Vol. B4 (1973) ACTA PHYSICA POLONICA No 4

INTEGRATION OF CLASSICAL NONLINEAR RELATIVISTIC WAVE EQUATIONS BY THE METHOD OF LIE SERIES*

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(Received February 8, 1973)

We applied the method of Lie series for the construction of local solutions of classical nonlinear relativistic wave equations. Next we gave the explicit form of solutions for the classical massive Thirring equation, nonlinear Heisenberg equation and the nonlinear equation in $\lambda\Phi^4$ theory.

1. Introduction

We considered in a previous papers [1] nonlinear relativistic wave equations for quantum scalar fields $\Phi(x)$ of the form

$$(\Box+m^2)\,\Phi(x) = \lambda F(\Phi). \tag{1.1}$$

Using a new technique for the expansion of operators in orthogonal operator bases, we reduced Eq. (1.1) to a corresponding nonlinear relativistic wave equation for the c-number generalized Fourier transform of operator $\Phi(x)$. Thus we obtain as many solutions for the quantum field $\Phi(x)$, as we are able to construct for the corresponding classical equation.

One can reduce Eq. (1.1) by introduction of new unknown functions to the partial nonlinear differential evolution equations of the form

$$\frac{\partial Z_i(t, \boldsymbol{x})}{\partial t} = \vartheta_i(\boldsymbol{x}, Z, Z^{(1)}, \ldots, Z^{(m)}), \tag{1.2}$$

where $Z_i^{(k)}(t, \boldsymbol{x})$ denotes in general the k-th partial derivative of the function $Z_i(t, \boldsymbol{x})$ with respect to the variables $\boldsymbol{x}$. The evolution equation (1.2) was recently treated extensively by the methods of semi-group theory, using the formalism of Banach spaces [2]. However, the main effort has so far been concentrated on the problem of finding a proper

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Banach space in which there exists a unique solution of Eq. (1.2) satisfying an initial Cauchy condition and continuous with respect to the initial data. This is of course not sufficient for an analysis of physical properties of systems described by nonlinear differential equations like (1.1) or (1.2).

We present in this paper the constructive approach to nonlinear partial differential equations based on the technique of Lie series. This method yields for a fairly general class of nonlinear P. D. E. an explicit power series solution. The method is also convenient for the construction of approximate solutions, what might be useful in applications in various problems of science and technology.

We present in Section 2 the basic properties of Lie series. Section 3 is devoted to the construction of the solution of systems of P. D. E. of the form

$$\frac{\partial Z_i(t, x)}{\partial x^\mu} = \vartheta_{\mu i}(Z), \qquad \begin{array}{l} i = 1, 2, \ldots, n, \\ \mu = 0, 1, \ldots, s. \end{array} \tag{1.3}$$

We construct the solutions of nonlinear evolution equations (1.2) in Section 4. The application of the present method to the solution of the classical nonlinear relativistic wave equation to the massive Thirring model is considered in Section 5. Section 6 contains the solution for the spinor field in the nonlinear Heisenberg spinor theory. The solution to classical $\lambda\Phi^4$ theory is contained in Section 7. Section 8 contains a discussion of results and various generalizations.

The method of Lie series was invented by Groebner to solve some problems in algebraic geometry [3]. Later on it was used by Groebner and collaborators [4] and independently by Filatov [5] in the theory of general nonlinear ordinary differential equations and some special P. D. E. Using convergence-improving methods Groebner and his collaborators showed that the method of Lie series is more effective than the well-known approximation methods of Adams, Cowell or Runge-Kutta-Fehlberg-Shanks [4, b]. In particular the Lie series method was used for the calculation of the trajectory of the soft landing of the moon rocket with the minimal use of fuel [6].

The method of Lie series was used also in the statistical mechanics for construction of formal solutions of Liouville evolution equations (see *e. g.* [7]).

2. *Properties of Lie series*

Let D denote a linear differential operator in the form

$$D(z) = \vartheta_1(z)\frac{\partial}{\partial z_1} + \vartheta_2(z)\frac{\partial}{\partial z_2} + \ldots + \vartheta_n(z)\frac{\partial}{\partial z_n} \equiv \vartheta_i(z)\frac{\partial}{\partial z_i}, \tag{2.1}$$

where $\vartheta_i(z) = \vartheta_i(z_1, z_2, \ldots, z_n)$ are holomorphic functions of the complex variables $z_1, \ldots, z_n$ in a certain domain $G \subset C^n$. The exponential operator defined by the series

$$e^{tD} \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu \tag{2.2}$$

is called the Lie series. We shall summarize in this section the basic properties of Lie series. Let

$$\vartheta_i(z) = \sum c^{(i)}_{k_1,\dots,k_n} z_1^{k_1} \dots z_n^{k_n} \tag{2.3}$$

be a power series expansion for $\vartheta_i(z)$. Let ϱ_i denote the common radius of convergence for all functions $\vartheta_i(z)$, $i = 1, \dots, n$ and a certain holomorphic function $F(z_1, \dots, z_n) \equiv F(z)$ and let

$$N_i = \max\{|C^{(i)}_{k_1,\dots,k_n}| \varrho^{k_1+\dots+k_n}\}, \qquad N = \max\{N_i\}. \tag{2.4}$$

The following theorem describes the main properties of Lie series (2.2).

Theorem 1. Let G be a finite closed domain of C^n in which the differential operator (2.1) and a function $F(z)$ are holomorphic. Then the Lie series

$$e^{tD}F(z) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu F(z) \tag{2.5}$$

converges absolutely and uniformly at least for

$$|t| < T = \frac{\varrho}{(n+1)N} \tag{2.6}$$

throughout the entire domain G where it thus represents a holomorphic function of $n+1$ complex variables $z_1, \dots, z_n, t$. Moreover, in the interior of $G \times \mathscr{T}, \mathscr{T} = [0, T)$ we have

$$\frac{\partial^\mu}{\partial t^\mu} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu F = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^{\nu+\mu} F \tag{2.7}$$

and

$$\frac{\partial^\mu}{\partial z_k^\mu} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu F = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \frac{\partial^\mu}{\partial z_k^\mu} D^\nu F; \tag{2.8}$$

for the proof *cf.* [4, a], Ch. I. The following theorem plays a fundamental role in the theory of partial nonlinear differential equations.

Commutation Theorem. Let $F(z)$ be a holomorphic function in a neighborhood of a point $z = (z_1, \dots, z_n)$ and let the point $Z = (Z_1, \dots, Z_n)$ with

$$Z_i = e^{tD} z_i, \tag{2.9}$$

where

$$D = \vartheta_k(z) \frac{\partial}{\partial z_k},$$

be still in the domain of holomorphy of F. Then

$$e^{tD}F(z) = F(e^{tD}z) \tag{2.10}$$

i. e., the operator $U_t = e^{tD}$ and the symbol F of a function commute. (For the proof *cf.* [4, a], Ch. I.).

In the following there will often appear the Lie series of the form

$$\Psi(t, x, z) = e^{tD(x,Z)}F(Z)|_{Z_i=z_i} \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \left[\vartheta_k(x, Z) \frac{\partial}{\partial Z_k} F(Z) \right]_{Z_i=z_i}. \tag{2.11}$$

For the simplification of notation we shall write Lie series (2.11) in the form

$$\Psi(t, x, z) = e^{tD(x,z)}F(z) \tag{2.12}$$

with

$$D(x, z) = \vartheta_k(x, z) \frac{\partial}{\partial z_k}. \tag{2.13}$$

3. Systems of nonlinear partial differential equations

A higher order nonlinear partial differential equation may often be reduced, by the introduction of new variables, to a system of the following type

$$\frac{\partial Z_i(t, x)}{\partial x^\mu} = \vartheta_{\mu i}(Z), \quad \mu = 0, 1, \ldots, s, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

with the initial conditions

$$Z_i(t, x)|_{t=t_0} = z_i(x) \in C^1, \quad i = 1, \ldots, n. \tag{3.2}$$

We give here the solution of Eq. (3.1) in the form of the Lie series (2.2).

Theorem 1. If $\vartheta_{\mu i}(Z)$ and $z_i(x)$, $\mu = 0, 1, \ldots, s$, $i = 1, 2, \ldots, n$, are entire functions of their arguments then the functions

$$Z_i(t, x) = e^{(t-t_0)D_0(z)} z_i, \tag{3.3}$$

where

$$D_0(z) = \vartheta_{0k}(z) \frac{\partial}{\partial z_k}, \tag{3.4}$$

satisfy Eq. (3.1) and the initial conditions (3.2) at least in the region

$$|t - t_0| < T = \frac{\varrho}{(n+1)N}, \quad x \in R^s, \tag{3.5}$$

where $\varrho < \infty$ and N is defined by Eq. (2.4).

If $\vartheta_{\mu i}(Z)$ and $z_i(x)$ are analytic functions of their arguments with the common radius of convergence ϱ_ϑ and ϱ_z respectively, then the solution (3.3) converges at least in the region $\Omega = \Omega_t \times \Omega_x \subset R^{s+1}$ defined by the relations

$$\Omega_t = \{t \in R^1 : |e^{(t-t_0)D_0} z_i| \leqslant \varrho_\vartheta, i = 1, \ldots, n\},$$

$$\Omega_x = \{x \in R^s : |z_i(x)| \leqslant \min(\varrho_\vartheta, \varrho_z), i = 1, \ldots, n\}. \tag{3.6}$$

The solution (3.3) represents the analytic function of $(t, \boldsymbol{x})$ in the region (3.5) or (3.6) respectively.

Proof: Let $F(Z)$ be a differentiable function of the solution $Z = \{Z_i\}_1^n$ of Eq. (3.1). Then

$$\frac{\partial}{\partial x^{\mu}} F(Z) = \frac{\partial F}{\partial Z_k} \frac{\partial Z_k}{\partial x^{\mu}} = \vartheta_{\mu k}(Z) \frac{\partial}{\partial Z_k} F(Z) \equiv D_{\mu}(Z)F(Z). \tag{3.7}$$

The commutator of the operators $D_{\mu}(Z)$ and $D_{\nu}(Z)$ has the form

$$[D_{\mu}, D_{\nu}]\Psi(Z) = (\vartheta_{\mu i}\vartheta_{\nu k,i} - \vartheta_{\mu k,i}\vartheta_{\nu i}) \frac{\partial}{\partial Z_k} \Psi(Z).$$

The last condition is precisely the integrability condition for Eq. (3.1). Consequently

$$[D_{\mu}, D_{\nu}] = 0, \qquad \mu, \nu = 0, 1, \ldots, s. \tag{3.8}$$

Now, by virtue of Theorem 2.1 the power series

$$Z_i(t, \boldsymbol{x}) = e^{(t-t_0)D_0} z_i, \qquad i = 1, 2, \ldots, n, \tag{3.9}$$

converges for at least $|t - t_0| < T = \varrho_{\vartheta}/(n+1)N$, where $\varrho_{\vartheta} < \infty$ is the common radius of convergence of all $\vartheta_i(Z)$, $i = 1, \ldots, n$ and N is defined by formula (2.4). The functions $Z_i(t, \boldsymbol{x})$ given by Eq. (3.9) satisfy the Eq. (3.1). Indeed, using Eqs (3.7), (3.8) and Commutation Theorem we obtain

$$\frac{\partial Z_i(x)}{\partial x^{\mu}} = e^{(t-t_0)D_0(z)} D_{\mu}(z) z_i = e^{(t-t_0)D_0(z)} \vartheta_{\mu i}(z) =$$

$$= \vartheta_{\mu i}(e^{(t-t_0)D_0(z)} z) = \vartheta_{\mu i}(Z). \tag{3.10}$$

If $\vartheta_{\mu i}(Z)$ are entire functions of their arguments then Eq. (3.10) holds for all t satisfying $|t - t_0| < T$; otherwise Eq. (3.10) holds for t for which $|Z_i(x)| \leqslant \varrho_{\vartheta}$, $i = 1, \ldots, n$. Finally, by Theorem 2.1, the series (3.9) converges for $|z_i| < \varrho_{\vartheta}$. Consequently the series (3) converges for $\Omega_x = \{\boldsymbol{x} \in R^s : |z_i(\boldsymbol{x})| \leqslant \min(\varrho_{\vartheta}, \varrho_z),\ i = 1, \ldots, n\}$.

According to a well-known theorem an infinite series of analytic functions which converges uniformly in $\mathscr{X} = \{z \in C^n : |z_i| \leqslant \varrho_{\vartheta}\}$ is an analytic functions in the interior of $\mathscr{X}$ [8]. Because the composition of analytic functions is analytic the second assertion of Theorem 1 follows.

Let us note that for $s = 0$ the Theorem 1 provides the solution for autonomous as well as for nonautonomous system of ordinary nonlinear differential equations. Indeed we have

Corollary 1. Let $s = 0$. Then the autonomous system of ordinary nonlinear equations

$$\frac{dZ_i(t)}{dt} = \vartheta_i(Z), \qquad Z_i(t)_{|t=t_0} = z_i \in C^1, \qquad i = 1, \ldots, n, \tag{3.11}$$

has the solution

$$Z_i(t) = e^{(t-t_0)D(z)}z_i, \qquad i = 1, \ldots, n, \tag{3.12}$$

where

$$D(z) = \vartheta_k(z)\frac{\partial}{\partial z_k}. \tag{3.13}$$

The solution (3.12) is convergent at least for $|t-t_0| < T = \dfrac{\varrho}{(n+1)N}$.

Corollary 2. The nonautonomous system

$$\frac{dZ_i(t)}{dt} = \vartheta_i(t, Z), \qquad Z_i(t)_{|t=t_0} = z_i \in C^1, \qquad i = 1, \ldots, n, \tag{3.14}$$

has the solution

$$Z_i(t) = e^{(t-t_0)D(z)}z_i, \qquad i = 0, 1, \ldots, n, \tag{3.15}$$

where

$$Z_0(t) = t, \qquad Z_0(t)_{|t=t_0} = z_0 = t_0 \tag{3.16}$$

and

$$D(z) = \frac{\partial}{\partial z_0} + \sum_{k=1}^{n} \vartheta_k(z)\frac{\partial}{\partial z_k}. \tag{3.17}$$

The solution (3.17) is convergent at least for $|t-t_0| < T = \dfrac{\varrho}{(n+1)N}$.

Proof: Using Eq. (3.16) we obtain the following autonomous system

$$\frac{dZ_i(t)}{dt} = \vartheta_i(Z), \qquad i = 0, 1, \ldots, n, \tag{3.18}$$

where $\vartheta_0(Z) = 1$.

Applying now Theorem 1 we obtain the assertion of Corollary 2.

4. *Integration of nonlinear evolution equations*

We shall now elaborate a method for an explicit construction of solutions of nonlinear evolution equations of the form

$$\frac{\partial Z_i(t, x)}{\partial t} = \vartheta_i(x, Z^{(1)} \ldots, Z^{(m)}), \qquad i = 1, \ldots, n, \tag{4.1}$$

where $Z^{(1)}, \ldots, Z^{(m)}$ mean the derivatives of the form

$$Z_{i,k}^{(1)}(t, x) = \frac{\partial}{\partial x^k} Z_i(t, x), \quad i = 1, \ldots, n, \quad k, k_l = 1, \ldots, s,$$
$$\vdots$$
$$Z_{i,k_1,\ldots,k_s}^{(m)}(t, x) = \frac{\partial^m}{\partial x_1^{k_1} \ldots \partial x_s^{k_s}} Z_i(t, x). \tag{4.2}$$

These equations appear in many problems of Theoretical Physics. The higher order equations in time variable may be always reduced to Eq. (4.1) by the introduction of an additional variable associated with the lower order time derivatives. We start with the analysis of the first order nonlinear equations of the form

$$\frac{\partial Z_i(t, x)}{\partial t} = \vartheta_i(x, Z, Z^{(1)}), \quad i = 1, \ldots, n, \tag{4.3}$$

where $Z_i(t, x)$ satisfy the initial conditions

$$Z_i(t, x)_{|t=t_0} = Z_i(t_0, x) \equiv z_i(x), \quad i = 1, \ldots, n. \tag{4.4}$$

Let $\tilde{Z}_i(t, x) = Z_i(t, x) - z_i(x)$. We assume that the functions $\vartheta_i(x, Z, Z^{(1)})$ and the initial conditions (4.4) are such that the following condition is satisfied:

(i) There exists a positive number ϱ such that the functions

$$z_i(x) \text{ and } \tilde{\vartheta}_i(x, \tilde{Z}, \tilde{Z}^{(1)}) \equiv \vartheta_i(x, \tilde{Z}+z, \tilde{Z}^{(1)}+z^{(1)}), \quad i = 1, \ldots, n,$$

are analytic functions of their arguments in the region given by

$$|x_k| < \varrho, \quad |\tilde{Z}_\varrho| < \varrho, \quad |\tilde{Z}_{l,k}^{(1)}| < \varrho, \quad \begin{matrix} k = 1, \ldots, s, \\ l = 1, \ldots, n. \end{matrix} \tag{4.5}$$

Let

$$\tilde{\vartheta}_i(x, \tilde{Z}, \tilde{Z}^{(1)}) = \sum c^{(i)}_{k_1\ldots k_s, j_1\ldots j_n, l_{11}\ldots l_{ns}} x_1^{k_1} \ldots x_s^{k_s} \cdot \tilde{Z}_1^{j_1} \ldots \tilde{Z}_n^{j_n} \cdot \tilde{Z}_{1,1}^{l_{11}} \ldots \tilde{Z}_{n,s}^{l_{ns}}$$

be a Maclaurin expansion for functions $\tilde{\vartheta}_i(x, Z, Z^{(1)})$ and let

$$N_i = \max \left\{ |c^{(i)}_{k_1\ldots k_s, j_1\ldots j_n, l_{11}\ldots l_{ns}}| \varrho^{k_1+\ldots+k_s+j_1+\ldots j_n+l_{11}+\ldots+l_{ns}} \right\} \tag{4.6}$$

and

$$N = \max_i \{N_i\}. \tag{4.7}$$

The following theorem describes the main properties of solutions of evolution equation (4.3).

Theorem 1. The evolution equation (4.3) where $\vartheta_i(x, Z, Z^{(1)})$ and $z_i(x)$ satisfy condition (i) has the unique analytic solution satisfying the initial condition (4.4) given by the formula[1]

$$Z_i(t, x) = e^{(t-t_0)D} z_i, \tag{4.8}$$

[1] In order to get an explicit form of factors at $(t-t_0)n/n!$ in Eqs (4.8), (4.26), (5.12), (6.8) and (7.6), we use the fact that $[\partial_t^1, \partial_{x_k}] = [D, \partial_{x_k}] = 0$. This implies, *e. g.*,

$$D^2 z_i = \vartheta_k \frac{\partial}{\partial z_k} \vartheta_i(x, z, z^{(1)}) = \vartheta_k \frac{\partial \vartheta_i}{\partial z_k} + \frac{\partial \vartheta_i}{\partial z_{s,l}} \frac{\partial}{\partial x^l} \vartheta_s \text{ etc.}$$

where

$$D(x, z, z^{(1)}) = \vartheta_k(x, z, z^{(1)}) \frac{\partial}{\partial z_k}. \tag{4.9}$$

The solution (4.8) is analytic in the region

$$|x| < r\varrho, \quad |t-t_0| < T = \frac{(1-r)^2\varrho}{1+8nsN}, \tag{4.10}$$

where $|x| \equiv \sum_{i=1}^{s} |x_i|$, $0 < r < 1$ and N is given by Eq. (4.7).

Proof: Set

$$\tilde{Z}_i(t, x) = Z_i(t, x) - z_i(x), \quad i = 1, \ldots, n. \tag{4.11}$$

These functions satisfy the equation

$$\frac{\partial \tilde{Z}_i(t, x)}{\partial t} = \tilde{\vartheta}_i(x, \tilde{Z}, \tilde{Z}^{(1)}) \equiv \vartheta_i(x, \tilde{Z}+z, \tilde{Z}^{(1)}+z^{(1)}) \tag{4.12}$$

and the initial conditions

$$\tilde{Z}(t_0, x) = 0. \tag{4.13}$$

By virtue of Perron Theorem, there exists the unique solution $Z_i(t, x)$ of Eq. (4.12) satisfying initial condi ion (4.13) which is analytic in the region Ω given by

$$|x| < r\varrho, \quad |t-t_0| < T = \frac{(1+r)^2\varrho}{1+8nsN}, \tag{4.14}$$

where $0 < r < 1$, ϱ is the radius defined by condition (i) and N is given by Eq. (4.7) (*cf.* Perron [9], § 3 and Berstein [10], II, §10). Now using Eq. (4.11) and condition (i) we see that the functions $Z_i(t, x)$ have also the region of analyticity given by Eq. (4.14). The power series expansion for $Z_i(t, x)$ has the form

$$Z_i(t, x) = \sum_{\nu=0}^{\infty} \frac{(t-t_0)^\nu}{\nu!} Z_i^{(\nu)}(x),$$

where coefficients $Z_i^{(\nu)}(x)$ represent the ν-th time derivatives of $Z_i(t, x)$ at $t = t_0$. Consequently in the region $|t-t_0| < T$ we have

$$Z_i(t, x) = e^{(t-t_0)\vartheta_{t_0}} Z_i(t_0, x). \tag{4.15}$$

Now let $F(Z)$ be a differentiable function of solutions $Z_i(t, x)$. Then using Eq. (4.3) we obtain

$$\partial_t F(Z) = \vartheta_k(x, Z, Z^{(1)}) \frac{\partial}{\partial Z_k} F(Z). \tag{4.16}$$

Thus, in the space of solutions the operator

$$D(x, Z, Z^{(1)}) \equiv \vartheta_k(x, Z, Z^{(1)}) \frac{\partial}{\partial Z_k} \tag{4.17}$$

corresponds to the operator ∂_t. Consequently, by virtue of Eqs (4.15), (4.16) and (4.17) one obtains

$$Z_i(t, x) = e^{(t-t_0)D} z_i,$$

where

$$D(x, z, z^{(1)}) = \vartheta_k(x, z, z^{(1)}) \frac{\partial}{\partial z_k}.$$

The function (4.8) provides the solution of the evolution equation (4.3) which is expressed in terms of initial conditions $z_i(x)$ and of the functions $\vartheta_i(x, z, z^{(1)})$ representing the degree of nonlinearity of the problem. This implies in particular that the solution of the nonlinear relativistic wave equation is a function of asymptotic $\Phi_{in}(x)$ field only.

Remark 1. One may convert the Minkowski space X and the space $\{Z_i(x)\}_{i=1}^n$ of solutions into a Banach space by introducing, for instance, the norms of the form

$$||V|| = (\sum_{k=1}^{m} |V_k|^2)^{1/2}.$$

One can then apply the technique of Banach spaces to get much more detailed information on the region of analyticity of solutions $Z_i(t, x)$, growth of solutions and their derivatives, *etc.* (*cf. e. g.*, [11]). In particular, one obtains the following additional result which gives important information on dependence of solution on coupling constant:
If $\vartheta_i(\lambda, x, Z, Z^{(1)})$, $i = 1, \ldots n$, are analytic functions of a parameter λ, $|\lambda| < 1$, x, Z, and $Z^{(1)}$ then the solution $Z_i(t, x)$ of Eq. (4.8) is an analytic function of λ for $|\lambda| < 1$ (*cf.* [11] Theorem 3).

The technique of Banach spaces is particularly convenient in cases where the dimension of coordinate space X or the space of solutions or both are infinite.

We shall now consider second order nonlinear equations of the form

$$\frac{\partial^2 Z(t, x)}{\partial t^2} = \vartheta(x, Z, Z^{(1)}, Z^{(2)}), \tag{4.18}$$

where

$$Z_\mu^{(1)}(t, x) = \frac{\partial Z(t, x)}{\partial x^\mu}, \quad \mu = 0, 1, \ldots, s, \tag{4.19}$$

and

$$Z_{\mu,i}^{(2)}(t, x) = \frac{\partial^2 Z}{\partial x^\mu \partial x^i}, \quad \mu = 0, 1, \ldots, s, \quad i = 1, \ldots, s. \tag{4.20}$$

We suppose that the function $Z(t, \boldsymbol{x})$ satisfies the following initial conditions

$$Z(t, \boldsymbol{x})_{|t=t_0} = Z(t_0, \boldsymbol{x}) \equiv z_1(\boldsymbol{x}), \quad \dot{Z}(t, \boldsymbol{x})_{|t=t_0} = \dot{Z}(t_0, \boldsymbol{x}) \equiv z_2(\boldsymbol{x}), \tag{4.21}$$

where $z_i(\boldsymbol{x})$, $i = 1, 2$ are analytic functions of their arguments. Introducing new variables

$$\begin{gathered} Z_0 = \partial_t^2 Z, \quad Z_1 = Z, \quad Z_2 = \partial_t Z, \\ Z_{l+2} = \partial_{x^l} Z_1, \quad Z_{l+s+2} = \partial_{x^l} Z_2, \quad l = 1, \ldots, s, \\ Z_{2s+3} = \partial_{x^1}\partial_{x^1} Z_1, \ldots, \quad Z_{s(s+1)/2+2s+2} = \partial_{x^s}\partial_{x^s} Z_1, \end{gathered} \tag{4.22}$$

we may cast Eq. (4.18) into the system of first order equations of the form

$$\frac{\partial Z_i(t, \boldsymbol{x})}{\partial t} = \vartheta_i(\boldsymbol{x}, Z, Z^{(1)}), \quad i = 0, 1, \ldots, s(s+1)/2+2s+2 \equiv K, \tag{4.23}$$

where

$$\begin{gathered} \vartheta_1 = Z_2, \quad \vartheta_2 = \vartheta(\boldsymbol{x}, Z), \\ \vartheta_{l+2} = \partial_{x^l} Z_2, \ldots, \vartheta_{l+s+2} = \partial_{x^l} Z_0, \quad l = 1, \ldots, s, \\ \vartheta_{1+2s+2} = \partial_{x^1} Z_{s+3}, \ldots, \vartheta_K = \partial_{x^s} Z_{2s+2}, \\ \vartheta_0 = \sum_{k=1}^{K} \frac{\partial \vartheta}{\partial Z_l} \vartheta_l. \end{gathered} \tag{4.24}$$

The solutions $Z_i(t, \boldsymbol{x})$ of Eq. (4.23) satisfy the following initial conditions defined by relations (22) for $t = t_0$:

$$Z_i(t, \boldsymbol{x})_{|t=t_0} \equiv z_i(\boldsymbol{x}), \quad i = 0, 1, \ldots, K. \tag{4.25}$$

Notice that all functions ϑ_i, $i = 0, 1, \ldots, K$ with the possible exception of ϑ_0 and ϑ_2 are entire functions of their arguments.

The following theorem describes the properties of solutions $Z(t, \boldsymbol{x})$ of second order equation (4.18).

Theorem 2. Let the functions ϑ_0 and ϑ_2 of the system (4.23) and initial conditions satisfy condition (i). Then the unique analytic solution $Z(t, \boldsymbol{x})$ of Eq. (4.18) satisfying the the initial conditions (4.21) is given by the formula (see previous footnote)

$$Z(t, \boldsymbol{x}) = e^{(t-t_0)D} z_1, \tag{4.26}$$

where

$$D(\boldsymbol{x}, z, z^{(1)}) = \sum_{i=0}^{K} \vartheta_i(\boldsymbol{x}, z, z^{(1)}) \frac{\partial}{\partial z_i} \tag{4.27}$$

and the functions $\vartheta_i(\boldsymbol{x}, z, z^{(1)})$ are given by Eq. (4.24). The solution (4.26) is analytic at least in the region

$$|\boldsymbol{x}| < r\varrho, \; |t-t_0| < T = \frac{(1+r)^2\varrho}{1+8(K+1)sN}, \tag{4.28}$$

where $0 < r < 1$ and N is defined by Eq. (4.7).

If the function $\vartheta(\boldsymbol{x}, Z^{(1)}, Z^{(2)})$ does not depend on derivatives with respect to time, then the solution $Z(t, \boldsymbol{x})$ can be represented in the form

$$Z(t, \boldsymbol{x}) = e^{(t-t_0)D} z_1, \tag{4.29}$$

where

$$D(\boldsymbol{x}, z, z^{(1)}, z^{(2)}) = \sum_{i=1}^{2} \vartheta_i(\boldsymbol{x}, z, z^{(1)}, z^{(2)}) \frac{\partial}{\partial z_i} \tag{4.30}$$

with

$$\vartheta_1 = z_2, \quad \vartheta_2 = \vartheta(\boldsymbol{x}, z, z^{(1)}, z^{(2)}). \tag{4.31}$$

The region of analyticity of the solution (4.29) is also given by formula (4.28).

Proof: The functions $\vartheta_i(\boldsymbol{x}, Z, Z^{(1)})$ and $z_i(\boldsymbol{x})$, $i = 0, 1, \ldots, K$, satisfy all the assumptions of Theorem 1. Consequently, there exists a unique analytic solution $Z_i(t, \boldsymbol{x})$, $i = 0, 1, \ldots, K$ of Eq. (4.23) satisfying initial conditions (4.25) which is analytic in the region (4.28). The Lie series form of solution $Z_i(t, \boldsymbol{x})$ is given by Eq. (4.8). If the function ϑ in Eq. (4.18) does not depend on derivatives with respect to time of $Z(t, \boldsymbol{x})$ then Eq. (4.18) can be represented as the following evolution equation

$$\frac{\partial Z_i(t, \boldsymbol{x})}{\partial t} = \vartheta_i(\boldsymbol{x}, Z, Z^{(1)}, Z^{(2)}), \quad i = 1, 2, \tag{4.32}$$

where

$$Z_1(t, \boldsymbol{x}) = Z(t, \boldsymbol{x}), \quad Z_2(t, \boldsymbol{x}) = \partial_t Z(t, \boldsymbol{x})$$

$$Z_1(t_0, \boldsymbol{x}) \equiv z_1(\boldsymbol{x}), \quad Z_2(t_0, \boldsymbol{x}) \equiv z_2(\boldsymbol{x})$$

and

$$\vartheta_1 = Z_2, \quad \vartheta_2 = \vartheta(\boldsymbol{x}, Z, Z^{(1)}, Z^{(2)}).$$

Using the same arguments as in the second part of proof of Theorem 1 we find that the solution $Z_i(t, \boldsymbol{x})$ of Eq. (4.18) satisfying the initial conditions (4.21) has the following form

$$Z_i(t, \boldsymbol{x}) = e^{(t-t_0)D} z_i, \quad i = 1, 2,$$

where

$$D(\boldsymbol{x}, z, z^{(1)}, z^{(2)}) = \sum_{i=1}^{2} \vartheta_i(\boldsymbol{x}, z, z^{(1)}, z^{(2)}) \frac{\partial}{\partial z_i}.$$

Clearly the solution (4.29) has the region of analyticity given by Eq. (4.28).

The form (4.8), (4.26) and (4.28) of solutions of evolution equation (4.3) or (4.18) respectively, shows that the manifold of solutions is in one to one correspondence with the manifold $\mathscr{Z}$ of initial date. The generators D of solution (4.8), (4.26) or (4.29) act non-linearly in $\mathscr{Z}$.

The Theorems 1 and 2 provide the form of solution for all known classical nonlinear relativistic wave equations like nonlinear Heisenberg spinor equation, equation of massive Thirring model, classical $\lambda\Phi^n$ equations, *etc.*

It is evident that this theory may be generalized to any higher order nonlinear analytic equation like (4.1) in finite or infinite numbers of variables x_μ and components Z_i.

5. *Quantum-mechanical massive Thirring model*

We shall now consider the massive Thirring model in two-dimensional space-time. The wave equation has the form [12]

$$i\gamma^\mu\partial_\mu\Psi - m\Psi - \lambda(\overline{\Psi}\gamma^\mu\Psi)\gamma_\mu\Psi = 0, \qquad \overline{\Psi} = \Psi^*\gamma_0, \tag{5.1}$$

with the initial condition

$$\Psi(t, x)|_{t=t_0} = \begin{pmatrix}\Psi_1(t, x)\\ \Psi_2(t, x)\end{pmatrix}_{t=t_0} \equiv \begin{pmatrix}\Psi_{01}(x)\\ \Psi_{02}(x)\end{pmatrix}, \tag{5.2}$$

where $\Psi_{0i}(x)$ are analytic functions of x for $|x| < \varrho$. Taking the representation of γ_μ-matrices in the form

$$\gamma_0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{5.3}$$

we obtain

$$\begin{pmatrix}\partial_t & 0\\ 0 & \partial_t\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} + \begin{pmatrix}\partial_x & m\\ -m & -\partial_x\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} + 2i\lambda\begin{pmatrix}\Psi_1|\Psi_2|^2\\ |\Psi_1|^2\Psi_2\end{pmatrix} = 0. \tag{5.4}$$

Set now

$$Z_i = \mathrm{Re}\,\Psi_i, \qquad Z_{i+2} = \mathrm{Im}\,\Psi_i, \qquad i = 1, 2, \tag{5.5}$$

$$z_i = \mathrm{Re}\,\Psi_{0i}, \qquad z_{i+2} = \mathrm{Im}\,\Psi_{0i}, \qquad i = 1, 2. \tag{5.6}$$

Using Eqs (5.5) and (5.6) we can write now the wave equation (5.1) in the standard form (4.3)

$$\frac{\partial Z_i(t, x)}{\partial t} = \vartheta_i(\lambda, Z, Z^{(1)}), \tag{5.7}$$

where

$$\vartheta_1(\lambda, Z, Z^{(1)}) = -mZ_2 + 2\lambda Z_3(Z_2^2 + Z_4^2) - Z_{1,x}, \tag{5.8}$$

$$\vartheta_2(\lambda, Z, Z^{(1)}) = mZ_1 + 2\lambda Z_4(Z_1^2 + Z_3^2) + Z_{2,x}, \tag{5.9}$$

$$\vartheta_3(\lambda, Z, Z^{(1)}) = -mZ_4 - 2\lambda Z_1(Z_2^2 + Z_4^2) - Z_{3,x}, \tag{5.10}$$

$$\vartheta_4(\lambda, Z, Z^{(1)}) = mZ_3 - 2\lambda Z_2(Z_1^2 + Z_3^2) + Z_{4,x}. \tag{5.11}$$

We see that all functions $\vartheta_i(\lambda, Z, Z^{(1)})$, $i = 1, \ldots, 4$, are entire functions of their arguments. Therefore, by virtue of Theorem 4.1 the solution of Eq. (5.4) has the form

$$\Psi_k(t, x) = e^{(t-t_0)D}(z_k + iz_{k+2}), \quad k = 1, 2, \tag{5.12}$$

where

$$D(\lambda, z, z^{(1)}) = \vartheta_i(\lambda, z, z^{(1)}) \frac{\partial}{\partial z_i}. \tag{5.13}$$

The solution (5.12) is an analytic function of t and x at least in the region

$$|x| < r\varrho, \quad |t - t_0| < T = \frac{(1-r)^2 \varrho}{1+32N}, \tag{5.14}$$

where $0 < r < 1$ and

$$N = \max\{\varrho, m\varrho, 2\lambda\varrho^3\}. \tag{5.15}$$

By virtue of Remark 4.1 the solution (5.12) is the analytic function of coupling constant λ at least for $|\lambda| < 1$.

6. *Heisenberg nonlinear spinor field theory*

The Heisenberg nonlinear spinor wave equation in fourdimensional space-time has the form (*cf.* [13], Ch. III)

$$i\sigma^\nu \frac{\partial \chi(x)}{\partial x^\nu} + l^2 \sigma^\nu \chi(x) (\chi^*(x) \sigma_\nu \chi(x)) = 0, \tag{6.1}$$

where $\boldsymbol{\sigma}^\nu = (I, \boldsymbol{\sigma})$ are conventional Pauli matrices, l^2 is a coupling constant of the dimension of a length and $\chi(x)$ is the two-dimensional Weyl spinor

$$\chi(x) = \begin{pmatrix} \chi_{1\beta}(x) \\ \chi_{2\beta}(x) \end{pmatrix}. \tag{6.2}$$

The first index of $\chi_{\alpha,\beta}(x)$ refers to Lorentz space and the second one to the isotopic space. Because the structure of the equation (6.1) is the same for all isotopic components of $\chi(x)$ we shall not write in what follows the isotopic indices explicitly.

We assume that the spinors $\chi(t, \boldsymbol{x})$ satisfy the following initial conditions

$$\chi(t, \boldsymbol{x})|_{t=t_0} = \begin{pmatrix} \chi_{01}(\boldsymbol{x}) \\ \chi_{02}(\boldsymbol{x}) \end{pmatrix}, \tag{6.3}$$

where $\chi_{0i}(\boldsymbol{x})$, $i = 1, 2$ are analytic functions of $\boldsymbol{x}$ for $|x_k| < \varrho$, $k = 1, 2, 3$.

Set now

$$Z_i(t, \boldsymbol{x}) = \operatorname{Re} \chi_i(t, \boldsymbol{x}), \quad Z_{i+2}(t, \boldsymbol{x}) = \operatorname{Im} \chi_i(t, \boldsymbol{x}), \quad i = 1, 2, \tag{6.4}$$

$$z_i(\boldsymbol{x}) = \operatorname{Re} \chi_{0i}(\boldsymbol{x}), \quad z_{i+2}(\boldsymbol{x}) = \operatorname{Im} \chi_{0i}(\boldsymbol{x}), \quad i = 1, 2. \tag{6.5}$$

Using Eqs (6.4) and (6.5) we can write Eq. (6.1) in the standard form (4.3)

$$\frac{\partial Z_i(t, x)}{\partial t} = \vartheta_i(\lambda, Z, Z^{(1)}), \tag{6.6}$$

where

$$\vartheta_1(\lambda, Z, Z^{(1)}) = -\partial_{x^1}Z_2-\partial_{x^2}Z_4-\partial_{x^3}Z_1+l^2\{-AZ_3+CZ_4+DZ_2+BZ_3),$$

$$\vartheta_2(\lambda, Z, Z^{(1)}) = -\partial_{x^1}Z_1+\partial_{x^2}Z_3+\partial_{x^3}Z_2+l^2\{-AZ_4+CZ_3-DZ_1-BZ_4),$$

$$\vartheta_3(\lambda, Z, Z^{(1)}) = -\partial_{x^1}Z_4+\partial_{x^2}Z_2+\partial_{x^3}Z_3+l^2\{AZ_1-CZ_2+DZ_4-BZ_1),$$

$$\vartheta_4(\lambda, Z, Z^{(1)}) = -\partial_{x^1}Z_3-\partial_{x^2}Z_1+\partial_{x^3}Z_4+l^2\{AZ_2-CZ_1-DZ_3+BZ_2), \tag{6.7}$$

and

$$A = Z_1^2+Z_2^2+Z_3^2+Z_4^2, \quad B = Z_1^2-Z_2^2+Z_3^2-Z_4^2,$$

$$C = 2(Z_1Z_2-Z_3Z_4), \quad D = 2(Z_2Z_3-Z_1Z_4).$$

We see that all functions $\vartheta_i(\lambda, Z, Z^{(1)})$, $i = 1, 2, 3, 4$ are entire functions of their arguments. Therefore, by virtue of Theorem 4.1 the solution of Eq. (6.1) has the form

$$\chi_k(t, x) = e^{(t-t_0)D}(z_k+iz_{k+2}), \tag{6.8}$$

where

$$D(z, z^{(1)}) = \vartheta_i(z, z^{(1)})\frac{\partial}{\partial z_i}. \tag{6.9}$$

The solution (6.8) is analytic function of t and x at least in the region

$$|x| < r\varrho, \quad |t-t_0| < T = \frac{(1-r)^2\varrho}{1+96NM}, \tag{6.10}$$

where $|x| = \sum_{k=1}^{3} |x_k|$, $0 < r < 1$, $N = \max\{\varrho, 2l^2\varrho^3\}$ and M is the dimension of isotopic space. The solution (6.8) is the analytic function of coupling constant l^2 at least for $|l^2| < 1$.

7. *Classical $\lambda\Phi^4$ theory*

Let $\Phi(t, x)$ be a wave function for a scalar particle satisfying in two-dimensional space-time the following nonlinear relativistic wave equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2\right)\Phi(t, x) = -\lambda\Phi^3(t, x) \tag{7.1}$$

with the initial conditions

$$\Phi(t, x)|_{t=t_0} = z_1(x), \quad \dot{\Phi}(t, x)|_{t=t_0} = z_2(x), \tag{7.2}$$

which are analytic functions of x.

Setting

$$\Phi(t, x) = Z_1(t, x),\ \dot{\Phi}(t, x) = Z_2(t, x), \tag{7.3}$$

we obtain

$$\frac{\partial Z_1}{\partial t} = Z_2,$$

$$\frac{\partial Z_2}{\partial t} = -m^2 Z_1 - \lambda Z_1^3 + Z_{1,xx}. \tag{7.4}$$

Thus in notation of Section 4 we have

$$\vartheta_1(Z, Z^{(2)}) = Z_2, \quad \vartheta_2(Z, Z^{(2)}) = -m^2 Z_1 - \lambda Z_1^3 + Z_{1,xx}. \tag{7.5}$$

We see that the functions $\vartheta_i(Z, Z^{(2)})$ are analytic functions of their arguments. Thus, using Theorem 4.2 we find that the solution of Eq. (7.1) satisfying the analytic initial conditions (7.2) has the form

$$\Phi(t, x) = e^{(t-t_0)D} z_1, \tag{7.6}$$

where

$$D(z, z^{(2)}) = \sum_{k=1}^{2} \vartheta_k(z, z^{(2)}) \frac{\partial}{\partial z_k}. \tag{7.7}$$

To calculate the region of analyticity of solution (7.6) we pass to the canonical system of first order equations (4.3).

Using the variables (4.22)

$$Z_0 = \partial_t^2 Z, \quad Z_1 = Z, \quad Z_2 = \partial_t Z,$$

$$Z_3 = \partial_x Z_1, \quad Z_4 = \partial_x Z_2, \quad Z_5 = \partial_x^2 Z_1, \tag{7.8}$$

we cast Eq. (7.1) into the following equivalent system of the first order equations

$$\frac{\partial Z_0}{\partial t} = (-m^2 - 3\lambda Z_1^2) Z_2 + \partial_x Z_4,$$

$$\frac{\partial Z_1}{\partial t} = Z_2, \quad \frac{\partial Z_2}{\partial t} = -m^2 Z_1 - \lambda Z_1^3 + Z_5,$$

$$\frac{\partial Z_3}{\partial t} = \partial_x Z_2, \quad \frac{\partial Z_4}{\partial t} = \partial_x Z_0, \quad \frac{\partial Z_5}{\partial t} = \partial_x Z_4. \tag{7.9}$$

This system has the initial conditions given by (4.25). We see that all functions $\vartheta_i(\lambda, Z, Z^{(1)})$ defined by r. h. s. of Eq. (7.9) are entire functions of their arguments. Consequently the condition 4 (i) is satisfied at least for ϱ equal to the radius of analyticity of initial conditions (4.25).

Using now Theorem 4.2 we conclude that the solution (7.6) of Eq. (7.1) is analytic function at least in the region Ω

$$|x| < r\varrho, \qquad |t-t_0| < T = \frac{(1-r)^2\varrho}{1+48N}, \tag{7.10}$$

where $0 < r < 1$ and

$$N = \max\{\varrho, m^2\varrho, 3\lambda\varrho^3\}. \tag{7.11}$$

By virtue of Remark 4.1 the solution (7.5) is the analytic function of the coupling constant λ at least for $|\lambda| < 1$. In fourdimensional space-time the nonlinear wave equation has the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_2^2} + m^2\right)\Phi(t, x) = -\lambda\Phi^3(t, x). \tag{7.12}$$

The solution of this equation satisfying the analytic initial conditions (7.2) is also given by formula (7.6) and (7.7) with

$$\vartheta_1(Z, Z^{(2)}) = Z_2,$$

$$\vartheta_2(Z, Z^{(2)}) = -m^2Z_1 - \lambda Z_1^3 + \partial_{x_1}^2 Z_1 + \partial_{x_2}^2 Z_1 + \partial_{x_3}^2 Z_1. \tag{7.13}$$

It is interesting that there are special solutions of Eq. (7.12) which can be expressed in terms of known special functions. For instance, for the plane-wave-like solutions of the form

$$\Phi(t, x) = \varphi(px), \qquad px = p_\mu x^\mu \equiv \sigma, \qquad p^2 = m^2, \tag{7.14}$$

Eq. (7.12) reduces to the following ordinary differential equation

$$\frac{d^2\varphi}{d\sigma^2} + \varphi + (\lambda/m^2)\varphi^3 = 0. \tag{7.15}$$

Multiplying Eq. (7.11) by $d\varphi/d\sigma$ we obtain

$$\frac{d}{d\sigma}\left\{\frac{1}{2}\left(\frac{d\varphi}{d\sigma}\right)^2 + \frac{1}{2}\varphi^2 + (\lambda/4m^2)\varphi^4\right\} = 0. \tag{7.16}$$

Consequently

$$\frac{d\varphi}{d\sigma} = \sqrt{2E - \varphi^2 - (\lambda/2m^2)\varphi^4} \tag{7.17}$$

or

$$\int \frac{d\varphi}{\sqrt{2E - \varphi^2 - (\lambda/2m^2)\varphi^4}} = \sigma - \sigma_0. \tag{7.18}$$

The l. h. s. is the integral which represents the inverse function to the Jacobi elliptic function cn (x, k). Thus

$$\Phi(x) = \varphi(px) = A\,\mathrm{cn}\,(\Omega px + B), \tag{7.19}$$

where the constants A, B, Ω and elliptic modulus k depend on initial conditions (7.2). Solution (7.15) provides a representation for the creation and annihilation operators in the $\lambda\Phi^4(x)$ theory of quantum scalar fields (*cf.* [1, a], § VI).

8. *Discussion*

1. It is well known that the number of nonlinear differential equations, which may be expressed in terms of known functions, is very limited (*cf. e. g.*, [14]). Consequently, in case of an arbitrary nonlinearity one may expect at most a power series form of solutions. Thus the forms (3.8), (4.26) and (4.29) are representative of the general result which might be achieved in treating problems.

2. It should be stressed that the theory of Lie series is well suited for a calculation of approximate solutions of nonlinear ordinary or partial D. E. To illustrate this we give the main approximation theorem for Lie series ([4, a], I.3).

Theorem 1. Let the fuctions $\vartheta_i(z_1, \ldots, z_n)$ in the operator

$$D(z) = \vartheta_k(z)\frac{\partial}{\partial z_k} \tag{8.1}$$

be all holomorphic in the same n-circle $|z_j - a_j| \leqslant \varrho_j$ and let

$$|\vartheta_i(z)| \leqslant N_i \quad \text{for} \quad |z_j - a_j| = \varrho_j, \quad i, j = 1, \ldots, n. \tag{8.2}$$

Then if the Lie series

$$Z_i(t, z) = e^{tD} z_i|_{z_j = a_j} \tag{8.3}$$

are broken off after the m-th term then the modulus of resulting error is below the following bound

$$R_m = \left|\sum_{\nu=m}^{\infty} \frac{t^\nu}{\nu!}(D^\nu z_i)_{z_j = a_j}\right| < \frac{[(n+1)N|t|]^m}{(n+1)m\varrho^{m-2}[\varrho - (n+1)N|t|]}, \quad m = 2, 3, \ldots \tag{8.4}$$

where

$$\varrho = \min\{\varrho_j\}_{j=1}^n, \qquad N = \max\{N_i\}_{i=1}^n. \tag{8.5}$$

In order that the error $R_m < \varepsilon\varrho$, $0 < \varepsilon < 1$ it is sufficient to restrict t by

$$|t| < \frac{\varrho}{(n+1)N} \cdot \frac{\varepsilon_1}{1+\varepsilon_1}, \qquad \varepsilon_1 = \sqrt[m]{\varepsilon m(n+1)}\,. \tag{8.6}$$

This theorem might be very useful in the analysis of properties of approximate solutions in various problems of applied science and technology.

It should also be mentioned that there are elaborated special effective convergence-improving methods for the Lie series ([4, b]).

3. Recently various types of nonpolynomial Lagrangian field theories with interaction Lagrangian of the type (see *e. g.* [15])

$$L_{\text{int}} = \frac{1}{1+\lambda\Phi^2} \tag{8.7}$$

or, so-called chiral $SU(2)\times SU(2)$ Lagrangian

$$L_{\text{int}} = \frac{(\partial_\mu\Phi)(\partial^\mu\Phi)}{(1+\lambda\Phi^2)^2} \tag{8.8}$$

have been considered.
It is evident that equations of motion derived from Lagrangian (8.7) or (8.8) are second order P. D. E. of the form (4.18) with analytic coefficients $\vartheta(\Phi, \Phi^{(1)}, \Phi^{(2)})$. Thus, using Theorem 4.2 we may obtain a solution of corresponding equations of motion in the form (4.29) with a region of analyticity given by Eq. (4.28).

The nonlinear coupled spinor field equations *e. g.* of Federbush type [16]

$$i\gamma^\mu\partial_\mu\psi - m_1\psi - \lambda(\bar{\varphi}\gamma^\mu\varphi)\gamma_\mu\psi = 0, \tag{8.9}$$

$$i\gamma^\mu\partial_\mu\varphi - m_2\varphi - \lambda(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\varphi = 0, \tag{8.10}$$

may be reduced using formula (5.5) to the system of first order P. D. E. of the form (4.3), with entire coefficients $\vartheta_i(Z, Z^{(1)})$. The solution of the coupled system (8.9) and (8.10) will be given by Eq. (4.8) with the region of analyticity given by Eq. (4.10).

It is evident that the present approach provides a class of analytic solutions for any Lagrangian $L_{\text{int}} = L_1(\Phi)$ or $L_2(\Psi, \bar{\Psi})$ where L_1 and L_2 are analytic functions of their arguments.

4. We gave in our works [1] and [17] a method of a reduction of nonlinear P. D. equations for quantum fields to the corresponding nonlinear P. D. equations for the c-number generalized Fourier transforms; the present method, by providing explicit solutions for classical equations, opens new possibilities for the analysis of extensive classes of relativistic models in Quantum Field Theory.

5. Notice that the present method also provides the solution of Eqs (3.1) and (4.1) in cases when the functions $\vartheta_{\mu i}$ on the r. h. s. of corresponding equations depend explicitly on the variable t. In this case the generator $D_0(\boldsymbol{x}, z, z^{(1)}, z^{(2)})$ should be replaced by

$$D_0(t_0, \boldsymbol{x}, z, z^{(1)}, z^{(2)}) = \partial_{t_0} + \vartheta_{0i}(t_0, \boldsymbol{x}, z, z^{(1)}, z^{(2)})\frac{\partial}{\partial z_i}.$$

6. The analysis of solutions of nonlinear P. D. E. (3.1) and (4.1) shows that in general the region of analyticity of solutions $Z_i(t, \boldsymbol{x})$ in $(t, \boldsymbol{x})$ space is much smaller than a region of analyticity of functions $\vartheta_i(\boldsymbol{x}, Z, Z^{(1)}, Z^{(2)})$ and $z_i(\boldsymbol{x})$, $i = 1, \ldots, n$.

The following simple example well illustrate this general phenomenon. Let

$$\frac{dZ(t)}{dt} = \vartheta(Z) = 1+Z^2 \tag{8.11}$$

with

$$Z(0) = z.$$

The solution of Eq. (8.11) is given by

$$Z(t) = e^{t(1+z^2)d/dz}z = \text{tg}\,(t+\gamma), \qquad \gamma = \text{arctg}\, z. \tag{8.12}$$

Now although $\vartheta(Z)$ and z are entire functions, the radius of convergence of solution (8.12) might be arbitrarily small by taking $\gamma = \pi/2-\varepsilon$, ε arbitrarily small. Thus in general a large radius of convergence of functions $\vartheta_i(x, Z, Z^{(1)}, Z^{(2)})$ and $z_i(x)$ does not imply a large radius of analyticity of a solution $Z_i(t, x)$.

7. It is interesting that we may cast every nonlinear relativistic wave equation considered in Sec. 5, 6 and 7 into the equation of a type (3.1), if we restrict ourselves to a certain subclass of solutions. For instance, let us look for the particular solutions in $\lambda\Phi^4$ theory for which the Lagrangian

$$L = \frac{1}{2}\Phi'^{\mu}\Phi_{,\mu} - \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4$$

is invariant with respect to the scale transformation

$$x^{\mu} \to x'^{\mu} = \xi x^{\mu}, \qquad \Phi(x) \to \Phi'(x') = \xi\Phi(x),$$

$$m \to m' = \xi^{-1}m, \qquad \lambda \to \lambda' = \xi^{-4}\lambda.$$

This invariance of Lagrangian implies the existence of a new integral of motion. Using this additional relation between $\Phi(x)$ and $\Phi_{,\mu}(x)$ and the wave equation (7.1), we may reduce our problem to a system of partial differential equations of the form (3.1).

This method might be used also for other nonlinear relativistic wave equations.

8. Recently there were obtained very interesting global results for particular types of relativistic wave equation by Chadam [18] and Morawetz and Strauss [19]. It was proven in those works an existence of global solutions of equations like (5.1), (7.1), (8.9) and (8.10). In addition, it was shown the existence of nontrivial unitary classical scattering operator $\hat{S}$ [19]. All these results are in a sense existence type theorems: hence the present method which is local, but constructive provides useful complementary information about solutions of nonlinear relativistic wave equations.

I would like to thank Professors A. O. Barut, W. E. Brittin, J. Dreitlein, M. Flato, L. Girardello, J. Simon, and W. Wyss for interesting discussions and suggestions. The author is grateful to Professors A. O. Barut and W. E. Brittin for the hospitality at the University of Colorado, and to the National Science Foundation for the financial support.

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