mathscr{H}_k = Q([1+(\nabla\xi_k)^2][1+(c\pi_k/Q)^2])^{1/2} =$$

$$= Q[1+(\nabla\xi_k)^2] = Q + \tfrac{1}{2}[(\nabla\xi_k)^2 + (c\pi_k/Q)^2]. \tag{4.8}$$

We end up with a Hamiltonian which is linear in the sense that the energy contributions from different n-values are additive. Rewrite (4.7) as

$$\xi_k = \sum_{n>0} \left(\frac{\hbar c}{2Qnk}\right)^{1/2} (a_{nk}\varphi^*_{nk} + a^*_{nk}\varphi_{nk}).$$

Expressing the Hamiltonian and the momentum associated with the field in terms of a_{nk}, a^*_{nk} we obtain

$$H_k = \int d^3x \mathscr{H}_k = E_0 + \sum_{n>0} \hbar cnka^*_{nk}a_{nk}, \tag{4.9}$$

where $E_0 = QV + \frac{1}{2}\sum_{n>0} \hbar c\, nk$,

$$P_k = \int d^3x(-\nabla\xi_k)\pi_k = \sum_{n>0} \hbar n\boldsymbol{k}a^*_{nk}a_{nk} + \tfrac{1}{2}\sum_{n>0} \hbar n\boldsymbol{k}. \tag{4.10}$$

We now postulate that each plane wave solution becomes in the quantum theory a quantized harmonic oscillator in the same way as in linear field theories, that is, a^*_{nk}, a_{nk} are regarded as creation and annihilation operators with the usual boson commutation rules

$$[a_{nk}, a_{n'k}] = [a^\dagger_{nk}, a^\dagger_{n'k}] = 0; \qquad [a_{nk}, a^\dagger_{n'k}] = \delta_{nn'}.$$

H_k, P_k are now operators. We see that if we confine ourselves to the field excitation which is formed by plane waves travelling in the same direction, we can perform the quantization which leads to the exact solution of various problems about the particular field excitation. The procedure may be applied to any specific $\boldsymbol{k}$. Observe that the present results agree with the corresponding ones obtained by the variational method. Indeed the two theories reinforce each other.

A general picture in the quantum theory begins to emerge after we carry out the above quantization procedure for all $\boldsymbol{k}$ values.

In an adequate quantum theory of the nonlinear field we would expect that the linear vector space of all quantum states should contain special subspaces, the subspace associated with $\boldsymbol{k}$ being spanned by basis vectors $a^\dagger_{nk}|0\rangle$, $a^\dagger_{n_1k}a^\dagger_{n_2k}|0\rangle$, $a^\dagger_{n_1k}a^\dagger_{n_2k}a^\dagger_{n_3k}|0\rangle \ldots$

In such subspaces the field behaves like a free field consisting of noninteracting particles moving along the same direction. The next step is to explore the "unknown" region outside those "known" subspaces and this presents great difficulties.

4.5. Two-particle states, II

When two particles are travelling in the same direction, there is no interaction. Hence we only have to consider two particles moving at angle with each other, *i. e.* $\boldsymbol{k}_1 \neq$ positive constant $\times$ $\boldsymbol{k}_2$ where $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ are the two $\boldsymbol{k}$-vectors specifying the states of the two particles concerned. With such $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ we can always effect a Lorentz transformation to the centre-of-momentum frame of reference where the two vectors will be seen to be equal in magnitude but exactly opposite in direction. Therefore it is sufficient to investigate two-particle states with $\boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$ without loss of generality. A trial wave function which readily comes into one's mind is

$$\Phi_{k,-k} = \frac{2}{(M(M-2)\sigma^2)^{1/2}} \sum_{n_1 \neq n_2} \exp\,[i(n_1 - n_2) \cdot kd]\xi_{n_1}\xi_{n_2}\Phi_0.$$

This function is orthogonal to Φ_0, Φ_k and is a null eigenvector of the linear momentum operator and it also gives the correct two-particle energy for the massless Klein-Gordon field. However this trial wave function is not satisfactory because calculation shows that it does not lead to any interaction between the two particles. We have to try some other trial wave functions. Since the state involved is of momentum zero it is not unreasonable to combine the two null momentum eigenvectors Φ_0 and $\Phi_{k,-k}$ to form a new trial wave function

$$\varphi = a\Phi_0 + b\Phi_{k,-k}, \tag{4.11}$$

where a, b are constants to be regarded as two independent variational parameters in addition to the original σ in Φ_0 and $\Phi_{k,-k}$. The optimization of

$$E = (\varphi, H\varphi)/(\varphi, \varphi),$$

with respect to a, b leads to the eigenequation

$$\begin{pmatrix} (\Phi_0, H\Phi_0) & (\Phi_0, H\Phi_{k,-k}) \\ (\Phi_0, H\Phi_{k,-k})^* & (\Phi_{k,-k}H\Phi_{k,-k}) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix},$$

where use has been made of the orthonormality property of Φ_0 and $\Phi_{k,-k}$. The eigenvalues are

$$\begin{aligned} E^{\pm} &= \tfrac{1}{2}\,\{[(\Phi_0, H\Phi_0) + (\Phi_{k,-k}, H\Phi_{k,-k})] \pm \\ &\pm ([(\Phi_{k,-k}, H\Phi_{k,-k}) - (\Phi_0, H\Phi_0)]^2 + 4|\,(\Phi_0, H\Phi_{k,-k})|^2)^{1/2}\}, \\ E^+ - E^- &= \{[(\Phi_{k,-k}, H\Phi_{k,-k}) - (\Phi_0, H\Phi_0)]^2 + 4|(\Phi_0, H\Phi_{k,-k})|^2\}^{1/2}. \end{aligned}$$

The corresponding trial wave functions φ^+, φ^- are orthogonal to each other. We then regard E^- as an estimate for the vacuum energy and E^+ as an estimate of the energy of the two-particle state with linear momentum zero. Then

$$\Delta E = E^+ - E^- > (\Phi_{k,-k}, H\Phi_{k,-k}) - (\Phi_0, H\Phi_0),$$

will serve as an estimate of the two-particle energy relative to the vacuum which may well bring in nonvanishing interaction energy. There is still an unspecified parameter σ to be determined. The obvious choice which is in harmony with the procedure adopted in section 4.3 is to employ the σ which optimizes ΔE. Appendix G gives

$$(\Delta E)_{\min} \approx \sqrt{2}\,\hbar c/\pi d,$$

which diverges as $d \to \infty$. Some other choices of σ are tried without avail in Appendix G. However there is reason to believe that again it is the trial wave function which is at fault. To see this we can apply the trial wave function (4.11) to the massless Klein-Gordon field.

We find that the corresponding expression is (see Appendix G for details)

$$\Delta E_{KG} = E^{+}_{KG} - E^{-}_{KG} > (\Phi_{k,-k}, H_{KG}\Phi_{k,-k}) - (\Phi_0, H\Phi_0),$$

leading to a spurious interaction energy which also diverges like $1/d$. Under these circumstances we cannot reach any definite conclusion about the interaction between two particles.

5. *Concluding remarks*

The variational approach we studied so far has been able to lead to some positive results for one-particle states. The exact reason for its failure in the two-particle case is an open question. Some variants of the two-particle trial wave functions and of the variational procedures have been studied without much success. It is quite possible that we have just not hit upon a sufficiently good trial wave function. The appearance of a divergent spurious interaction energy of order $1/d$ between two massless Klein-Gordon particles lends support to this view — the trial wave functions are just not good enough to give complete cancellation of all the divergent terms. In the linear case it is certainly true that a more sophisticated trial wave function will effect such cancellation and yield the correct result. However the situation may not be so simple in the nonlinear case. It may well be that there are no such things as two-particle states and we are quite wrong in attempting to simulate such a state by our choice of trial wave function. Further work is needed to resolve this problem. Although the variational method in its present form has only limited success for our model field, there is no reason why the method cannot be further developed into a general theory applicable for other field theories, especially those of more conventional types. Obviously there is much scope for further development.

APPENDIX A

Properties of $U_{Kn}(x)$

(A) Fourier transform:

$$U_{Kn}(k) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx U_{Kn}(x) \exp(-ikx) =$$

$$= \begin{cases} (L/2\pi)^{\frac{1}{2}} \exp(-iknL) \text{ if } K-\pi/L < k < K+\pi/L, \\ (^1/_2)(L/2\pi)^{\frac{1}{2}} \exp(-iknL) \text{ if } k = K \pm \pi/L, \\ 0 \text{ otherwise.} \end{cases}$$

(B) Orthogonality: $\int_{-\infty}^{\infty} dx U^*_{Kn}(x) U_{K'n'}(x) = \delta_{KK'}\delta_{nn'}$.

(C) Completeness: $\sum_{K,n} U^*_{Kn}(x) U_{Kn}(x') = \delta(x-x')$.

(D) $\dfrac{\partial U_{Kn}}{\partial x} = iKU_{Kn} + \sum_{m \neq n} \dfrac{(-1)^{m-n}}{L(m-n)} U_{Km}$.

(E) $I^{K'-K}_{pnn'} = \int_{-\infty}^{\infty} dx U^*_{0p} U^*_{Kn} U_{K'n'}$.

Case I: $K'-K=0$.

$$I^{0}_{pnn'} = 3/4\sqrt{L} \qquad \text{if } p=n=n',$$

$$= \frac{1}{2\pi^2\sqrt{L}(p-n')^2}(1-(-1)^{p-n'}) \qquad \text{if } p=n\neq n',$$

$$= \frac{(-1)^{p+n+n'}}{2\pi^2\sqrt{L}}\left\{\frac{(-1)^p}{(p-n)(p-n')}+\frac{(-1)^n}{(n-p)(n-n')}+\frac{(-1)^{n'}}{(n'-p)(n'-n)}\right\} \text{ if } p\neq n\neq n'.$$

Case II: $K'-K=2\pi/L$.

$$I^{2\pi/L}_{pnn'} = 1/8\sqrt{L} \qquad \text{if } p=n=n',$$

$$= \frac{(-1)^{p-n'}-1}{4\pi^2\sqrt{L}(p-n)^2}+\frac{i(-1)^{p-n}}{4\pi\sqrt{L}(p-n')} \qquad \text{if } p=n\neq n',$$

$$= -\frac{(-1)^{p+n+n'}}{4\pi^2\sqrt{L}}\left\{\frac{(-1)^p}{(p-n)(p-n')}+\frac{(-1)^n}{(n-p)(n-n')}+\frac{(-1)^{n'}}{(n'-p)(n'-n)}\right\}$$

$$\text{if } p\neq n\neq n'.$$

Case III:

$I^{K'-K}_{pnn'}=0$ in all other cases apart from those obtainable by the symmetry properties of the expression $I^{K'-K}_{pnn'}$.

Symmetry properties of $I^{K'-K}_{pnn'}$:

(a) $I^{K-K'}_{pnn'} = (I^{K'-K}_{pnn'})^*$

(b) The order of the indices pnn' is irrelevant to the value of

$I^{K'-K}_{pnn'}$, *e.g.*,

$I^{K'-K}_{pnn'} = I^{K'-K}_{nn'p} = I^{K'-K}_{n'pn}$ and so on,

$I^{K'-K}_{ppn'} = I^{K'-K}_{n'pp}$ and so on.

Our present set of functions have certain similarities with the Bloch functions expressed in terms of the Wannier functions in solid state physics [9]. But they are in fact quite different.

APPENDIX B

The theorem

$$I=\int_{-\infty}^{\infty} dx F(\theta_a(x)) \approx \sum_n \left(LF(\langle\theta_a\rangle_n)+\frac{1}{2}\left(\frac{\partial^2 F}{\partial\theta_a\partial\theta_b}\right)_n \sum_{K\neq 0}\theta^*_{aKn}\theta_{bKn}\right).$$

Proof:

Expanding the number 1 in terms of the complete orthonormal set U_{Kn} we obtain

$$1=\sqrt{L}\sum_n U_{0n}(x);$$

so

$$I = \int dx F(\theta_a(x)) = \sqrt{L} \sum_n \int d^3x U_{0n} F(\theta_a) = \sum I_n,$$

$$I_n = \sqrt{L} \int dx U_{0n} F(\theta_a).$$

We assume θ_a to be slowly varying functions of x so that the main contribution to I_n comes from the values of θ_a near $x_n = nL$ in which region $\theta'_{an} = \theta_a(x) - \langle\theta_a\rangle_n \ll 1$.

Let $F(\theta_a) = F(\langle\theta_a\rangle_n + \theta'_{an})$, then a Taylor expansion gives (no summation over n)

$$F(\theta_a) = F(\langle\theta_a\rangle_n) + \left(\frac{\partial F}{\partial \theta_a}\right)_n \theta'_{an} + \frac{1}{2}\left(\frac{\partial^2 F}{\partial\theta_a \partial\theta_b}\right)_n \theta'_{an}\theta'_{bn} + \dots,$$

$$I_n = LF(\langle\theta_a\rangle_n) + \sqrt{L}\left(\frac{\partial F}{\partial\theta_a}\right)_n \int U_{0n}\theta'_{an} dx + \frac{1}{2}\sqrt{L}\left(\frac{\partial^2 F}{\partial\theta_a\partial\theta_b}\right)_n \int dx U_{0n}\theta'_{an}\theta'_{bn} + \dots.$$

Now our assumption that I_n depends mainly on the values of θ_a near $x_n = nL$ will mean that the same applies to each integral over a Taylor series term. Let $\theta'_{ab}(x)$ be of the order of, say $\lambda \ll 1$, for x near $x_n = nL$, then we can roughly estimate that $\int dx U_{0n}\theta'_{an}\theta'_{bn}\theta'_{cn}$ is smaller than $\int dx U_{0n}\theta'_{an}\theta'_{bn}$ by a factor λ. In this way we are able to establish a series approximation to I_n.

Now the proof of the theorem rests on a

Lemma: If f_n is a slowly varying real function of n, then

$$\sqrt{L}\sum_n f_n \int U_{0n}\theta'_{an}\theta'_{bn} dx \approx \sum_{n, K\neq 0} f_n \theta^*_{aKn}\theta_{bKn}.$$

Proof:

$$\sqrt{L}\sum_n f_n \int U_{0n}\theta_a\theta_b dx = \sum_{\substack{p,n,n',\\ K,K'}} \sqrt{L} f_p \theta^*_{aKn}\theta_{bK'n'} \int dx U_{0p} U^*_{Kn} U_{K'n'}.$$

The property (E) of U_{Kn} as given in Appendix A is used to evaluate the right-hand expression which gives

$$\text{RHS} = \sum_K \sqrt{L}\,[(1)+(2)+(3)+(4)+(5)],$$

where (no summation over K)

$$(1) = \sum_{p=n=n'} f_n\left[\tfrac{3}{4}\,\theta^*_{aKn}\theta_{bKn} + \tfrac{1}{8}\,\theta^*_{aKn}\theta_{bK+2\pi/Ln} + \tfrac{1}{8}\,\theta^*_{aKn}\theta_{bK-2\pi/Ln}\right],$$

$$(2) = \sum_{p=n\neq n'} f_n \left\{\frac{1-(-1)^{n-n'}}{2\pi^2(n-n')^2}\,\theta^*_{aKn}\theta_{bKn'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2(n-n')^2} + \frac{i(-1)^{n-n'}}{4\pi(n-n')}\right]\theta^*_{aKn} \times \right.$$

$$\left. \times\,\theta_{bK+2\pi/Ln'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2(n-n')^2} - \frac{i(-1)^{n-n'}}{4\pi(n-n')}\right]\theta^*_{aKn}\theta_{bK-2\pi/Ln'}\right\},$$

$$(3) = \sum_{p=n'\neq n} f_{n'} \left\{ \frac{1-(-1)^{n-n'}}{2\pi^2(n-n')^2} \theta^*_{aKn}\theta_{bKn'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2(n-n')^2} - \frac{i(-1)^{n-n'}}{4\pi(n-n')} \right] \theta^*_{aKn} \times \right.$$

$$\left. \times \theta_{bK+2\pi/Ln'} + \left[-\frac{1-(-1)^{n-n'}}{4\pi^2(n-n')^2} + \frac{i(-1)^{n-n'}}{4\pi(n-n')} \right] \theta^*_{aKn}\theta_{bK-2\pi/Ln'} \right\},$$

$$(4) = \sum_{n=n'\neq p} f_p \left\{ \frac{1-(-1)^{p-n}}{2\pi^2(p-n)^2} \theta^*_{aKn}\theta_{bKn} + \left[-\frac{1-(-1)^{p-n}}{4\pi^2(p-n)^2} - \frac{i(-1)^{p-n}}{4\pi(p-n)} \right] \theta^*_{aKn} \times \right.$$

$$\left. \times \theta_{bK+2\pi/Ln} + \left[-\frac{1-(-1)^{p-n}}{4\pi^2(p-n)^2} + \frac{i(-1)^{p-n}}{4\pi(p-n)} \right] \theta^*_{aKn}\theta_{bK-2\pi/Ln} \right\}.$$

Since f_p is slowly-varying with p we may make the assumption

$$\sum_{p\neq n} f_p \frac{1-(-1)^{p-n}}{(p-n)^2} \approx f_n \sum_{p\neq n} \frac{1-(-1)^{p-n}}{(p-n)^2} = \frac{\pi^2}{2} f_n;$$

$$\sum_{p\neq n} f_p \frac{(-1)^{p-n}}{p-n} \approx f_n \sum_{p\neq n} \frac{(-1)^{p-n}}{p-n} = 0.$$

Hence the $f_{n'}$ in (3) may be replaced by f_n and

$$(4) \simeq \sum_n f_n [\tfrac{1}{4} \theta^*_{aKn}\theta_{bKn} - \tfrac{1}{8} \theta^*_{aKn}\theta_{bK+2\pi/Ln} - \tfrac{1}{8} \theta^*_{aKn}\theta_{bK-2\pi/Ln}].$$

Similarly

$$(5) \simeq \sum_{n\neq n'} \frac{-f_n}{\pi^2(n-n')^2} (1-(-1)^{n-n'}) (\theta^*_{aKn}\theta_{bKn'} -$$

$$- \tfrac{1}{2} \theta^*_{aKn}\theta_{bK+2\pi/Ln'} - \tfrac{1}{2} \theta^*_{aKn}\theta_{bK-2\pi/Ln'}).$$

Adding up we obtain finally

$$\sqrt{L} \sum_n f_n \int dx U_{0n} \theta_a \theta_b \approx \sum_{Kn} f_n \theta^*_{aKn} \theta_{bKn}.$$

Some remarks on the above approximation are worth mentioning. Let $y(x)$ be a function of x, then we can readily show

$$\text{(A)} \; L \left\langle \frac{\partial y(x)}{\partial x} \right\rangle_n = \sum_{n'} \langle y \rangle_{n'} \frac{(-1)^{n-n'}}{(n-n')},$$

$$\text{(B)} \; L^2 \left\langle \frac{\partial^2 y(x)}{\partial x^2} \right\rangle_n = - \left[\sum_{n'\neq n} \frac{2(-1)^{n-n'}}{(n-n')^2} + \frac{\pi^2}{6} \langle y \rangle_n \right],$$

where

$$\langle y \rangle_n = (L)^{-1/2} \int y(x) U_{0n} dx.$$

Therefore our approximation used in proving the Lemma is equivalent to the aproximation

$$\langle \partial F(x)/\partial x \rangle_n \approx 0 \text{ and } \langle \partial^2 F(x)/\partial x^2 \rangle_n \approx 0,$$

where

$$F(x) = \sum_n f_n U_{0n}^{(x)}.$$

Now to apply the Lemma to prove the Theorem we have

$$\sqrt{L} \sum_n f_n \int dx U_{0n} \theta'_{an} \theta'_{bn} = \sqrt{L} \sum_n f_n (\int U_{0n} \theta_a \theta_b dx - \sqrt{L} \langle \theta_a \rangle_n \langle \theta_b \rangle_n) \approx$$

$$\approx \sum_{n, K \neq 0} f_n \theta^*_{aKn} \theta_{bKn}, \text{ by the Lemma.}$$

Applying this result to

$$I = \int_{-\infty}^{\infty} dx F(\theta_a(x)) = \sum_n I_n,$$

we immediately obtain the Theorem.

APPENDIX C

Evaluation of the Hamiltonian

$$H = \int d^3x \mathscr{H}, \text{ where } \mathscr{H} = Q([1+(\nabla\xi)^2]\,[1+(c\pi/Q)^2])^{1/2}.$$

Firstly approximation may be made for the derivatives of ξ.

$$\xi_{,1} = \sum_{Km} q_{Km} \left[iK^1 U_{Km}(x) + \sum_{m'^1 \neq m^1} \frac{(-1)^{m'^1 - m^1}}{L(m'^1 - m^1)} U_{Km'^1 m^2 m^3} \right] =$$

$$= \sum_{Km} q_{Km1} U_{Km}(x),$$

where

$$q_{Km1} = iK^1 q_{Km} + \sum_{m'^1 \neq m^1} \frac{(-1)^{m'^1 - m^1}}{(m'^1 - m^1)} q_{Km'^1 m^2 m^3}.$$

Since $q_{Km'^1 m^2 m^3}$ is assumed to be slowly-varying with m'^1, the second sum in the above expression is very small and may be neglected in order to be consistent with the approximations made in Appendix B. Now apply the Theorem to evaluate H:

$$H \approx \sum_n L^3([1+\langle \nabla\xi \rangle_n^2]\,[1+\langle c\pi/Q \rangle_n^2])^{1/2} + \frac{1}{2} \sum_n \left(\frac{\partial^2 \mathscr{H}}{\partial\theta_a \partial\theta_b} \right)_n \sum_{K \neq 0} \theta^*_{aKn} \theta_{bKn},$$

where $\theta_a \equiv (\xi_{,1}, \xi_{,2}, \xi_{,3}, c\pi/Q)$.

$\partial^2 \mathscr{H} / \partial\theta_a \partial\theta_b$ may be expressed in terms of the metric tensors $g_{\tau\kappa}$ and $g^{\tau\kappa}$.

$$\partial^2\mathscr{H}/\partial\pi^2 = (c^2/Q)[1+(\nabla\xi)^2]^{1/2}[1+(c\pi/Q)^2]^{-3/2} = c^2\sqrt{-g}/(Qg^{00});$$

$$\partial^2\mathscr{H}/\partial\pi\partial\xi_{,j} = (c^2\pi\xi_{,j}/Q)\,([1+(\nabla\xi)^2]\,[1+(c\pi/Q)^2])^{-1/2} = -cg^{0j}/g^{00};$$

$$\partial^2\mathscr{H}/\partial\xi_{,j}\partial\xi_{,k} = Q\left[\frac{1+(c\pi/Q)^2}{1+(\nabla\xi)^2}\right]^{1/2}\left(\delta_{jk}-\frac{\xi_{,j}\xi_{,k}}{1+(\nabla\xi)^2}\right) =$$

$$= -\frac{Q}{\sqrt{-g}}\left(g^{jk}-\frac{g^{0j}g^{0k}}{g^{00}}\right).$$

$$H \simeq \sum_n QL^3([1+\langle\nabla\xi\rangle_n^2]\,[1+\langle c\pi/Q\rangle_n^2])^{1/2}+\frac{1}{2}\sum_{\substack{n,j\\K\neq 0}}\frac{g_n^{0j}c}{g_n^{00}}(q_{Kn}^{\dagger}iK^jp_{Kn}-p_{Kn}^{\dagger}iK^jq_{Kn})+$$

$$+\sum_n\frac{1}{2}\frac{c^2}{Q}\frac{\sqrt{-g_n}}{g_n^{00}}\sum_{K\neq 0}p_{Kn}^{\dagger}p_{Kn}+$$

$$+\frac{1}{2}\sum_{\substack{n,j,k\\K\neq 0}}\frac{(-Q)}{\sqrt{-g_n}}\left(\frac{g_n^{0j}g_n^{0k}}{g_n^{00}}-g_n^{jk}\right)\sum_{K\neq 0}K^jK^kq_{Kn}^{\dagger}q_{Kn},$$

where $g_n^{\tau\kappa} = [g^{\tau\kappa}]_{\pi=\langle\pi\rangle_n}$, $\xi_{,j} = \langle\xi_{,j}\rangle_n$.

APPENDIX D

The approximately classical nature of $g_n^{\tau\kappa}$

Two assumptions are made in relation to the everywhere-slowly-varying nature of the field.

(1) $\theta'_{an}(x) = \theta_a(x)-\langle\theta_a\rangle_n \ll \theta_a(x)$ and $\theta'_{an}(x) \ll 1$ in a neighbourhood of $x = nL$ of dimensions of the order L. Note that θ_a stand for the dimensionless quantities $\xi_{,j}$, $c\pi/Q$ so that above inequalities are independent of the units employed.

(2) L can be large, or more precisely, we require that $\hbar c/QL^4 \ll 1$, where $\hbar c/QL^4$ is a dimensionless quantity. In CGS units with $Q = 10^{-7}$ erg cm$^{-3} \approx$ the mean energy density of the universe, we get $L \gg \hbar c/Q)^{\frac{1}{4}} \approx 10^{-2.5}$ cm.

Firstly these assumptions enable us to count the orders of smallness of a quantity (see Appendix B). Secondly they lead to the result that $g_n^{\tau\kappa}$ may be regarded as c-numbers. This results is seen in the following analysis based on a finite difference method:

$$\xi_{,1}(x) \approx (\xi(x^1+L, x^2, x^3)-\xi(x^1-L, x^2, x^3))/2L.$$

Hence $\langle\xi_{,1}\rangle_n \approx 1(q_{0n+}-q_{0n-})/2L^{5/2}$, where $\boldsymbol{n}+ = (n^1+1, n^2, n^3)$, $\boldsymbol{n}- = (n^1-1, n^2, n^3)$;

$$\langle c\pi/Q\rangle_{n'} = cP_{0n'}/QL^{3/2}.$$

The commutator

$$[\langle\xi_{,1}\rangle_n, \langle c\pi/Q\rangle_{n'}] \approx (\hbar c/2QL^4)\,(i\delta_{n+n'}-i\delta_{n-n'}) \to 0 \text{ as } \hbar c/QL^4 \to 0.$$

Since $\langle E_{,j}\rangle_n$, $\langle c\pi/Q\rangle_{n'}$ also commute with all the other operators appearing in the theory, we conclude that $\langle \xi_{,j}\rangle_n \langle c\pi/Q\rangle_{n'}$ can be treated as c-numbers. The set of quantities $g_n^{\tau\kappa}$ depends solely on $\langle \xi_{,j}\rangle_n$, $\langle c\pi/Q\rangle_n$. Hence they may be similarly taken as c-numbers.

APPENDIX E

The vacuum expectation value

(1) $R_0^{(n)} = (\Phi_{0n}, [1+(cP_n/Qd^3)^2]^{1/2}\Phi_{0n}) = (\sigma Qd^3/2\sqrt{\pi}\, ch)\exp(\lambda)\,[K_0(\lambda)+K_1(\lambda)].$

$\Phi_{0n} = (\pi\sigma^2)^{-1/4}\exp(-\xi_n^2/2\sigma^2).$

(2) $\Delta_0^{(n)} = (\Phi_0, \sqrt{1+(\Delta\xi_n)^2}\,\Phi_0) = (\prod_{m\neq n}\Phi_{0m}, \sqrt{1+(\Delta\xi_n)^2}\prod_{m\neq n}\Phi_{0m}) =$

$= (\sqrt{2/\pi}\, d/\sigma)\exp(\beta)K_1(\beta).$

(3) $(\Phi_0, H_n\Phi_0) = Qd^3\Delta_0^{(n)}R_0^{(n)} =$

$= (Q^2d^7/\sqrt{2}\,\pi ch)\exp(\lambda+\beta)K_1(\beta)\,[K_0(\lambda)+K_1(\lambda)].$

(4) Minimization of $(\Phi_0, H_n\Phi_0)$ in the limit of vanishing d.

There are only 5 possibilities for the behaviour of the optimum λ and β as $d\to 0$: $\lambda\to\infty$, $\beta\to 0$; $\lambda\to 0$, $\beta\to 0$; $\lambda\to 0$, $\beta\to\infty$; $\lambda\to$ finite and non-zero, $\beta\to 0$; $\lambda\to 0$, $\beta\to$ finite and non-zero.

Using the known asymptotic behaviour of the relevant Bessel functions one can verify that a consistent minimization is possible only for the case $\lambda_{\text{opt}}\to 0$, $\beta_{\text{opt}}\to 0$. From numerical computation we also know that $(\Phi_0, H_n\,\Phi_0)$ possesses a minimum for small d. Hence we conclude that as $d\to 0$, the minimum occurs with $\lambda_{\text{opt}}\to 0$, $\beta_{\text{opt}}\to 0$ and we have

$$(\Phi_0, H\Phi_0)_{\min}\approx\frac{Q^2d^7M}{\sqrt{2}\,\pi\hbar c}\left(\frac{2}{\varrho^2}+\frac{2\sqrt{2\ln(1/\varrho)}}{\varrho}+\dots\right).$$

APPENDIX F

One-particle states

(1) $R_1^{(n)} = (\xi_n\Phi_{0n}, [1+(cP_n/Qd^3)^2]^{1/2}\xi_n\Phi_{0n}) = (\sigma^3Qd^3/2\sqrt{\pi}\, ch)\exp(\lambda)K_1(\lambda).$

(2) $\Delta_{mm'}^{(n)} = (\xi_m\Phi_0, \sqrt{1+(\Delta\xi_n)^2}\,\xi_{m'}\Phi_0) =$

$$=\begin{cases}(d\sigma/6\sqrt{2\pi})\exp(\beta)[2\beta(K_0(\beta)-K_1(\beta)+7K_1(\beta)] & \text{if } \boldsymbol{m}=\boldsymbol{m}'\in S_{\pm}^{(n)},\\ -(d\sigma/6\sqrt{2\pi})\exp(\beta)[2\beta(K_0(\beta)-K_1(\beta))+K_1(\beta))] & \text{if } \boldsymbol{m}, \boldsymbol{m}'\in P_{\pm}^{(n)},\\ (d\sigma/\sqrt{2\pi})\exp(\beta)K_1(\beta) & \text{if } \boldsymbol{m}=\boldsymbol{m}'\in S_{\pm}^{(n)},\\ 0 & \text{otherwise,}\end{cases}$$

where $S_{\pm}^{(n)}$ is the following set

$$(n_1\pm 1, n_2, n_3);\ (n_1, n_2\pm 1, n_3);\ (n_1, n_2, n_3\pm 1),$$

and $P_{\pm}^{(n)}$ is the following set of pairs

$$(n_1+1, n_2, n_3), (n_1-1, n_2, n_3); (n_1-1, n_2, n_3), (n_1+1, n_2, n_3); (n_1, n_2+1, n_3), (n_1, n_2-1, n_3);$$

$$(n_1, n_2+1, n_3), (n_1, n_2-1, n_3); (n_1, n_2, n_3+1), (n_1, n_2, n_3-1); (n_1, n_2, n_3-1), (n_1, n_2, n_3+1).$$

Let $\Delta_1^{(n)} = \Delta_{m,m}^{(n)}$ with $\boldsymbol{m} \in S_{\pm}^{(n)}$,

$\Delta_{1\pm}^{(n)} = \Delta_{m,m'}^{(n)}$ with $\boldsymbol{m}, \boldsymbol{m}' \in P_{\pm}^{(n)}$.

Then

$$\Delta_1^{(n)} + \Delta_{1+}^{(n)} = (2\pi)^{-1/2} d\sigma \exp(\beta) K_1(\beta) = \sigma^2 \Delta_0/2.$$

(3) $$(\Phi_k, H_n\Phi_k) = (2/M\sigma^2) \sum_{m \neq m'} \exp(i[\boldsymbol{m}'-\boldsymbol{m}] \cdot \boldsymbol{k}d)(\xi_m\Phi_0, H_n\xi_{m'}\Phi_0) =$$
$$= (2Qd^3/M\sigma^2)[6R_0\Delta_1 + 6R_0\Delta_{1\pm}Z_k + R_1\Delta_0 - 7\sigma^2\Delta_0R_0/2] =$$
$$= (\Phi_0, H_n\Phi_0) + (Q^3d^7/\sqrt{2\pi}\, c\hbar M) \exp(\lambda+\beta)(A+B),$$

where

$$A = K_1(\beta)[2K_1(\lambda) - (K_0(\lambda)+K_1(\lambda))Z_k] > 0,$$
$$B = 2\beta(1-Z_k)[K_0(\beta) - K_1(\beta)][K_0(\lambda)+K_1(\lambda)] < 0.$$

(4) The minimization of

$$\langle H\rangle^k = (\Phi_k, H\Phi_k) - (\Phi_0, H\Phi_0),$$

in the limit of vanishing d may be carried out in the same way as for the case of the vacuum expectation value. A consistent minimization is possible only if $\lambda_{\text{opt}} \to \infty$, $\beta_{\text{opt}} \to 0$. Using the known asymptotic properties of the relevant Bessel functions we obtain

$$\langle H^k\rangle \approx (2\pi)^{-\frac{1}{2}} Qd^4(1/\varrho^2\sigma + 4k^2\sigma/3).$$

Hence $\langle H\rangle_{\min}^k \approx (8/3\pi)^{1/2}\hbar ck$.

APPENDIX G

Two-particle states II

The trial wave function $a\Phi_0 + b\Phi_{k,-k}$, gives

$$E^{\pm} = \tfrac{1}{2}\{[(\Phi_{k,-k}, H\Phi_{k,-k}) + (\Phi_0, H\Phi_0)] \pm$$
$$\pm([\Phi_{k,-k}, H\Phi_{k,-k}) - (\Phi_0, H\Phi_0)]^2 + 4|(\Phi_0, H\Phi_{k,-k})|^2)^{1/2}\},$$
$$\Delta E = E^+ + E^- = ([(\Phi_{k,-k}, H\Phi_{k,-k}) - (\Phi_0, H\Phi_0)]^2 + 4|(\Phi_0, H\Phi_{k,-k})|^2)^{1/2}.$$

(1) The nonlinear field

$$(\Phi_{k,-k}, H\Phi_{k,-k}) \approx 2(\Phi_k, H\Phi_k) - (\Phi_0, H\Phi_0);$$

$$(\Phi_0, H\Phi_{k,-k}) = (M/(M-1))^{1/2} 12Qd^3R_0^{(n)}\Delta_{1\pm}^{(n)}Z_k/\sigma^2 \approx 12Qd^3R_0^{(n)}\Delta_{1\pm}^{(n)}Z_k/\sigma^2.$$

Clearly

$$\Delta E \approx 2(\langle H\rangle^k + |(\Phi_0, H\Phi_{k,-k})|^2)^{1/2} \geqslant 2\langle H\rangle^k,$$

hence $(\Delta E)_{\min} \geqslant 2(8/3\pi)^{1/2}\hbar ck$.

To actually estimate $(\Delta E_{\min})$ let

$$Y_k = 1-Z_k \to 2d^2k^2/3+O(d^4) \quad \text{as } d\to 0.$$

Now

$$\langle H\rangle^k = (2Qd^3/\sigma^2)\,[6R_0\Delta_1+6R_0\Delta_{1\pm}Z_k+R_1\Delta_0-(7/2)\sigma^2\Delta_0R_0] =$$
$$= (2Qd^3/\sigma^2)\,[(R_1-\sigma^2R_0/2)\Delta_0-6Y_kR_0\Delta_{1\pm}].$$

The rather tedious procedure of minimization of ΔE may be carried out as before. The only consistent minimization occurs when

$$\lambda_{\rm opt}\to 0;\ \beta_{\rm opt}\to 0 \quad \text{as } d\to 0.$$

The result is

$$(\Delta E)_{\min} \simeq \sqrt{2}\,\hbar c/\pi d, \quad \text{which diverges as } d\to 0.$$

Two other assignments of the value of ΔE are also considered. The first alternative is to take ΔE calculated with the value of σ which minimizes E^-. Obviously this also leads to a divergent energy since $\Delta E > (\Delta E)_{\min}$. The other alternative is to consider

$$\Delta E = E^+_{\min}-E^-_{\min}.$$

This again may readily be shown to be bigger than $(\Delta E)_{\min}$, and hence divergent.

(2) The massless Klein-Gordon field

Let us apply the above trial wave function to the real massless Klein-Gordon field to estimate the corresponding two-particle energy. Then

$$\Delta E_{KG} = ([(\Phi_{k,-k}, H_{KG}\Phi_{k,-k})-(\Phi_0, H_{KG}\Phi_0)]^2+4|(\Phi_0, H_{KG}\Phi_{k,-k})|^2)^{1/2}.$$

Calculation gives

$$(\Phi_0, H_{KG}\Phi_{k,-k}) = -3Qd\sigma^2Z_k/4,$$

and

$$(\Delta E_{KG})_{\min} \approx \sqrt{3}\,\hbar c/d,$$

which gives a spurious and divergent interaction energy. The other two choices of the value of σ in ΔE only lead to values bigger than this one.

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[2] Notation: $x^0 = ct$, $\boldsymbol{x} = (x^1, x^2, x^3)$. Lower case Greek indices take the values 0, 1, 2, 3, and lower case Latin indices the values 1, 2, 3. The summation convention for repeated indices is adopted.

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