ON THE EXPANSION OF NUCLEAR ROTATIONAL ENERGY IN POWERS OF ANGULAR VELOCITY OF ROTATION

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The coefficients α, β, γ and δ appearing in the expansion of the nuclear rotational energy in powers of the square of angular velocity are calculated by a microscopic method. Two simplified nuclear two level models are used: (*i*) with use of the BCS approximation, and (*ii*) based on the exact diagonalization of the nuclear rotational Hamiltonian including the short-range pairing forces acting between the nucleons.

1. Introduction

It is well known that the nuclear moment of inertia increases with rotational angular momentum I. Nuclear rotational energy can be expanded in powers of I. However, it has been observed [1], [2] that the expansion in powers of ω^2 where ω denotes angular velocity of the rotational motion leads to a much faster convergence and is, therefore, more useful in fitting the observed energy spectra [3], [4]. Writting down the energy expansion

$$E = \alpha\omega^2+\beta\omega^4+\gamma\omega^6+\delta\omega^8+\ldots \tag{1.1}$$

together with the selfconsistency condition [1]

$$\frac{\partial E}{\partial\omega} = \hbar\omega\frac{\partial}{\partial\omega}\sqrt{I(I+1)} \tag{1.2}$$

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derived from the cranking model of Inglis [5], [6] we obtain the following expression for the nuclear moment of inertia

$$\mathcal{I} \equiv \frac{\hbar}{\omega}\sqrt{I(I+1)} = 2\alpha + \frac{4}{3}\beta\omega^2 + \frac{6}{5}\gamma\omega^4 + \frac{8}{7}\delta\omega^6 + \ldots. \tag{1.3}$$

The coefficients α, β, γ, δ ... appearing in the above expressions could be in principle calculated from the microscopic theory involving parameters characteristic for the nuclear structure. However, up to now only the parameter α equal to one half of $\mathcal{I}_0$ (where $\mathcal{I}_0$ is the limiting value of the moment of inertia $\mathcal{I}$ when $\omega \to 0$) has been calculated. The cranking model formulae for the higher-order parameters β, γ, δ are complicated [1], [7].

In this paper we attempt to present microscopic calculations of the parameters α, β, γ, δ in case of two simple solvable models of the nucleus where the single particle structure is replaced by a schematic model with a two level spectrum with high degeneracy. We shall first use the extended cranking model and treat the nuclear superfluidity in the framework of BCS theory [8] and second employ an exact model of particles coupled to a rotor. The latter model has been developed recently in connection with the investigation of the phase transition in a rotating nucleus [9], [10].

2. *Cranking model method with the BCS approximation*

Instead of calculating directly the total energy of the nucleus as a function of the angular momentum I it is convenient sometimes to perform a contact (Legendre type) transformation (see for ex. Ref. [11]) using the moment of inertia $\mathcal{I}$ as the original variable. Thus, we may introduce the moment of inertia through the relation

$$\hbar^2/2\mathcal{I} = dE/d(I(I+1)), \tag{2.1}$$

this equation being fully consistent with Eqs (1.2) and (1.3). Now instead of considering $E(I)$ as function of $I(I+1)$ we may use

$$V(\mathcal{I}) = E(I) - \hbar^2 I(I+1)/2\mathcal{I}. \tag{2.2}$$

This is essentially in line with the VMI (= variable moment of inertia) model considered in Ref. [3] where the intrinsic nuclear energy V is a function of $\mathcal{I}$. In case when the short range pairing interaction acting between the nucleons is important one may use the energy gap parameter Δ instead of $\mathcal{I}$ as an independent variable assuming the existence of a well defined monotonic function $\mathcal{I} = \mathcal{I}(\Delta)$. Then Eq. (2.2) may be written as

$$E = V(\Delta) + \frac{\hbar^2 I(I+1)}{2\mathcal{I}(\Delta)}; \tag{2.3}$$

where the new function $V(\Delta)$ is now

$$V(\Delta) = V(\mathcal{I}(\Delta)). \tag{2.4}$$

We shall now assume that the usual form of the cranking model expression

$$\mathscr{I} = 2\hbar^2 \sum_i \frac{|\langle i|j_x|0\rangle|^2}{E_i - E_0} \tag{2.5}$$

for the nuclear moment of inertia $\mathscr{I}$ is valid at any value of I (or ω) and the dependence on I (or ω) enters only through the intrinsic parameters of the rotating system such as energy gap Δ, deformation *etc.* In this section we shall be dealing only with the pairing correlations, neglecting the other degrees of freedom. Now, the usual gap equation for Δ is not valid but instead the expression (2.3) should be minimized (*cf.* Refs [12], [13] and [14]) separately for each I. The condition

$$(\delta E)_{I-\text{fixed}} = 0 \tag{2.6}$$

leads to the relation (*cf.* Ref. [14])

$$(dV/d\Delta)/(d\mathscr{I}/d\Delta) = dV/d\mathscr{I} = \omega^2/2. \tag{2.7}$$

Here the relation

$$\omega = \hbar\sqrt{I(I+1)}/\mathscr{I} \tag{2.8}$$

has been employed (*cf.* Eq. (1.3)).

Now, let us assume that the nuclear spectrum consists of two levels only with the pair degeneracy Ω, each. Let us denote the level splitting by 2ε and assume that number of particles is equal to 2Ω (*i. e.* to one half of the number of available states). The standard BCS theory employed together with the cranking model method leads to the following expressions for V and $\mathscr{I}$ the notation is standard (*cf.* for example Ref. [15])

$$V = \sum_\nu \varepsilon_\nu 2v_\nu^2 - G(\sum_\nu u_\nu v_\nu)^2 - G\sum_\nu v_\nu^4 =$$

$$= -2\Omega\varepsilon^2/\sqrt{\varepsilon^2+\Delta^2} - G\Omega^2\Delta^2/(\varepsilon^2+\Delta^2), \tag{2.9}$$

where the term containing v_ν^4 has been neglected.

Now

$$\mathscr{I} = 2\hbar^2 \sum_{\mu\nu} |\langle\nu|j_x|\mu\rangle|^2(\mu_\mu v_\nu - v_\mu u_\nu)^2(E_\mu+E_\nu)^{-1} =$$

$$= \varepsilon^3\mathscr{I}_{\text{rig}}(\varepsilon^2+\Delta^2)^{-3/2}, \tag{2.10}$$

where

$$\mathscr{I}_{\text{rig}} = \frac{\hbar^2}{\varepsilon}\sum_{\mu\nu}|\langle\nu|j_x|\mu\rangle|^2 \tag{2.11}$$

denotes the rigid-body moment of inertia in the model. Now, using Eqs (2.9) and (2.10) we can express V as a function of $\mathscr{I}$

$$V = -2\Omega\varepsilon(\mathscr{I}/\mathscr{I}_{\text{rig}})^{1/3} - G\Omega^2(1-(\mathscr{I}/\mathscr{I}_{\text{rig}})^{2/3}). \tag{2.12}$$

This equation is the particular form of the dependence (2.4) valid for the two level model. Hence, according to relation (2.7)

$$\omega^2 = -\tfrac{4}{3}\,\Omega\varepsilon\mathscr{I}_{\mathrm{rig}}^{-1}(\mathscr{I}/\mathscr{I}_{\mathrm{rig}})^{-2/3} + \tfrac{4}{3}\,G\Omega^2(\mathscr{I}/\mathscr{I}_{\mathrm{rig}})^{-1/3}\mathscr{I}_{\mathrm{rig}}^{-1}. \tag{2.13}$$

This equation can be solved for $\mathscr{I}$:

$$\mathscr{I} = \frac{8\varepsilon^3\mathscr{I}_{\mathrm{rig}}}{G^2\Omega^3\{1\mp(1-3\mathscr{I}_{\mathrm{rig}}\varepsilon\omega^2/G^2\Omega^2)^{1/2}\}^3}. \tag{2.14}$$

In case of

$$G/G_c > 2 \tag{2.15}$$

relation (2.14) leads to the multivalued expression for $\mathscr{I}$ as a function of ω^2 giving a "back bending" curve and can be related [14] to the existence of the rapid phase transition [16], [17], [18]. Here G_c denotes the critical value of the pairing force strength G, below which no superfluid solution exists in the framework of BCS theory. In the two level model we have

$$G_c = \varepsilon/\Omega. \tag{2.16}$$

For our purpose it suffices to consider only the plus sign in expression (2.14) and to expand it in powers of ω^2. Comparison with Eq. (1.3) gives

$$\alpha = \frac{1}{2}\left(\frac{G_c}{G}\right)^3 \mathscr{I}_{\mathrm{rig}}, \tag{2.17}$$

$$\beta = \frac{27}{16}\left(\frac{G_c}{G}\right)^4 \mathscr{I}_{\mathrm{rig}}^2/(\Omega^2 G), \tag{2.18}$$

$$\gamma = \frac{135}{32}\left(\frac{G_c}{G}\right)^5 \mathscr{I}_{\mathrm{rig}}^3/(\Omega^4 G^2), \tag{2.19}$$

$$\delta = \frac{1323}{128}\left(\frac{G_c}{G}\right)^6 \mathscr{I}_{\mathrm{rig}}^4/(\Omega^6 G^3). \tag{2.20}$$

Instead of the parameters α, β, γ, δ, ... it is convenient to use moment of inertia $\mathscr{I}_0$ at $\omega \to 0$, the nuclear softness parameter σ [3], and the two parameters c, d defining the curvature in the curve $\mathscr{I}(\omega^2)$ (*cf.* [4]). They are related to α, β, γ and δ in the following way

$$\mathscr{I}_0 = 2\alpha, \tag{2.21}$$

$$\sigma = \beta/(6\alpha^2), \tag{2.22}$$

$$c = \alpha\gamma/\beta^2 \tag{2.23}$$

$$d = \delta\alpha^2/\beta^3. \tag{2.24}$$

The parameters σ, c and d are dimensionless.

It is worth to notice that the parameter c is constant in this model:

$$c = \alpha\gamma/\beta^2 = 20/27. \tag{2.25}$$

On the other hand the parameter d is not constant:

$$d = \frac{392}{729}\left(\frac{G}{G_c}\right)^3 \mathscr{I}_{\mathrm{rig}}^{-1}. \tag{2.26}$$

3. Particles coupled to rotor

Let us now turn our attention to a different model of a rotating nuclear system. It is based on the coupling between a system of $N = 2\Omega$ valence particles distributed ower two levels split by an energy difference of 2ε with a pair degeneracy 2Ω each, as before, and a rotor characterized by the moment of inertia $\hbar^2/a$. This model has been described in detail earlier [9], [10]. Here, we shall only briefly recall the most essential points.

We shall assume that each pair of states in the upper level together with the corresponding pair of states in the lower level form a split $j = 3/2$ multiplet (Fig. 1) and we shall

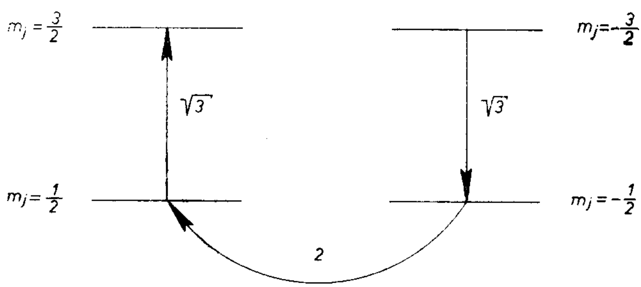

Fig. 1. The two level model referred to in the discussion in Section 3. The numbers at the arrows indicate magnitudes of matrix elements between the single particle states

use index α to list the four possibilities: $m = +3/2$ ($\alpha = 1$), $m = -3/2$ ($\alpha = 2$), $m = +1/2$ ($\alpha = 3$) and $m = -1/2$ ($\alpha = 4$) as indicated in Fig. 1. Now, if $c_{k\alpha}^\dagger$ denotes a creation operator in the state k ($k = 1, 2, \ldots, \Omega$) corresponding to given value of $\alpha = 1, 2, 3, 4$, we can construct 16 operators

$$N_{\alpha\beta} = \sum_{k=1}^{\Omega} c_{k\alpha}^\dagger c_{k\beta} \tag{3.1}$$

together with the 12 operators

$$B_{\alpha\beta}^\dagger = \sum_{k=1}^{\Omega} c_{k\alpha}^\dagger c_{k\beta}^\dagger \tag{3.2}$$

and

$$B_{\alpha\beta} = (B_{\alpha\beta}^\dagger)^\dagger\,. \tag{3.3}$$

One can easily see that the commutation relations of the operators (3.1) to (3.3) close.

The set of 28 operators $N_{\alpha\beta}$, $B^\dagger_{\alpha\beta}$ and $B_{\alpha\beta}$ forms therefore an algebra of R(8), the group of rotations in 8 dimensional space. The Hamiltonian of the system containing the rotational term, pairing forces and single particle splitting has the form

$$H' = \frac{a}{2}(I-j)^2 + H_P + H_{\text{s.p.}}. \tag{3.4}$$

The pairing forces H_P can be written in terms of the operators (3.3):

$$H_P = -G(B^\dagger_{12}+B^\dagger_{34})(B_{12}+B_{34}), \tag{3.5}$$

while the single particle Hamiltonian $H_{\text{s.p.}}$ is simply

$$H_{\text{s.p.}} = 2\varepsilon(N_{11}+N_{22}-N_{33}-N_{44}). \tag{3.6}$$

In this paper we shall employ the simplified version of the model corresponding to the introduction of a two dimensional rotor that can only rotate about a fixed axis (say, x-axis) perpendicular to the nuclear symmetry axis (z-axis). In this case the Hamiltonian H' reduces to a simpler form

$$H = \frac{a}{2}(I-j_x)^2 + H_P + H_{\text{s.p.}} \tag{3.7}$$

where H and $H_{\text{s.p.}}$ are given as before by Eqs (3.5) and (3.6), respectively. Now, the generators entering (3.7) form under the commutation relations a subalgebra of the full R(8) algebra. This can be shown [10] to be an algebra of R(6). Its 15 generators are

$$\begin{aligned}
K_+ &= B^\dagger_{12},\ K_- = B_{12},\\
\Lambda_+ &= B^\dagger_{34},\ \Lambda_- = B_{34},\\
\Gamma_+ &= \tfrac{1}{2}(B^\dagger_{14}+B^\dagger_{23}+B^\dagger_{13}+B^\dagger_{24}),\ \Gamma^- = \tfrac{1}{2}(B_{13}+B_{23}+B_{13}+B_{24}),\\
\Delta_+ &= \tfrac{1}{2}(B^\dagger_{14}+B^\dagger_{23}-B^\dagger_{13}-B^\dagger_{24}),\ \Delta_- = \tfrac{1}{2}(B_{14}+B_{23}-B_{13}-B_{24}),\\
h_+ &= \tfrac{1}{2}(N_{13}-N_{24}+N_{23}-N_{14}),\ h_- = \tfrac{1}{2}(N_{31}-N_{42}+N_{32}-N_{41}),\\
e_+ &= \tfrac{1}{2}(N_{13}-N_{24}-N_{23}+N_{14}),\ e_- = \tfrac{1}{2}(N_{31}-N_{42}-N_{32}+N_{41}),\\
\hat{N} &= \tfrac{1}{2}(N_{11}+N_{22}+N_{33}+N_{44})\\
\hat{V} &= \tfrac{1}{2}(N_{11}+N_{22}-N_{33}-N_{44}),\\
\hat{D} &= \tfrac{1}{2}(N_{12}+N_{21}-N_{34}-N_{43}).
\end{aligned} \tag{3.8}$$

The operator j_x can now be expressed [10] in terms of generators (3.8)

$$j_x = \frac{\sqrt{3}}{2}(h_+ + e_+ + h_- + e_-) + \hat{D}. \tag{3.9}$$

It can be checked that the set of the operators (3.8) closes under the commutation rotations. The lowest energy states of the Hamiltonian (3.7) correspond to the symmetric representations of R(6). Matrix elements of various terms in (3.7) in these representations are found with use of the Gel'fand Tsetlin method [19] and the Hamiltonian (3.7) is then diagonalized numerically for several values of I. Simultaneously, the angular velocity

$$\omega = dE/dI = a(I - \langle j_x \rangle) \tag{3.10}$$

is calculated for the same set of I values. In this way the total energy E is determined numerically as function of ω^2 and the parameters α, β, γ, δ, ... are found with use of Eq. (1.1). Another more convenient version of the calculation consists in computing numerically the moment of inertia

$$\mathscr{I} = \hbar I/\omega \tag{3.11}$$

and using Eq. (1.3) instead of (1.1) for the determination of the parameters α, β, γ, δ,...

It is worth noticing that formulae (3.10) and (3.11) contain I instead of $\sqrt{I(I+1)}$. This is the deficiency following from the assumption of the two dimensional rotation.

4. *Discussion*

We have performed the calculation of the coefficients α, β, γ and δ in the expansion of the rotational energy in powers of the square of the angular velocity. Two simple models have been employed in the calculation. Both of them are based on the assumption that the single particle spectrum of the deformed nucleus can be replaced approximately by two highly degenerate levels. The first of the two models (model I) discussed in Section 2 employs the BCS approximation. Now, the expression (2.5) for the moment of inertia $\mathscr{I}$ contains only the first term of the cranking model expansion, while the higher order terms (considered for example in Refs [1], [7]) have been neglected. Consequently, the increase of the moment of inertia with ω (or I) is caused only by the change in pairing correlation. Thus Eqs (2.17) to (2.20) for the coefficients α, β, γ and δ are limited only to the contributions coming from the antipairing effect. On the other hand the particle-coupled-to-rotor model (model II) based on the exact diagonalisation of the Hamiltonian (3.7) contains the sum of the contributions coming both from the antipairing effect as well as all the higher order terms in the cranking model (*i. e.* terms proportional to ω^4, ω^6, ... in expansion (1.1)). Furthermore, model II describes the motion of 2Ω external particles coupled to a rotor, while in model I the rotor does not exist and the motion is determined by the 2Ω particles only.

The above reasons make rather difficult a strict comparison between the two models. Nevertheless, it seems interesting to look at the properties of the parameters α, β, γ and δ resulting from the two approaches. In the discussion below we try to compensate partly for the nonexistence of the rotor in model I by assuming the degeneracy Ω slightly higher as compared to Ω in model II (see Figs 2, 3, 4 and 5 below).

Fig. 2 presents the variation of the quantity $2\alpha = \mathscr{I}_0$ as a function of the pairing force strength G. We can see from this figure that model II based on the exact diagonalisation of the Hamiltonian gives a much slower variation of $\mathscr{I}$ with respect to G as compared to the BCS theory (model I). This may be partly caused by the fact that at $G \to \infty$ the quantity $\mathcal{I}_0$ goes to the finite limit equal to $\hbar^2/a$, the moment of inertia of the rotor in model II. Nevertheless one may suspect that the BCS theory underestimates the moment of inertia in the physical region (*i. e.* for G slightly above G_c).

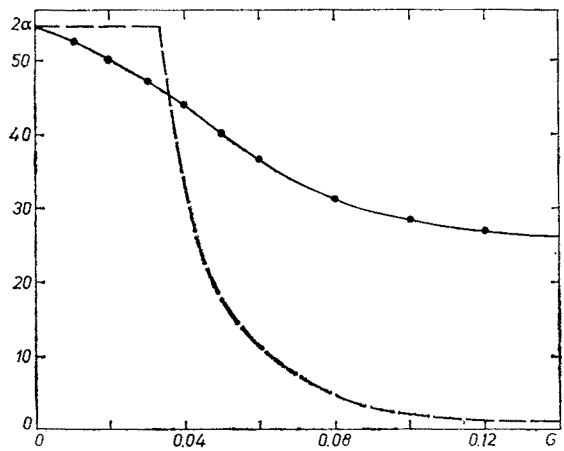

Fig. 2. The parameter $2\alpha = \mathcal{I}_0$ plotted *versus* the pairing force strength G. Solid line corresponds to model II discussed in Section 3 with the parametrs $a = 0.04$, $\varepsilon = 0.2$ and $\Omega = 4$. Dashed line represents the results of model I Section 2 using $\varepsilon = 0.2$ and $\Omega = 6$. The parameter $\mathcal{I}_{\text{rig}}$ appearing in model I (see Eq. (2.17)) has been chosen as to fit both the curves for $G = 0$

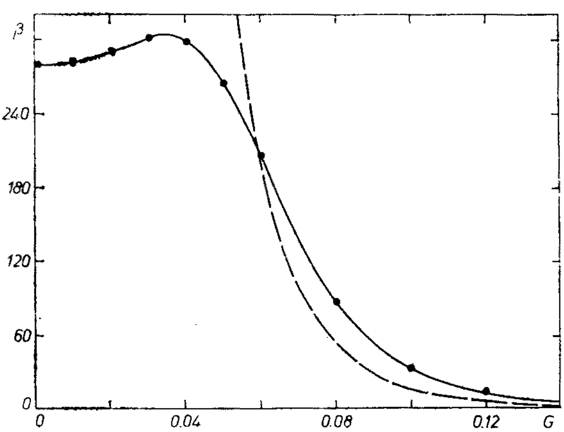

Fig. 3. The parameter β plotted *versus* the pairing force strength G corresponding to model II (solid line) and I (dashed line). For the parameters used see caption to Fig. 2

Figs 3, 4 and 5 illustrate the variation of the coefficients β, γ and δ. One can see that both the models give roughly the same G dependence of these coefficients in the region of large G (*i. e.* for G/G_c considerably exceeding unity). On the other hand, in the region of $G \sim G_c$ or $G/G_c < 1$ the BCS approximation is useless and we can only discuss the result of model II.

The two parameters γ and δ determine the curvature of the function $\mathscr{I}(\omega^2)$. We can see that the BCS formulae (2.19) and (2.20) always give positive values for these coefficients. However, it remains an open question what would be the signs of the higher-order cranking model contributions in the framework of the BCS approximation. On the other

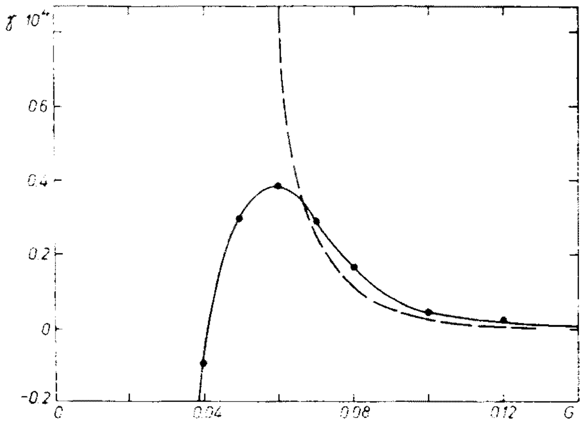

Fig. 4. The parameter γ plotted *versus* the pairing force strength G corresponding to model II (solid line) and I (dashed line). For the parameters used see caption to Fig. 2

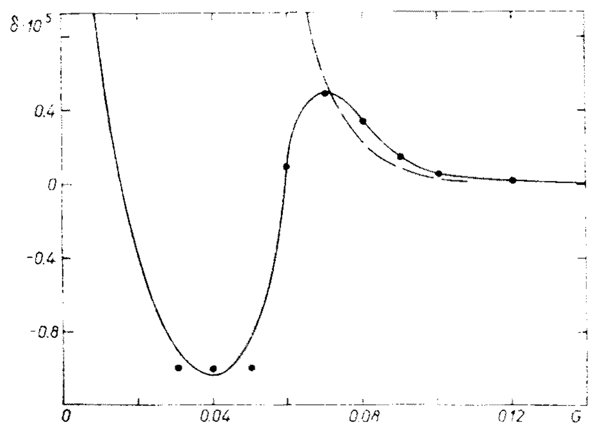

Fig. 5. The parameter δ plotted *versus* the pairing force strength G corresponding to model II (solid line) and I (dashed line). For the parameters used see caption to Fig. 2

hand model II shows that γ and δ can vary very rapidly and even change signs. The values coming from fitting of Eqs (1.1) to (1.3) to the experimental spectra seem to indicate that γ is negative and δ — positive in the region of actual G [4]. Fig. 4 and 5 show that this can occur in our model II only at the region of very small G ($G/G_c \ll 1$). However, the discussion of the detailed properties of these two parameters clearly requires a more careful investigation exceeding perhaps the simplified two-level model.

The total moment of inertia calculated in model II is shown in Fig. 6 as function of ω^2. The increase of $\mathscr{I}$ in this plot is caused by a combined effect of the antipairing contribution and the higher order terms in the cranking model (solid line). The dashed-and-dotted line in Fig. 6 illustrates the calculation in model II with $G = 0$ (*i. e.* containing only the higher-order cranking model terms without the antipairing contributions). The considerable increase of $\mathscr{I}$ in this case proves that model II is by no means limited to the antipairing effect. It may be seen that the calculated moment of inertia $\mathscr{I}$ exceeds the value of $\mathscr{I}_{rig}$

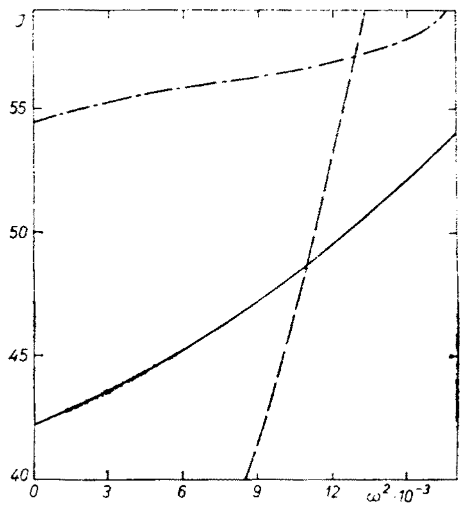

Fig. 6. The nuclear moment of inertia $\mathscr{I}$ plotted *versus* ω^2. Solid line corresponds to model II calculated with $G = 0.045$. Dashed line represents calculation in model I for the same value of G. The dashed-and--dotted line illustrates the calculation for $G = 0$ in model II. For the other parameters used see caption to Fig. 2

considerably at high angular momenta. The dashed line in Fig. 6 represents the results obtained in model I based on the BCS theory including only the antipairing effect without the higher-order cranking model terms. The curve is much steeper in this case. This reflects probably the deficiency of the BCS approximation in the rotating nucleus.

On the other hand let us notice that in model II (Section 3) the distinction between the external particles and the rotating core is somewhat arbitrary. This is an obvious imperfection of this model. Very roughly, one has to assume that the nucleons close to the Fermi surface (and, therefore, interacting *via* pairing force most effectively) are treated explicitly as external in Eqs (3.4) or (3.7) while the remaining contribute to the core. Only in this case one may hope that the quantity a in Eqs (3.4) or (3.7) depends very weakly on G and therefore this dependence may be neglected.

Although the above calculation does not answer the interesting question why is the expansion in powers of ω^2 so much superior to the expansion in $I(I+1)$, we hope that it can be useful in the quantitative estimates of the expansion parameters α, β, γ, δ.

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REFERENCES

[1] S. M. Harris, *Phys. Rev.*, **138**, B509 (1965).
[2] A. Bohr, *XV-me Conseil Int. de Physique*, Brussels, Belgium, Sept. 28 — Oct. 2, 1970; B. R. Mottelson, *The Nuclear Structure Symposium of the Thousand Lakes*, Joutsa, Finland, Aug. 1970.
[3] M. A. J. Mariscotti, G. Scharff-Goldhaber, B. Buck, *Phys. Rev.*, **178**, 1864 (1969).
[4] Ø. Saethre, S. A. Hjorth, A. Johnson, S. Jägare, H. Ryde, Z. Szymański, to be published.
[5] D. R. Inglis, *Phys. Rev.*, **96**, 1059 (1954); **97**, 701 (1955).
[6] A. Bohr, B. R. Mottelson, *Dan. Mat. Fys. Medd.*, **30**, no 1 (1955).
[7] C. W. Ma, J. Rasmussen, *Phys. Rev.*, **C2**, 798 (1970).
[8] J. Bardeen, L. N. Cooper, J. R. Schrieffer, *Phys. Rev.*, **108**, 1175 (1957).
[9] J. Krumlinde, Z. Szymański, *Phys. Letters*, **36B**, 157 (1971).
[10] J. Krumlinde, Z. Szymański, to be published.
[11] H. Goldstein, *Classical Mechanics*, Addison-Wesley Press Inc., Cambridge, Massachusetts 1951, Chapter 7.
[12] J. Krumlinde, *Nuclear Phys.*, **A121**, 306 (1968).
[13] D. R. Bés, S. Landowne, M. A. J. Mariscotti, *Phys. Rev.*, **166**, 1045 (1968).
[14] R. A. Sorensen, *Proc. of the Orsay Colloquium on Intermediate Nuclei*, July 1971, and Research Institute for Physics, Stockholm, Annual Report, 1970.
[15] S. T. Belyaev, *Dan. Mat. Fys. Medd.*, **31**, no 11 (1959).
[16] A. Johnson, H. Ryde, J. Sztarkier, *Phys. Letters*, **34B**, 605 (1971).
[17] A. Johnson, H. Ryde, S. A. Hjorth, *Nuclear Phys.*, **A179**, 753 (1972).
[18] B. R. Mottelson, J. G.. Valatin, *Phys. Rev. Letters*, **5**, 511 (1960).
[19] I. M. Gel'fand, M. L. Tsetlin, *Dokl. Akad. Nauk SSSR*, **71**, 1017 (1950).