GENERAL RELATIVISTIC FLUID SPHERES. VI. ON PHYSICALLY MEANINGFUL STATIC MODEL SPHERES IN ISOTROPIC COORDINATES

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Physical conditions which are imposed on the matter in a static fluid sphere are expressed in terms of differential inequalities. These inequalities are relating the two functions appearing in the general formal solution (in isotropic coordinates) to such conditions as the positive definite character of density or pressure or adiabatic stability. The standardized form of the set of inequalities admits a future computer-aided study on the influence of physical conditions upon the solutions of Einstein equations. Examples of an analysis of the physical behaviour of exact solutions are given.

1. *On testing the applicability of exact solutions*

Since the method outlined in two previous papers (Kuchowicz 1971, 1972a) is able to yield practically all possible exact solutions for spheres of perfect fluid in isotropic coordinates, it might be useful to present some statements concerning the applicability of these solutions. Such exact solutions are expected to provide simplified but easy to handle models of static relativistic objects. Though these latter objects seem to be in fact not merely static spheres (*cf.* the oblique rotator model for pulsars), and a thorough study of their features necessitates to go beyond the apparently simplified picture of a structureless sphere, our models may be treated just as a first-order approximation to the real case. The advantage of this approach, like in any case we are using simplified models, lies in the fact that we operate with relatively simple, analytical expressions for some of the relevant physical quantities. In the present case, we have to do essentially with two sets of formulae which were given in the previous paper of this series (Kuchowicz 1972a): Eq. (1.8) providing the general formal solution, and Eq. (1.10) for energy density and pressure inside the sphere.

After deriving a general expression for the exact solution of Einstein equations,

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there arises the question of fitting various demands. Let us make some brief remarks on this point.

It is an obvious demand to wish that the solution is joined in a continuous way to the external Schwarzschild solution. The resulting four boundary conditions for isotropic coordinates are given in Section 2 of the preceding paper (Kuchowicz 1972a). Sometimes it may be impossible to fulfil these conditions; in this case it may be useful to adapt a method which was suggested for canonical coordinates (Kuchowicz 1968): If the solution has the desired physical properties in the centre of the fluid sphere, it may be joined at some internal boundary to another solution which is physically meaningful near the surface. An example of this procedure will be provided in this paper.

Since we are using the isotropic form of the metric

$$ds^2 = e^\nu dt^2 - e^\lambda[dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \tag{1.1}$$

it is a trivial condition to demand that the two functions e^ν and e^λ are nonnegative. Also pressure and density should neither be negative, nor should they increase outwards. If it is impossible to fulfil all these conditions everywhere inside the fluid sphere, it may be possible that they are fulfilled inside some spherical layer, and the resulting solution for the whole sphere may be composed of several physically meaningful solutions, each valid in a certain layer, and joined at its boundary to another one.

Another condition which should be imposed upon a material sphere is the adiabatic stability (Bondi 1964) which may be given the form

$$0 \leqslant \left(\frac{dp}{d\varrho}\right)_{\text{adiabat}} = a^2 \leqslant 1, \tag{1.2}$$

where a is the velocity of sound which cannot, of course, exceed the velocity of light (hence the upper limit of 1 in our units in which we have $c=1$). This condition should be fulfilled at least in the envelope region of the sphere.

Not all of the conditions mentioned above should hold simultaneously. It is the better, of course, the more of them hold for the solution which is just being tested. It is necessary to find for each of the derived solutions in which spherical shells various conditions are fulfilled; one may expect that only few exact solutions are physically meaningful for all values of $r \leqslant r_f$. Here r_f denotes the radius of the fluid sphere in isotropic coordinates. An analysis of whether a given exact solution of the Einstein equations fulfils certain conditions can be done in a straightforward way, starting from the expressions for density and pressure; examples will be provided in Sections 3 and 4 of this paper. Such an analysis may be also formalized by dealing only with a set of differential inequalities which are given in the next Section.

2. *A set of differential inequalities representing various physical conditions inside the sphere*

The conditions mentioned in the preceding section, together with some other ones, have been considered by several authors. It is very easy to state these conditions verbally, yet it proves to be a tremendous task to make a general use of them. Several general inequalities representing these conditions have been derived by some authors (Buchdahl 1959, Bondi 1964, Islam 1969, 1970), yet they do not suggest any effective work. It may be added here that perhaps this slow progress was due to the fact that only the canonical coordinate system was used, surely because of the simple form of the external Schwarzschild solution in it. Now, experience teaches us that giving up the canonical coordinates might profit in some questions: a simple example is just the application of the Kruskal metric.

While a long time the canonical coordinates have yielded useful results, recently the general formal solution obtained in isotropic coordinates (Kuchowicz 1971, 1972 a) proves to be promising. We give this solution in a form which differs slightly from Eq. (1.7) and (1.8) of the preceding paper (Kuchowicz 1972 a):

$$e^{\nu} = C_1 \exp\left[\int [w(\xi)+u(\xi)]d\xi\right],$$

$$e^{\lambda} = C_2 \exp\left[-\int w(\xi)d\xi\right], \tag{2.1}$$

where u and w fulfil the following first-order equation:

$$\frac{du}{d\xi} = -\frac{1}{2}u^2-2uw-w^2 \tag{2.2}$$

and C_i are integration constants. We do not give the expressions for energy density and pressure as they result from those given in the preceding paper (Kuchowicz 1972a) by a substitution $w = v-u$. As it has been communicated earlier (Kuchowicz 1972b), it is now possible to present in a standardized form all the physical conditions which are imposed upon the matter in stellar interior. All these conditions reduce to simple differential inequalities involving two functions u and w of one nonnegative variable $\xi = r^2$.

From the expression for energy density there follows immediately the condition of a non-negative density:

$$6w+\xi(4w'-w^2) \geqslant 0. \tag{2.3}$$

In an analogous way the condition of a non-negativity of pressure reads:

$$2u-\xi(w^2+2uw) \geqslant 0. \tag{2.4}$$

Primes denote here differentiation with respect to the auxiliary variable ξ.

Though the radial variable r is not simply the distance from the centre, it is sometimes meaningful to wish that the density should not raise with increasing r, *i.e.* also with increasing ξ. This leads to the following condition:

$$5w^2+10w'+\xi(2ww'+4w''-w^3) \leqslant 0. \tag{2.5}$$

In a similar way, the pressure does not raise with increasing ξ when the following inequality holds:

$$u^2+3w^2+4uw+\xi(u+w)(2w'-uw-w^2) \geqslant 0. \tag{2.6}$$

If the two preceding inequalities are fulfilled, the condition of adiabatic stability leads to the additional inequality:

$$10w' + 8w^2 + 4uw + u^2 + \xi[4w'' + \\ + 2w'(u+2w) - 2w^3 - 2uw^2 - u^2 w] \leqslant 0. \tag{2.7}$$

The sign of the inequality (2.7) is reversed when both Eq. (2.5) and (2.6) do not hold.

Some authors consider also other types of conditions, resulting from general investigations on the energy-momentum tensor, equation of state, *etc.* Thus very frequently it is demanded that the trace of the energy-momentum tensor should be non-negative, *i.e.* $p \leqslant \frac{1}{3}\varrho$. This leads to the following inequality:

$$3(w-u) + \xi(2w' + w^2 + 3uw) \geqslant 0. \tag{2.8}$$

Zeldovich does not agree with this condition which is regarded by him rather as a prejudice. In this opinion, we have to recognize only the most rigid of the equation of state (Zeldovich 1961) which leads to the condition $p \leqslant \varrho$. This is less rigorous than the preceding condition, and is expressed in mathematical form by the inequality:

$$3w - u + \xi(2w' + uw) \geqslant 0. \tag{2.9}$$

A discussion of the most rigid form of the equation of state is given in the recent monograph on gravitation theory and stellar evolution (Zeldovich and Novikov 1971, Chapter 6, § 12).

In all the inequalities above there appear only the derivatives of w, as the use of Eq. (2.2) eliminates the derivatives of u. Now, it may be added that the latter equation may be looked upon as giving in an explicit way the function w in terms of the till now arbitrary function u:

$$w = -u \pm \sqrt{\frac{u^2}{2} - u'}\,. \tag{2.10}$$

While making use of Eq. (2.10) and inserting it everywhere into our set of inequalities, we may treat our set of conditions as a single set of inequalities for a single function $u(\xi)$. From a general form of this set it is not easily recognizable whether and how many functions fulfilling all these conditions do exist. Yet it may be argued that such functions are known, though they may satisfy not all inequalities for a whole interval of positive values of ξ. Now, in the search for exact solutions of Einstein equations it may be reasonable to look from the beginning how the various conditions (represented by inequalities) are satisfied by the initial functions u and w. The form of the inequalities is standardized, with only two (or one, after we use Eq. (2.10)) functions of one variable, which contrasts much with the inequalities given *e.g.* in the classical paper by Buchdahl (1959). This should make it easy to apply computer programs to an analysis of the set of inequalities:*

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* Unfortunately, no such programs are known to the author.

3. *An exact solution of astrophysical interest*

With the following "Ansatz":

$$u = \frac{\sqrt{2}-1}{\xi} - \frac{2\sqrt{2}\,D\xi^{\frac{\sqrt{2}}{2}-1}}{1+D\xi^{\frac{\sqrt{2}}{2}}}, \tag{3.1}$$

where D is a constant, we obtain from Eq. (2.2):

$$w = \frac{2-\sqrt{2}}{2\xi} + \frac{\sqrt{2}\,D\xi^{\frac{\sqrt{2}}{2}-1}}{1+D\xi^{\frac{\sqrt{2}}{2}}}. \tag{3.2}$$

Performing integration indicated in Eq. (2.1) we have the two metric functions:

$$e^{\nu} = \frac{C_1\xi^{\frac{\sqrt{2}}{2}}}{[1+D\xi^{\frac{\sqrt{2}}{2}}]^2}, \quad e^{\lambda} = \frac{C_2\xi^{\frac{\sqrt{2}}{2}-1}}{[1+D\xi^{\frac{\sqrt{2}}{2}}]^2}. \tag{3.3}$$

The three constants D, C_1, C_2 have to be determined from the four boundary conditions at the sphere radius r_f:

$$e^{\lambda(\xi_f)} = (1+\alpha)^4, \quad u(\xi_f) = \frac{2\alpha^2}{\xi_f(1-\alpha^2)},$$

$$e^{\nu(\xi_f)} = \left(\frac{1-\alpha}{1+\alpha}\right)^2, \quad w(\xi_f) = \frac{2\alpha}{\xi_f(1+\alpha)}, \quad \xi_f = r_f^2, \tag{3.4}$$

where the parameter α is the mass concentration for isotropic system. It is defined in Section 2 of the preceding part of this series (Kuchowicz 1972a), and may assume values from the following interval:

$$0 < \alpha < 1. \tag{3.5}$$

The four boundary conditions (3.4) are fulfilled for only one value of the parameter α: $\alpha = 2-\sqrt{3}$. The following values of the three constants are obtained:

$$D = (5-\sqrt{24})r_f^{-\sqrt{2}},$$

$$C_1 = 4(5-\sqrt{24})r_f^{-\sqrt{2}}, \quad C_2 = 432(5-\sqrt{24})(7-\sqrt{48})r_f^{2-\sqrt{2}}. \tag{3.6}$$

We have the following expressions for density and pressure:

$$8\pi\varrho = \frac{1}{2C_2}[D^2\xi^{\frac{\sqrt{2}}{2}} + 14D + \xi^{-\frac{\sqrt{2}}{2}}],$$

$$8\pi p = \frac{1}{2C_2}[D^2\xi^{\frac{\sqrt{2}}{2}} - 5D + \xi^{-\frac{\sqrt{2}}{2}}]. \tag{3.7}$$

What may be said now about the various conditions inside the sphere? We may find immediately from the two formulae above that the density and pressure are non-negative everywhere for $\xi \leqslant \xi_f$, with the values of the constants given by Eq. (3.6). Also the conditions of ϱ and p as non-increasing with r and ξ do not impose any restrictions upon the range of admissible values of ξ. The condition of adiabatic stability is automatically fulfilled everywhere inside the sphere.

The condition $p \leqslant \varrho$ is always fulfilled inside the sphere. The other condition: $p \leqslant \frac{1}{3}\varrho$ gives

$$2y^2 - 29y + 2 \leqslant 0, \tag{3.8}$$

with $y = D\xi^{\frac{\sqrt{2}}{2}}$. This is fulfilled for

$$0.07 \approx \frac{29-\sqrt{825}}{4} \leqslant y \leqslant \frac{29+\sqrt{825}}{4} \approx 14.44. \tag{3.9}$$

While the upper limit of allowed y imposes no restriction upon the values of ξ, from the lower limit we get

$$r \geqslant 0.773. \tag{3.10}$$

The equation of state is of an ultrarelativistic type:

$$8\pi p = 8\pi\varrho - \frac{19D}{2C_2}. \tag{3.11}$$

This model of relativistic sphere, though valid only for a single value of the mass concentration α, may prove useful in applications, *e.g.* in studies on the geodesic motion of neutrinos along lines presented earlier (Kuchowicz 1967). It is interesting to mention here that a geodesic motion of massless neutrinos is possible in the whole interior of the sphere described by the metric (3.3). We may also introduce the notion of the total mass M_{obs} as observed in the external gravitational field, and the proper mass M_{pr} defined as the integral of the density over the volume of our relativistic sphere. The invariant quantity $(M_{\mathrm{pr}} - M_{\mathrm{obs}})/M_{\mathrm{obs}}$ is called the gravitational packing fraction (Zeldovich and Novikov 1971, Chapter 10, § 6), and is a characteristic of the total gravitational mass defect. It amounts to *ca* 1/2 in our case, which is the highest possible value.

4. *The case of a solution which does not have everywhere the desired behaviour*

In the preceding section we had a solution with desirable properties, which did not violate our conditions. In most cases it is not so. An example of a solution which can be used only in some regions is provided by the solution which is obtained from a simple separation of our equation

$$\frac{du}{d\xi} \equiv \frac{du}{dv} \cdot \frac{dv}{d\xi} = \frac{u^2}{2} - v^2 \tag{4.1}$$

into the following two ones:

$$\frac{du}{dv} = \frac{u^2}{2v^2} - 1, \qquad \frac{dv}{d\xi} = v^2. \tag{4.2}$$

The first of these gives the following expression for u in terms of v:

$$u = \frac{v(1+\sqrt{3})+(\sqrt{3}-1)Dv^{\sqrt{3}+1}}{1-Dv^{\sqrt{3}}}. \tag{4.3}$$

Here we have to introduce for v the solution of the second of these equations

$$v = \frac{1}{E-\xi}. \tag{4.4}$$

The metric is now

$$e^{\nu} = \frac{C_1}{E-\xi}, \qquad e^{\lambda} = \frac{C_2}{|E-\xi|^{\sqrt{3}}(1-y)^2}, \tag{4.5}$$

and the energy density and pressure are:

$$8\pi\varrho = \frac{1}{C_2}\,\xi|E-\xi|^{\sqrt{3}-2}[(4\sqrt{3}-3)y^2-30y-3-4\sqrt{3}]-$$

$$-\frac{6\sqrt{3}}{C_2}\,|E-\xi|^{\sqrt{3}-1}(1-y^2),$$

$$8\pi p = \frac{1}{C_2}\,\xi|E-\xi|^{\sqrt{3}-2}[(3-2\sqrt{3})y^2+6y+3+2\sqrt{3}]+$$

$$+\frac{2}{C_2}\,|E-\xi|^{\sqrt{3}-1}[(1-\sqrt{3})y^2-2y+1+\sqrt{3}], \tag{4.6}$$

where

$$y = Dv^{\sqrt{3}} = \frac{D}{(E-\xi)^{\sqrt{3}}}.$$

If we regard this solution as being valid everywhere inside the material sphere, then we have from the boundary conditions

$$D = \frac{\alpha-1-\sqrt{3}}{\alpha-1+\sqrt{3}}\left[\frac{(1-\alpha^2)\xi_f}{2\alpha}\right]^{\sqrt{3}} < 0, \qquad E = \frac{1+2\alpha-\alpha^2}{2\alpha}\,\xi_f > 0 \tag{4.7}$$

and $C_i > 0$.

These precisely determined values of the integration constants are a hindrance to the use of our solution in the whole interior of the sphere. It is an easy task to find that with our values of D and E the condition for non-negative central density $|y_c| > 1$ is fulfilled only for those relatively small values of α which satisfy the inequality

$$\frac{1+\sqrt{3}-\alpha}{\sqrt{3}-1+\alpha}\left[\frac{1-\alpha^2}{1+2\alpha-\alpha^2}\right]^{\sqrt{3}} > 1. \tag{4.8}$$

We have thus a negative central density for α down to 1/3, while for $\alpha = 0.1$ the condition (4.8) is fulfilled, and the central density is positive.

Another problem is the correct behaviour of density even if it is positively definite; it should diminish with increasing ξ. When we consider the behaviour in the centre only, we find that we have to do with diminishing density only for the following interval of the central values of the quantity y,

$$8\sqrt{3}-14 < y_c < 2. \tag{4.9}$$

Now, with $D < 0$, we have $y_c < 0$, and the condition (4.9) cannot be satisfied simultaneously with the preceding condition (of non-negative central density). Though no difficulties arise here from the physical conditions imposed upon the pressure, the case with the behaviour of the density points to the necessity to have at least other values of the constants D and E, in order to be able to secure the proper behaviour of the physical quantities near the centre. Now, other values of these constants may be obtained if we restrict our solution to the core of the sphere only, and if it is joined at an internal boundary $\xi = \xi_i$ to another solution which behaves well in the envelope region. We may thus join our solution to the internal Schwarzschild solution which was given in Section 3.5 of the preceding paper (Kuchowicz 1972a). A dependence of D and E, through the constants of the Schwarzschild internal solution, on the quantities α and ξ_f is established; it is important that we have at disposal the additional parameter ξ_i. We may also study a composite sphere which is made up of our solution in the envelope region, and of another type of solution in the core. The condition we must consider now with care is that of the adiabatic stability.

The constants which appear in the solution play an important role in the applicability of this solution as a model of a relativistic object. It may occur that in spite of some *a priori* adequate kind of dependence of the physical quantities on r we are unable to retain the solution since it does not fulfil some physical conditions near the centre, and other ones — near the surface of the sphere. If we are unable to retain the solution in either of these regions, there remains only the possibility of dealing with this solution in the intermediate region, in which it is joined to some core solution from one side, and to an envelope solution— from the other side. These core and envelope solutions might correspond to vanishingly small fractions of mass inside the sphere, yet they might act in a regularizing way upon the intermediate-region solution which would be responsible for the physical behaviour of the whole sphere. Perhaps in some future, in connection with the possibilities of computer-aided model simulations, it might be useful to have a supply of standard envelope and core solutions that might be used for computations.

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