PHYSICAL INTERPRETATION OF A MANIFESTLY COVARIANT HAMILTONIAN FORMALISM FOR DIRAC PARTICLE

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As a continuation of considerations concerning a covariant Hamiltonian formalism previously proposed by the author, the FW-representation of the homogeneous Dirac equation and the definitions of the "mean" position and spin operators have been reformulated in a manifestly covariant way. Problems of a consistent physical interpretation of the displayed mathematical scheme have been discussed and some of its controversional aspects explained.

1. Introduction

Possibilities and limitations of a consistent formulation of "relativistic quantum mechanics" have been discussed from various points of view by many authors in recent years. Although every such scheme possesses an approximate and limited character only (being *e.g.* restricted to those physical situations which guarantee that the number of particles remains constant) it seems that an "autonomic" sense can be attributed to this problem, rather than the sense of a "by-product" of more general field-theoretical considerations[1]. The former point of view adopted in our considerations (and supported by some further arguments) requires a more detailed analysis of whether, and under what conditions, such a "particle description" can consistently be introduced. However, the problem is, as yet, far from an ultimate solution. The main difficulties arise in consequence of the non-unique way of generalizing into the relativistic domain the notions of position and spin[2]. Currently, this problem occupies an important place in many publications on the subject. Moreover, the distinctive role played in every quantum-mechanical formulation

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[1] In order to avoid a misunderstanding caused by different interpretations of the term "relativistic quantum mechanics" in different contexts, the sense in which this term will be used in our paper ought to be specified. By it we will denote every scheme based on the notion of "particle-observables", including that of position.

[2] of a particle, or of the centre of mass for a system of particles, as shown many years ago by Pryce ([1], [2]).

by the time variable, as compared to that of the space coordinates, is also the source of considerable complication. Attempts to introduce a time (or proper time) operator are as frequent as those trying to avoid this assumption. Our present considerations follow the latter path, using the well known covariant distinction between the timelike and spacelike directions in Minkowski space based on the notion of an arbitrary unit timelike vector and of a three-dimensional manifold orthogonal to the former. In particular, a covariant quantum-mechanical Hamiltonian formalism, previously proposed by us (Hanus [3], [4]) and applied to the special case of the Dirac particle (Hanus, Słomiński [5]) will be used and further developed. A short recapitulation of our earlier results — which establish the mathematical background for the present considerations — will be given in Chapter 2. The formalism allows a further generalization to include the problems of the FW-representation of the Dirac equation (Foldy, Wouthuysen [6], see also Tani [7]) and of the "mean" operators immediately related to that representation. Owing to the important role played by these operators in the "particle-interpretation" of the Dirac equation, their formal covariant generalization, proposed in Chapter 3, gives an advantageous starting point for attempts at constructing a consistent scheme of physical interpretation related to the displayed mathematical formalism.

2. *Covariant Hamiltonian for the free Dirac particle*

According to the proposed formalism (see [4]), the position and momentum observables $\boldsymbol{x}$ and $\boldsymbol{p}$ of a particle possessing a mass $m \neq 0$ and described by a covariant first-order homogeneous wave equation have been generalized into simple bivectors (plane antisymmetrical pseudotensors of second rank) in the Minkowski space $\hat{X}_{\kappa\lambda}$ and $\hat{P}_{\kappa\lambda}$, respectively[3] defined in terms of the "formal" relativistic operators x_μ and p_μ fulfilling the standard commutation relations

$$[x_\mu, p_\nu] = i\delta_{\mu\nu}. \tag{1}$$

The "generalized particle-observables" related to the position and momentum of the particle have the form

$$\hat{X}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\kappa\lambda\mu\nu}X_{\mu\nu}, \qquad X_{\mu\nu} = n_\mu x_\nu - n_\nu x_\mu, \tag{2}$$

$$\hat{P}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\kappa\lambda\mu\nu}P_{\mu\nu}, \qquad P_{\mu\nu} = n_\mu p_\nu - n_\nu p_\mu, \tag{3}$$

while

$$x_N = -in_\mu x_\mu \tag{4}$$

[3] The notation introduced in [4] and [5] will be used in this paper. In particular, $\hbar = c = 1$, the Dirac symbols have their standard meaning, $\varepsilon_{\kappa\lambda\mu\nu}$ stands for the completely antisymmetrical pseudotensor of fourth rank (with $\varepsilon_{1234} = 1$), while an arbitrary unit timelike vector n_μ is defined by the conditions $n_\mu n_\mu = -1$, $n_4 = in_0$, $n_0 > 0$. The symbol "$\wedge$" denotes dual tensors.

has appeared to play the role of a "c-number timelike variable" and

$$p_N = -in_\mu p_\mu, \tag{5}$$

owing to the mass relation $p_\mu p_\mu = -m^2$, has been expressed in terms of the "spacelike quantities" $\hat{P}_{\kappa\lambda}$ and reinterpreted as the covariant Hamiltonian operator

$$\mathscr{H}_N = -ip_N(\hat{P}_{\kappa\lambda}, m), \tag{6}$$

leading to the Schrödinger equation

$$-d_N\psi = \mathscr{H}_N\psi, \qquad d_N = -in_\mu \frac{\partial}{\partial x_\mu}. \tag{7}$$

This equation plays the role of a subsidiary condition imposed on the solutions of the basic wave equation and selects the subspace of "particle states" $\psi(\hat{X}_{\kappa\lambda}x_N)$. Additional constraints and commutation rules obeyed by $\hat{X}_{\kappa\lambda}$ and $\hat{P}_{\kappa\lambda}$ have been deduced in [4], but they do not need to be quoted here. The division into "strong" and "weak" relations follows the idea of Dirac [8]. Similar, although somewhat more complicated formulae defining the covariant counterparts of the orbital, spin and total angular momenta of the Dirac particle $\boldsymbol{l}$, $\boldsymbol{s} = \frac{1}{2}\boldsymbol{\sigma}$ amd $\boldsymbol{j} = \boldsymbol{l}+\boldsymbol{s}$, successively, deduced in [5], read

$$\hat{L}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\kappa\lambda\mu\nu}L_{\mu\nu}, \qquad L_{\mu\nu} = n_\mu l'_\nu - n_\nu l'_\mu, \tag{8}$$

$$l'_\nu = -\frac{i}{2}\,\varepsilon_{\nu\varrho\tau\omega}l_{\varrho\tau}n_\omega, \qquad l_{\varrho\tau} = x_\varrho p_\tau - x_\tau p_\varrho, \tag{9}$$

$$\hat{S}_{\kappa\lambda} = \frac{1}{2}\hat{\Sigma}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\kappa\lambda\mu\nu}S_{\mu\nu}, \qquad S_{\mu\nu} = \frac{1}{2}\Sigma_{\mu\nu} = n_\mu s'_\nu - n_\nu s'_\mu, \tag{10}$$

$$s'_\nu = -\frac{i}{2}\,\varepsilon_{\nu\varrho\tau\omega}s_{\varrho\tau}n_\omega, \qquad s_{\varrho\tau} = \frac{1}{2}\sigma_{\varrho\tau} = \frac{1}{4i}(\gamma_\varrho\gamma_\tau - \gamma_\tau\gamma_\varrho), \tag{11}$$

$$\hat{J}_{\kappa\lambda} = \hat{L}_{\kappa\lambda} + \hat{S}_{\kappa\lambda}. \tag{12}$$

The characteristic result

$$l'_N = -in_\mu l'_\mu = 0, \qquad s'_N = -in_\mu s'_\mu = 0, \qquad j'_N = l'_N + s'_N = 0 \tag{13}$$

shows that, in contradistinction to the polar three-vectors, the axial vectors do not possess any timelike scalars as a relativistic supplements within the proposed formalism.

The covariant Dirac Hamiltonian has been expressed in the form

$$\mathscr{H}_N = \varrho_{\mathrm{III}}m + \varrho_{\mathrm{I}}\Theta_N, \tag{14}$$

where

$$\varrho_{\mathrm{III}} \equiv \gamma_N = -in_\mu\gamma_\mu, \qquad \varrho_{\mathrm{I}} = -\gamma_5, \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \qquad \varrho_{\mathrm{II}} = -i\varrho_{\mathrm{III}}\varrho_{\mathrm{I}}, \tag{15}$$

the ϱ_i thus being covariant generalizations of the well known Dirac operators ϱ_3, ϱ_1, ϱ_2, while

$$\Theta_N = \hat{S}_{\kappa\lambda}\hat{P}_{\kappa\lambda} = \tfrac{1}{2}\hat{\Sigma}_{\kappa\lambda}\hat{P}_{\kappa\lambda}, \tag{16}$$

$$\Theta_N^2 = \tfrac{1}{2}\hat{P}_{\kappa\lambda}\hat{P}_{\kappa\lambda} \equiv \hat{P}^2, \quad \hat{P} = (\hat{P}^2)^{1/2}, \tag{17}$$

$$\chi_N = \Theta_N(\Theta_N^2)^{-1/2} = \frac{\Theta_N}{\hat{P}}, \tag{18}$$

$$E_N = (\mathscr{H}_N^2)^{1/2} = \sqrt{m^2+\hat{P}^2}, \tag{19}$$

$$\lambda_N = \mathscr{H}_N(\mathscr{H}_N^2)^{-1/2} = \frac{\mathscr{H}_N}{E_N} \equiv \text{sign}\,\mathscr{H}_N. \tag{20}$$

Θ_N, $\hat{P}$, χ_N, E_N and λ_N are, obviously, covariant analogues of $\boldsymbol{\sigma}\cdot\boldsymbol{p}, p, \chi = (\boldsymbol{\sigma}\cdot\boldsymbol{p})/p$, $E = \sqrt{m^2+p^2}$ and λ = sign $\mathscr{H}$, respectively (for details see [3] and [5]). A close connection existing between Θ_N and the Bargmann-Wigner pseudovector w_μ is worth mentionning. It is given by

$$\Theta_N = 2iw_N, \quad w_N = -in_\mu w_\mu, \quad w_\mu = \frac{i}{2}\varepsilon_{\mu\nu\varrho\tau}p_\nu\frac{1}{2}\sigma_{\varrho\tau}. \tag{21}$$

The formula quoted in (1)-(20) correspond to the conventional interpretation of the Dirac equation, with $\boldsymbol{x}$ and $\dot{\boldsymbol{x}} = \boldsymbol{\alpha}$ in the roles of the position and velocity observables, respectively. (The explicit formula for a covariant generalization of the latter operator has been deduced in [5], but it will not be used in our further considerations).

3. The covariant FW-representation

The well known FW-representation of the Dirac equation immediately relates two important problems to one another: one of separating positive and negative energy states of the particle, and one of defining and interpreting the "mean" operators of its position and spin[4]. General ideas concerning the problem of localizability of relativistic elementary systems (hence, in particular of the Dirac particle) introduced by Newton and Wigner [9] have been extended by Wightman [10]. All these works imply the essentially non--covariant character of the mean-operators which give the localization in space at a given time, but not the space-time localization. The same point of view is represented in the papers of Matthews and Sankaranarayanan ([11], [12], [13]) where the distinctive physical meaning of the FW-mean-operators, as compared to other possible definitions, has been stressed[5]. This lack of covariance gives rise to the fact that instead of the mean-operators other candidates for position and spin observables, in particular the covariant operators

[4] These operators, introduced earlier by Pryce (in [1] and [2]) are, alternatively, denoted in the literature as "Pryce's operators case e" or "local operators of Newton and Wigner", or "mean-operators of Foldy, Wouthuysen and Tani". We shall use the term "mean-operators" closely related to their physical interpretation.

[5] for some generalization of these results see Hanus and others [14] and Janyszek, Rakowski [15].

(Pryce's case "d"; see also Hilgevoord, Wouthuysen [16]) are often preferred in relativistic considerations, despite the rather unsatisfactory algebraic properties of the latter operators[6].

The opinion that no covariant meaning can be attributed to the mean-operators is not generally accepted. Indeed, there is still one possibility, which does not contradict the previously quoted results: to try to define the mean-operators using the notion of localization on an arbitrary spacelike hyperplane (or, more generally, on a hypersurface). An interesting formalism following this idea has been developed in the years 1963–1970 by Fleming ([18]–[25]). In [18] and [19] the notion of an arbitrary spacelike hyperplane has been used, for the first time, in order to generalize investigations concerning position as a dynamical variable in relativistic quantum theory. Various possible definitions of this variable have been analysed, successively, by this author, in accordance with his opinion that there is no priviledged, "true" position observable, but that its different definitions are related to different dynamical properties of the described particle (the mean-position operator representing the centre of spin), and that all these definitions, if appropriately interpreted, remain consistent with the postulates of relativity. In [20]–[25] these results have been progressively extended into a wide quantum scheme comprising — besides the covariant description of dynamical variables defined on spacelike hyperplanes — also the general theory of the scalar field. It is worth mentionning that this is a scheme reaching far outside the scope of conventional field-theoretical formulations, as the seven-dimensional continuous manifold of the points (x_μ, n_λ) has been used for a geometrical basis and the formalism has become equivalent to a non-local version of quantized field theory.

Our considerations, although inspired, to some extent, by the papers of Fleming, have followed a different path: they have remained within the traditional geometrical scheme of the Minkowski space. Moreover, they have been limited from the outset to the treatment of relativistic quantum-mechanical problems. The latter restriction alone strongly implies the necessity of using a formalism which gives a covariant distinction between the timelike and spacelike directions, These remarks explain in more detail the main purpose of our present considerations. The extension of the covariant Hamiltonian formalism (summarized in Chapter 2) from the traditional to the new, modified method of describing the Dirac particle in the FW-representation and in terms of the mean operators, completes our earlier results in a natural way. Simultaneously, the incorporation of the fundamental problem of relativistic position and spin observables into our formalism gives the best test for verifying to what extent this mathematical scheme allows a consistent physical interpretation.

There are no formal obstacles against formulating in a manifestly covariant way the FW-transformation and the transformed Dirac observables. Similarly, as the non-covariant Dirac Hamiltonian

$$\mathscr{H} = \varrho_3 m + \varrho_1 \boldsymbol{\sigma} \cdot \boldsymbol{p}, \tag{22}$$

transformed by the unitary FW-transformation

$$U = e^{is}, \quad S = \frac{1}{2}\varrho_2 \chi \operatorname{arctg}\frac{p}{m} \tag{23}$$

[6] This path has been followed *e. g.* in the paper of Suttorp and de Groot [17].

assumes the form

$$\mathscr{H}' = U\mathscr{H}U^{-1} = \varrho_3 E, \tag{24}$$

its covariant counterpart (14) transormed by

$$U_N = e^{isN}, \qquad S_N = \frac{1}{2}\,\varrho_{\rm II}\chi_N \,\mathrm{arctg}\,\frac{\hat{P}}{m} \tag{25}$$

gives

$$\mathscr{H}'_N = U_N \mathscr{H}_N U_N^{-1} = \varrho_{\rm III} E_N. \tag{26}$$

(It is obvious that, owing to the close correspondence between (14) and (22) this form of the Dirac Hamiltonian is more suitable for our considerations than that containing $\boldsymbol{\alpha}$ and β.) The explicit form of other non-covariant FW-transformed observables is well known[7]. We have expressed them, however, in a form somewhat different from the standard one, but more suitable for our further covariant generalization. Using two formulae (which can easily be verified)

$$\boldsymbol{p}\times(\sigma\times\boldsymbol{p}) = p^2\sigma-(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}, \tag{27}$$

$$\boldsymbol{\sigma}\times\boldsymbol{p} = \frac{i}{2}\,\{\sigma(\boldsymbol{\sigma}\cdot\boldsymbol{p})-(\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma\} = \frac{i}{2}\,[\boldsymbol{\sigma},(\boldsymbol{\sigma}\cdot\boldsymbol{p})], \tag{28}$$

we obtain

$$\boldsymbol{p}' = \boldsymbol{p}, \tag{29}$$

$$\boldsymbol{x}' = \boldsymbol{x} - \frac{i}{2}\,\frac{[\boldsymbol{\sigma},(\boldsymbol{\sigma}\cdot\boldsymbol{p})]}{2E(E+m)} + \varrho_2\left\{\frac{\boldsymbol{\sigma}}{2E} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}}{2E^2(E+m)}\right\}, \tag{30}$$

$$\boldsymbol{l}' = (\boldsymbol{x}\times\boldsymbol{p})' = \boldsymbol{l} + \frac{p^2\boldsymbol{\sigma}-(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}}{2E(E+m)} + \varrho_2\,\frac{i}{2}\,\frac{[\boldsymbol{\sigma},(\boldsymbol{\sigma}\cdot\boldsymbol{p})]}{2E}, \tag{31}$$

$$\boldsymbol{s}' = \frac{1}{2}\,\boldsymbol{\sigma}' = \frac{1}{2}\,\boldsymbol{\sigma} - \frac{p^2\boldsymbol{\sigma}-(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}}{2E(E+m)} - \varrho_2\,\frac{i}{2}\,\frac{[\boldsymbol{\sigma},(\boldsymbol{\sigma}\cdot\boldsymbol{p})]}{2E}, \tag{32}$$

$$\boldsymbol{j}' = \boldsymbol{l}'+\boldsymbol{s}' = \boldsymbol{l}+\boldsymbol{s} = \boldsymbol{j}, \tag{33}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})' = \boldsymbol{\sigma}, \boldsymbol{p}, \tag{34}$$

$$E' = E, \tag{35}$$

$$\lambda' = \varrho_3. \tag{36}$$

Remembering that $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{l}$ and $\boldsymbol{\sigma}$ have been generalized into the covariant quantities $\hat{X}_{\kappa\lambda}$, $\hat{P}_{\kappa\lambda}$, $\hat{L}_{\kappa\lambda}$, and $\hat{\Sigma}_{\kappa\lambda}$, respectively, and taking into account that in this situation the expressions (27) and (28) also possess their covariant analogues, namely

$$p^2\boldsymbol{\sigma}-(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p} \rightarrow \hat{P}^2\hat{\Sigma}_{\kappa\lambda} - \Theta_N\hat{P}_{\kappa\lambda} \tag{37}$$

[7] see [6], [7], or anyone of numerous monographs on this subject.

$$\frac{i}{2}\,[\sigma, (\sigma \cdot p)] \to \frac{i}{2}\,[\hat{\Sigma}_{\kappa\lambda}, \Theta_N], \tag{38}$$

we can obtain, without difficulty, the manifestly covariant transcription of the FW-transformed observables (29)–(36). They read[8]

$$\hat{P}'_{\kappa\lambda} = \hat{P}_{\kappa\lambda}, \tag{39}$$

$$\hat{X}'_{\kappa\lambda} = \hat{X}_{\kappa\lambda} - \frac{i}{2}\,\frac{[\hat{\Sigma}_{\kappa\lambda}, \Theta_N]}{2E_N(E_N+m)} + \varrho_{\mathrm{II}}\left\{\frac{\hat{\Sigma}_{\kappa\lambda}}{2E_N} - \frac{\Theta_N\hat{P}}{2E_N^2(E_N+m)}\right\}, \tag{40}$$

$$\hat{L}'_{\kappa\lambda} = \hat{L}_{\kappa\lambda} + \frac{\hat{P}\hat{\Sigma}_{\kappa\lambda} - \Theta_N\hat{P}_{\kappa\lambda}}{2E_N(E_N+m)} + \varrho_{\mathrm{II}}\,\frac{i}{2}\,\frac{[\hat{\Sigma}_{\kappa\lambda}, \Theta_N]}{2E_N}, \tag{41}$$

$$\hat{S}'_{\kappa\lambda} = \frac{1}{2}\,\hat{\Sigma}'_{\kappa\lambda} = \frac{1}{2}\,\hat{\Sigma}_{\kappa\lambda} - \frac{\hat{P}\hat{\Sigma}_{\kappa\lambda} - \Theta_N\hat{P}_{\kappa\lambda}}{2E_N(E_N+m)} - \varrho_{\mathrm{II}}\,\frac{i}{2}\,\frac{[\hat{\Sigma}_{\kappa\lambda}, \Theta_N]}{2E_N}, \tag{42}$$

$$\hat{J}'_{\kappa\lambda} = \hat{L}'_{\kappa\lambda} + \hat{S}'_{\kappa\lambda} = \hat{L}_{\kappa\lambda} + \hat{S}_{\kappa\lambda} = \hat{J}_{\kappa\lambda}, \tag{43}$$

$$\Theta'_N = \Theta_N, \tag{44}$$

$$E'_N = E_N, \tag{45}$$

$$\lambda'_N = \varrho_{\mathrm{III}}. \tag{46}$$

The way toward an analogous covariant generalization of the mean position and spin operators is now straightforward: in the non-covariant formulation they have been defined as operators which become identical with $\boldsymbol{x}$ and $\boldsymbol{s} = \frac{1}{2}\,\boldsymbol{\sigma}$, respectively, in the FW-representation. In other words, their explicit form in the traditional "Dirac representation" results after applying to $\boldsymbol{x}$ and $\frac{1}{2}\boldsymbol{\sigma}$ the inverse FW-transformation. Analogous reasoning now leads to the necessity of expressing these two mean operators as simple bivectors (denoted by the symbols $\hat{\tilde{X}}_{\lambda\kappa}$ and $\hat{\tilde{S}}_{\kappa\lambda} = \frac{1}{2}\,\hat{\tilde{\Sigma}}_{\kappa\lambda}$, respectively). We have

$$\hat{\tilde{X}}'_{\kappa\lambda} = \hat{X}_{\kappa\lambda}; \quad \hat{\tilde{X}}_{\kappa\lambda} = U_N^{-1}\hat{X}_{\kappa\lambda}U_N =$$

$$= \hat{X}_{\kappa\lambda} - \frac{i}{2}\,\frac{[\hat{\Sigma}_{\kappa\lambda}, \Theta_N]}{2E_N(E_N+m)} - \varrho_{\mathrm{II}}\left\{\frac{\hat{\Sigma}_{\kappa\lambda}}{2E_N} - \frac{\Theta_N\hat{P}_{\kappa\lambda}}{2E_N^2(E_N+m)}\right\}, \tag{47}$$

$$\tfrac{1}{2}\,\hat{\tilde{\Sigma}}'_{\kappa\lambda} = \tfrac{1}{2}\,\hat{\Sigma}_{\kappa\lambda}; \quad \tfrac{1}{2}\,\hat{\tilde{\Sigma}}_{\kappa\lambda} = U_N^{-1}\hat{\Sigma}_{\kappa\lambda}U_N =$$

$$= \frac{1}{2}\,\hat{\Sigma}_{\kappa\lambda} - \frac{\hat{P}\hat{\Sigma}_{\kappa\lambda} - \Theta_N\hat{P}_{\kappa\lambda}}{2E_N(E_N+m)} + \varrho_{\mathrm{II}}\,\frac{i}{2}\,\frac{[\hat{\Sigma}_{\kappa\lambda}, \Theta_N]}{2E_N}. \tag{48}$$

The mean operators $\hat{\tilde{L}}_{\kappa\lambda}$ and $\hat{\tilde{J}}_{\kappa\lambda} = \hat{\tilde{L}}_{\kappa\lambda} + \hat{\tilde{S}}_{\kappa\lambda} = \hat{J}_{\kappa\lambda}$ can easily be calculated from (47) and

[8] The notation assumed is consistent owing to the fact that the operators $\hat{P}_{\kappa\lambda}$, $\hat{\Sigma}_{\kappa\lambda}$, $\hat{P}$, E_N and Θ_N commute with each other, except for $\hat{\Sigma}_{\kappa\lambda}$ with each other with Θ_N. (They all also commute with ϱ_{I}, ϱ_{II}, ϱ_{III}.)

(48), while for the mean-velocity operator $\hat{\hat{V}}_{\kappa\lambda}$ we obtain

$$\hat{\hat{V}}_{\kappa\lambda} \equiv i d_N \hat{\hat{X}}_{\kappa\lambda} = -i[\hat{\hat{X}}_{\kappa\lambda}, \mathscr{H}_N] = \lambda_N \frac{\hat{P}_{\kappa\lambda}}{E_N}, \tag{49}$$

$$\hat{\hat{V}}'_{\kappa\lambda} = \varrho_{\mathrm{III}} \frac{\hat{P}_{\kappa\lambda}}{E_N}. \tag{50}$$

In this way all essential results characterizing the FW-representation for the Dirac particle have been reformulated in a manifestly covariant way, in terms of the generalized particle observables.

It is possible to deduce (25), (26) and (39)–(48) (here obtained by arguments using (27) and (28)) in a more systematic way. A step by step covariant generalization of the original considerations of Foldy, Wouthuysen and Tani is necessary for this purpose. The explicit and rather lenghty calculations will be given separately (Rakowski [26]).

4. Physical interpretation of the displayed covariant formalism

The formulae (1)–(50) represent, so far, a purely formal scheme, allowing one to express in a manifestly covariant way the quantum-mechanical description of a particle — which, in particular, has been assumed to be the Dirac particle. The problem of constructing a meaningful and internally consistent scheme of physical interpretation on this mathematical basis is, however, not straightforward, as the known postulates of special relativity must be combined in a compatible way with those of quantum theory (including the problems of measurement of quantum dynamical variables). At first, the physical meaning of the vector n_μ must be defined, as various points of view on this subject are possible and encountered in the litterature. In particular, Fleming[9] has identified the set of all n_μ (filling the future cone) with all possible inertial frames ("observers", according to the terminology used in [27]). This assumption corresponds to the "active point of view" on the Lorentz transformation when transforming n_μ. Following, alternatively, the "passive point of view", one must treat n_μ as an arbitrary, but fixed vector. However attractive the former may of interpreting n_μ might seem to be, only the latter can remain consistent with our formalism, as only then $\hat{X}_{\kappa\lambda}$, $\hat{P}_{\kappa\lambda}$, $\hat{L}_{\kappa\lambda}$ and $\hat{S}_{\kappa\lambda}$ possess the postulated transformation character in the Minkowski space. Hence, n_μ, in analogy to the dynamical quantities, x_μ, p_μ, $l_{\mu\nu}$ and $s_{\mu\nu}$ must be understood a fixed vector (*i. e.* distinguishable for some physical reason). In contradistinction to the latter, n_μ has been treated, however, as a *c*-number quantity. In this situation the most natural (and, perhaps, the only possible) interpretation is that n_μ specifies the time axis in the "laboratory frame" (in which some macroscopic equipment, suitable for measuring observables of the described particle, remains at rest). We assume this interpretation of n_μ, obviously consistent with the well--known active role of measurement in quantum theory. This interpretation completely

[9] in the paper [20] already quoted; for some further considerations on this subject see also Hammer and others [27].

determines that of our generalized particle observables (discussed in the two preceding Chapters). Having the manifestly covariant form, they become, on the other hand, frame--dependent, as the choice of the laboratory system is inherent in their definition. In other words, the quantum-mechanical description of the Dirac particle can be "translated" from one Lorentz frame to another, in the geometrical sense, but no covariant generalization of the "instant measurement" in one specified frame into a "hyperplane measurement" (in the sense proposed by Fleming) can be obtained within our scheme of interpretation, as the choice of the laboratory frame has now become the physical, and not the geometrical problem. In this respect our results essentially deviate from those of Fleming. (For the same reason our generalization of the FW-transformation differs from those discussed in [20] and [27]). The explanation of this difference is simple, if we remembering that our considerations deal with the specific properties of relativistic quantum-mechanical systems, described in terms of their own observables (including the position observable, a notion completely different from that of space, or space-time coordinates). The generalized particle observables defined by us seem just to characterize peculiar difficulties of this description, rather different from those met in quantized field theory[10].

The implications of general quantum postulates for the above interpretation must now be analysed. Two important questions concern the Hermitian character of the generalized particle observables and the consistency of their algebraic properties (understood in the sense of "strong" relations) with the Schrödinger equation (6) playing the role of a subsidiary condition. This consistency has been established, as yet, only for a simplified model of a "relativistic quantum-mechanical system possesing a classical analogue" (since only commutation relations obtained by the correspondence arguments have been used in [4], in order to deduce (6)). It is also obvious that $\hat{X}_{\kappa\lambda}$ and $\hat{P}_{\kappa\lambda}$ can be treated as Hermitian operators (the presence of the imaginary unit in their space-time components arising merely from the notation used). The problem becomes more complicated for systems possessing internal degrees of freedom, as then additional observables appear in the Hamiltonian (5). The problem of the Dirac particle gives a good insight into the character of difficulties arising in such cases. Simple considerations show that in our case these difficulties are immediately related to the well known, specific property of the Dirac operators γ_μ. As four Hermitian operators, they form a vector in an abstract four-dimensional euclidean space, rather than in the Minkowski space with indefinite metrics[11]. This "pathological" geometrical structure of the vector γ_μ, and, in consequence, also of the tensor $\sigma_{\mu\nu} = -\frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ manifests itself by many characteristic features of the relativistic electron theory, *e. g.* by

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[10] All field operators (hence, also all local observables) are functions of the c-number variables x_μ. Therefore, the procedure of translating the physical description from one Lorentz frame to another resembles more closely that encountered in classical physics. On the other hand, it is well known (see *e. g.* Wigner [28]) that within field theory x_μ do not possess any immediate connection with the position of the particle in space-time.

[11] The explicit use of the signature $(+ - - -)$ does not change anything, as then $\gamma^{0+} = \gamma^0$, but $\gamma^{j+} = -\gamma^j$, $j = 1, 2, 3$ must be assumed.

the necessity of using $\bar{\psi}$ instead of ψ^* in the conjugate covariant Dirac equation (although ψ^* represents the state vector of the first quantization in the dual "bra" space), or by the appearance of an "imaginary electric dipole moment" in the second-order equation (describing the Dirac particle in the external electromagnetic field). That property of γ_μ also causes serious complications in the physical interpretation discussed in this Chapter. The covariant Dirac Hamiltonian (14), as well as other generalized particle observables, cannot be treated as a Hermitian operator, since $\gamma_N^+ \neq \gamma_N$, $\Theta_N^+ \neq \Theta_N$ (because of the factor i lacking in the fourth component of γ_μ). Moreover, as it can be easily verified, the consistency of the Schrödinger equation (6) with the algebraic ("strong") relations cannot be maintained in the presence of the Dirac operators[12]. A way out of the apparent discrepancy appearing at this stage of considerations is suggested by the already established frame-dependence of the generalized particle observables. Indeed, there is one and only one Lorentz frame in which the generalized observables become Hermitian, while the subsidiary and constraint relations, are satisfied identically. In this frame, specified by the condition

$$n_\mu = (0, 0, 0, i), \tag{51}$$

all our covariant formulae pass over into the usual, three-dimensional ones, so that the traditional scheme of interpretation of quantum mechanics becomes valid. As, in accordance with our basic assumption (51), specifies the laboratory frame, its distinction is of a physical character and does not contradict the principles of relativity. The covariant formalism manifests the possibility of a geometrical translation (in terms of the space-time coordinates) into an arbitrary Lorentz frame, of the physical results related to the arbitrary, but fixed laboratory frame.

The conclusion obtained makes possible a consistent physical interpretation related to the previously constructed quantum-mechanical covariant Hamiltonian formalism. On the other hand, this conclusion imposes considerable restrictions on the meaning in which Lorentz covariance can be understood within this formalism. Although the foregoing considerations have dealt only with the case of the Dirac particles, one may say, without going into detailed calculations, that similar complications are to be expected for bosons described on the "first-quantization" level. The best way to give such a description is to use the known β-formalism of Kemmer [29], based on a covariant first-order wave equation similar to the Dirac equation, but with the Duffin-Kemmer operators β_μ instead of γ_μ. As the symbolic four-vector β_μ possesses the same "pathological" geometrical structure as γ_μ, the same difficulties must appear in calculations leading to the covariant Hamiltonian formulation of the Kemmer equation (apart from some additional complications having their source in the singularity of β_μ — as then the Hamiltonian formulation is no longer equivalent to the initial wave equation).

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[12] It is impossible to eliminate γ_N, by expressing it in terms of some "spacelike quantities", as this has been done in the case of p_N, since γ_μ are mutually independent. On the other hand, because of their "pathological" properties, γ_N can hardly be treated as the "timelike component". At any rate, γ_N appears, inevitably, in the Hamiltonian causing the above-mentioned difficulties.

The limitation imposed on the meaning of Lorentz covariance in relativistic quantum--mechanical problems has appeared to be indispensable within the proposed Hamiltonian formalism. This limitation seems, however, to possess a more general character, as its source lies in the algebraic properties of the operators γ_μ or β_μ, rather than in the given formalism itself. It has already been stressed that the formalism does not require any such restrictions in the somewhat trivial case of a quantum system possessing a classical analogue, hence, not possessing any internal degrees of freedom (contrary to the more interesting cases of the Dirac or Kemmer particles). The presence of these internal degrees of freedom seems to be the main cause of the difficulties which have led to the necessity of a specific approach toward the problem of Lorentz covariance. One may relate this result to the known fact[13] that Lorentz transformations, whitch leave $\bar{\psi}\psi$ rather than $\psi^*\psi$ invariant, do not correspond to unitary transformations in the Hilbert space of relativistic quantum-mechanical states (of the Dirac or Kemmer particle), in contradistinction to field-theoretical problems where such a correspondence plays a fundamental role.

The covariance of the generalized mean-operators (defined by the formulae (47)–(50)) must be understood in the same restricted sense. This result provides an additional argument in the controversial problem of the covariance *vs* the non-covariance of these operators. On the other hand, the distinctive meaning of these operators (relegating to them the role of the "true particle observables") so suggestively implied by the papers already quoted ([6], [7], [9] and [10]) has been confirmed by our results, at least for relativistic quantum-mechanical problems (understood in the sense specified in this paper).

The interplay of the manifestly covariant mathematical formalism and of the restrictions imposed on the former for the sake of correct physical interpretation may be illustrated by the example of the covariant FW-transformation (formulae (25) ane (26)). The covariant triple $\varrho_{\mathrm{I}}, \varrho_{\mathrm{II}}, \varrho_{\mathrm{III}}$ possesses all the algebraic properties (except hermicity) of the non-covariant $\varrho_1, \varrho_2, \varrho_3$. (In the traditional formulation of the Dirac electron theory the latter are often treated as a three-vector $\boldsymbol{\varrho}$; however, contrary to the spin vector $\boldsymbol{\sigma}$, $\boldsymbol{\varrho}$ has nothing in common with the physical three-dimensional space, so that its covariant generalization is not related to the Minkowski space). Hence many interesting formal considerations, including the interpretation of the FW-transformation as a rotation in the ϱ-space may be immediately extended to the covariant case. Nevertheless, only in the laboratory frame (51) does this transformation become unitary and acquire the property of transforming the Dirac Hamiltonian into the "even" form (separating positive and negative energy states). It should be still stressed that, although the meaning attributed in our considerations to the Lorentz invariance deviates from the traditional point of view, it remains compatible with the axiomatic approach (setting up only one condition: that

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[13] This problem has been raised *e. g.* by Bollini and Giambiagi [30] in connection with their studies concerning some analogy between the homogeneous, special Lorentz transformations and the FW-transformations. (The problem has been, subsequently, discussed from various points of view in many papers not quoted here).

the proper, inhomogeneous Lorentz group be the symmetry group for an arbitrary physical theory consistent with special relativity). In particular, the consistency of the proposed physical interpretation with the Haag postulates (quoted *e. g.* in [20] and [28]) has been preserved.

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