# DETERMINATION OF STATISTICAL TENSORS IN SEQUENTIAL DECAYS

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It is shown how to determine statistical tensors from the angular distributions in the decays where one of the decay products decays in turn. All tensor components may be estimated by this method, even the components of tensors $T^L_M$ with odd $L$ (*e. g.* Im $\varrho_{10}$ for spin 1 particles) which cannot be measured by the usual method of moments. In addition, the decay amplitudes can be found if there are more than one coupling constants, like for spin 2 particles. Examples of decays of particles with the following spins are discussed: $1^+ \to 0^+ + (1^- \to 0+0)$, $2^- \to 0^- + (1^- \to 0+0)$, $\frac{3}{2}^- \to 0^- + (\frac{3}{2}^+ \to \frac{1}{2}+0)$, $2^- \to 0^- + (2^+ \to 0+0)$ and $1^+ \to 0^- + (2^+ \to 0+0)$.

## 1. Introduction

The polarization and alignment of particles produced in high-energy collisions can be conveniently described in terms of statistical tensors (for a review and a list of references *cf.* [1]) or equivalently, in terms of density matrices. The simplest way of estimating the statistical tensors is the method of moments: the average values of spherical harmonics over the angular distribution are proportional to the statistical tensor components (*cf.* Ref. [1])

$$\langle Y^L_M(\vartheta, \varphi)\rangle = F(L)T^L_M, \tag{1.1}$$

the coefficients $F(L)$ depending on the decay coupling constants.

In the most common case of decays into particles of spins $\frac{1}{2}+0$ or $0+0$ there is only one decay amplitude, which is eliminated by the normalization condition and coefficient $F(L)$ is then a constant. This enables one to use Eq. (1.1) for direct determination of tensors $T^L_M$ with $L$ even. On the other hand, the coefficients $F(L)$ vanish for $L$ odd, if parity is conserved in the decay. The coefficients $F(L)$ with odd $L$ may be different from zero only if the decaying particle interferes with another particle (or with the background) of opposite parity (*cf.* Ref. [2]). Therefore the components of tensors with odd $J$ are difficult to measure. An example of such a component (in the helicity frame) for spin 1 particle is $T^1_1$, equal to $i\sqrt{2}$ Im $\varrho_{10}$.

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In decays into particles of spins higher than $\frac{1}{2}+0$ or $0+0$ there is more than one coupling constant and coefficients $F(L)$ are *a priori* unknown. If nothing is known about the dynamics of the decay, it is impossible to determine then any tensor using formula (1.1). The only exception is the decay $J^P \to 1^{P_1}+0^{P_2}$ if the parities $P$, $P_1$ and $P_2$ satisfy the relation

$$PP_1P_2 = (-1)^J. \tag{1.2}$$

In this case there is also one decay amplitude. In the other case and for higher spins there are more independent decay amplitudes and $F(L)$ cannot be determined by kinematics. A similar situation is found in many body decays (*cf.* Ref. [3]).

To summarize, the simple method of moments is good for determining even $L$ tensors for resonances decaying into $0+0$, $\frac{1}{2}+0$, or into $1+0$ if Eq. (1.2) is satisfied. More sophisticated methods have to be used to estimate either odd $L$ tensors or any tensors at all for particles decaying into higher spin particles. For instance, the odd $L$ tensors may be evaluated for hyperon decays of the strange baryons (Ref. [4], *cf.* also Ref. [5]). Then an additional piece of information is available — the hyperon polarization. The odd $L$ tensors are then evaluated from the correlation between the baryon decay distribution and the hyperon polarization.

Another possibility of measuring odd $L$ tensors was indicated by Chung [6, 7] for the sequential decays where the additional information is obtained from the decay distribution of a product of the primary decay. The purpose of the present paper is to develop this idea and to show on several examples how it is possible to determine the tensors for any $L$ and the ratios and relative phases of the decay amplitudes from the moments over the sequential decay distributions.

The advantage of this method is that in most cases everything can be measured: even $L$ tensors, ratios of moduli of the decay amplitudes, odd $L$ tensors (up to the sign) and the relative phases of the decay amplitudes (up to the sign). Note that in the decays considered below the simple method of moments (1.1) fails.

In Section 2 we derive the basic equations and in Section 3 we discuss the symmetry properties of the double moments. Section 4 is devoted to the detailed discussion of an example: the decay $1^+ \to 1^-+0^-$ followed by $1^- \to 0+0$. All the important features of the method are illustrated here. Formulae for four other processes are given in the Appendix.

## 2. *Moments of the sequential decay distribution*

We consider here the sequential decays of particles with the following spins (and helicities)

$$J \to s+0, \tag{2.1}$$

$$s \to 0+0 \tag{2.2}$$

or

$$s \to \tfrac{1}{2}+0. \tag{2.3}$$

Here $J$ denotes the spin of the primary particle and $s$ denotes the spin of one of its decay products. Spin projections (helicities) will be denoted by $\Lambda$ for particle $J$ and by $\lambda$ for particle $s$. Further, helicity of the decay product of spin $\frac{1}{2}$ in Eq. (2.3) will be $\mu$. Of course, $\mu = 0$ in the case of decay (2.2). The angular decay distribution can be expressed by the density matrices of particle $J$ and the coupling constants $F^J_\lambda$ and $F^s_\mu$ of the decay (2.1) and (2.2) or (2.3)

$$W(\Theta, \Phi, \vartheta, \varphi) = \frac{(2J+1)(2s+1)}{16\pi^2} \sum_{\lambda\lambda'\Lambda\Lambda'\mu} F^J_\lambda (F^J_{\lambda'})^* \varrho_{\Lambda\Lambda'} \times$$

$$\times D^J(\Phi, \Theta, 0)^*_{\Lambda\lambda} D^J(\Phi, \Theta, 0)_{\Lambda'\lambda'} D^s(\varphi, \vartheta, 0)^*_{\lambda\mu} D^s(\varphi, \vartheta, 0)_{\lambda'\mu}. \tag{2.4}$$

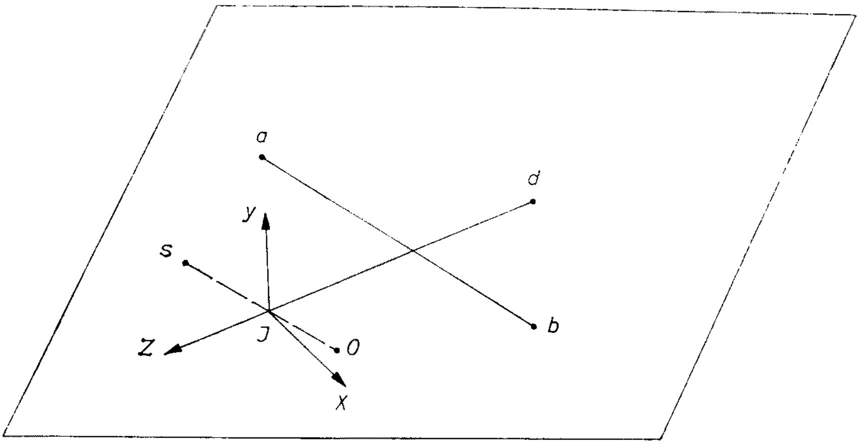


Fig. 1. The helicity decay frame of particle $J$ produced in the process $a+b \to J+d$, followed by $J \to s+0$. The $Z$ axis is directed opposite to the momentum of the c. m. system in the $J$ rest frame. The $Y$ axis is perpendicular to the production plane, *i. e.* $\boldsymbol{e}_Y \parallel \boldsymbol{p}_{\text{cms}} \times \boldsymbol{p}_a$ in the $J$ rest system

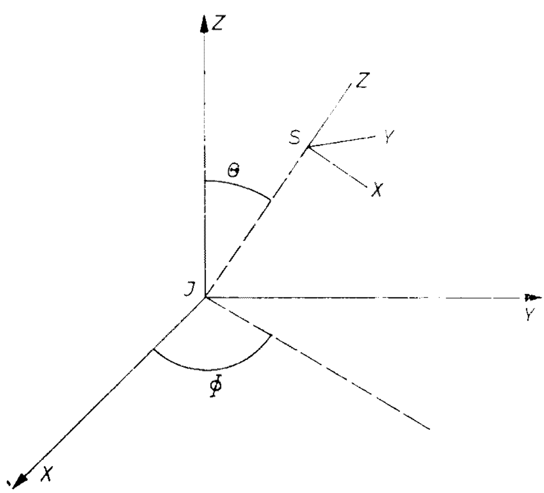


Fig. 2. The helicity decay frames of particles $J$ and $s$. The axes $Z$, $z$ and $x$ lie in the same plane. Angles $\Theta$, $\Phi$ define the orientation of the momentum of particle $s$ in the $XYZ$ frame. Angles $\vartheta$, $\varphi$ (not indicated in the figure) define the orientation of the momentum of one of the decay products of $s$ in the $xyz$ frame

The angles $\Theta$, $\Phi$ of the decay (2.1) and the angles $\vartheta$, $\varphi$ of the decay (2.2) or (2.3) are defined in Figs 1 and 2. The coefficient in front of Eq. (2.4) has been inserted in order to have

$$\int W(\Theta, \Phi, \vartheta, \varphi)\, d\cos\Theta d\Phi d\cos\vartheta\, d\varphi = 1. \tag{2.5}$$

Our notation and conventions are similar to those in Ref. [7]. The helicity coupling constants $F$ are normalized to unity

$$\sum_{\lambda} |F^J_{\lambda}|^2 = \sum_{\mu} |F^s_{\mu}|^2 = 1. \tag{2.6}$$

Hence it follows that $|F^s_{\mu}| = 1$ for process (2.2) and $|F^s_{\mu}| = 1/\sqrt{2}$ for process (2.3). The coupling constants are related to the reduced helicity decay amplitudes $M_J(\lambda, 0)$ of Ref. [8] in the following way

$$F^J_{\lambda} = M_J(\lambda, 0)/(\sum_{\lambda} |M_J(\lambda, 0)|^2)^{1/2}. \tag{2.7}$$

They are also related to the coupling constants in the $LS$ basis by the following equation

$$F^J_{\lambda} = \sum_{LS} \sqrt{\frac{2L+1}{2J+1}}\, C(L, 0; s, \lambda | J, \lambda) F^J_{LS}. \tag{2.8}$$

Formula (2.4) can be simplified by replacing the density matrix with the statistical tensors (*cf.* Ref. [1])

$$\varrho_{\Lambda\Lambda'} = \sum_{LM} (-1)^{J+\Lambda-L} C(J, -\Lambda; J, \Lambda' | L, M) T^L_M \tag{2.9}$$

and by coupling the $D$-functions with the Clebsch-Gordan coefficients. Then it is straightforward to calculate the double moments

$$H(l, m; L, M) = \langle D^l_{m0}(\varphi, \vartheta, 0) D^L_{Mm}(\Phi, \Theta, 0)\rangle \equiv$$
$$\equiv \int D^l_{m0}(\varphi, \vartheta, 0) D^L_{Mm}(\Phi, \Theta, 0) W(\Theta, \Phi, \vartheta, \varphi) d\cos\Theta d\Phi d\cos\vartheta d\varphi. \tag{2.10}$$

The result is

$$H(l, m; L, M) = \sqrt{\frac{2J+1}{2L+1}}\, H^l_m(L) H^s(l) (T^L_M)^*, \tag{2.11}$$

TABLE I

Coefficients $H^s(l)$ for a few decays

| Process | $l = 0$ | $l = 2$ | $l = 4$ |
|---|---|---|---|
| $\frac{1}{2} \to \frac{1}{2}+0$ | 1 | — | — |
| $1 \to 0+0$ | 1 | $-\sqrt{\frac{2}{5}}$ | — |
| $\frac{3}{2} \to \frac{1}{2}+0$ | 1 | $-\frac{1}{\sqrt{5}}$ | — |
| $2 \to 0+0$ | 1 | $-\sqrt{\frac{2}{7}}$ | $\sqrt{\frac{2}{7}}$ |

where

$$H^l_m(L) = \sum_{\lambda\lambda'} F^J_\lambda F^{J*}_{\lambda'} C(J, \lambda'; L, m|J, \lambda) C(s, \lambda'; l, m|s, \lambda) \tag{2.12}$$

and

$$H^s(l) = \sum_\mu c(s, \mu; l, 0|s, \mu), \tag{2.13}$$

where $\mu = 0$ for reaction (2.2) and $\mu = \pm\frac{1}{2}$ for reaction (2.3). Coefficients $H^s(l)$ are different from zero for $l$ even and not exceeding $2s$ (see Table I).

### *3. Symmetry properties of the moments*

We quote here the most important symmetry properties of the moments $H(l, m; L, M)$ defined by Eq. (2.10) (*cf.* also Ref. [7]).

From the hermiticity of the density matrix $\varrho_{\Lambda\Lambda'}$ it follows that

$$H(l, m; L, M) = (-1)^M H(l, -m; L, -M)^*. \tag{3.1}$$

If particle $J$ was produced in a collision of two unpolarized particles and if $M$ is the spin projection on an axis lying in the production plane (*e. g.* helicity of Gottfried-Jackson axis)

$$H(l, m; L, M) = (-1)^{L-M} H(l, m; L, -M). \tag{3.2}$$

This follows from the relation

$$T^L_{-M} = (-1)^{L-M} T^L_M \tag{3.3}$$

which holds under these assumptions (*cf.* Ref. [1]). If, on the other hand, $M$ is the projection on an axis perpendicular to the production plane (a transversity type frame), we have then instead of Eq. (3.2)

$$H(l, m; L, M) = 0 \text{ for } M \text{ odd}. \tag{3.4}$$

Finally, parity conservation in the decay $J \to s+0$ gives

$$F^J_\lambda = P_s P_0 P_J (-1)^{J-s_a} F^J_{-\lambda}, \tag{3.5}$$

where $P$'s are intrinsic parities. Hence we get

$$H(l, m; L, M) = (-1)^{L+l} H(l, -m; L, M). \tag{3.6}$$

From Eqs (3.2) and (3.6) we see that only the moments with $m \geqslant 0$ and $M \geqslant 0$ are relevant in the helicity-type frames, and that only the moments with $m \geqslant 0$ and $M$ even are relevant in the transversity-type frames. Furthermore, for $m = 0$ the non-vanishing moments must have $l+L$ odd.

### 4. *Example: decay* $1^+ \to 1^- + 0^-$ *followed by* $1^- \to 0+0$

From Eqs (2.10)–(2.12) we obtain the following non-vanishing moments with $m \geqslant 0$

$$H(2,0;0,0) = \tfrac{2}{5}(1-3|F_1|^2), \tag{4.1}$$

$$H(2,1;1,M) = -\frac{\sqrt{6}}{5}\, i\,\mathrm{Im}\,(F_1F_0^*)\,(T_M^1)^*, \tag{4.2}$$

$$H(0,0;2,M) = \frac{\sqrt{6}}{5}(3|F_1|^2-1)\,(T_M^2)^*, \tag{4.3}$$

$$H(2,0;2,M) = \frac{\sqrt{6}}{25}(3|F_1|^2-2)\,(T_M^2)^*, \tag{4.4}$$

$$H(2,1;2,M) = -\frac{3\sqrt{6}}{25}\,\mathrm{Re}\,(F_1F_0^*)\,(T_M^2)^*, \tag{4.5}$$

$$H(2,2;2,M) = -\frac{3\sqrt{6}}{25}\,|F_1|^2(T_M^2)^*. \tag{4.6}$$

These relations are valid both in the helicity-type and in the transversity-type frames. The only difference is that in the helicity-type frames we are interested in moments with $M \geqslant 0$ and in the transversity-type frames we are interested in moments with $M$ even.

The coupling constants satisfy here the relations

$$2|F_1|^2+|F_0|^2 = 1, \tag{4.7}$$

$$F_1 = -F_{-1}^*. \tag{4.8}$$

They can be related to the coupling constants for the $S$-wave and $D$-wave decays by the following equations (see (2.8))

$$F_1 = \frac{1}{\sqrt{3}}F_S + \frac{1}{\sqrt{6}}F_D, \tag{4.9}$$

$$F_0 = \frac{1}{\sqrt{3}}F_S - \sqrt{\frac{2}{3}}F_D. \tag{4.10}$$

It is easy to see that the above relations enable us to find all the quantities on the right-hand sides of Eqs (4.1)–(4.6) except for the signs of Im $(F_1F_0^*)$ and of $T_M^1$ (*cf.* Ref. [7]). Thus Eqs (4.1) and (4.7) may be used to find $|F_1|$ and $|F_0|$, then Eqs (4.3), (4.4) and (4.6) determine $T_M^2$. Then Eq. (4.5) gives us Re $F_1F_0^*$ and $|\mathrm{Im}\, F_1F_0^*|$ may be found from the relation

$$(\mathrm{Im}\, F_1F_0^*)^2 = |F_1|^2|F_0|^2-(\mathrm{Re}\, F_1F_0^*)^2. \tag{4.11}$$

Finally, $T_M^1$ may be determined (up to the sign) from Eq. (4.2).

Therefore, everything may be found except for the sign of $T_M^1$ and the relative sign of the helicity decay amplitudes, provided $T_M^2$ and Im $F_1F_0^*$ do not vanish for some dynamical reasons.

Note that the simple method of moments is equivalent to evaluating $H(0, 0; s, M)$ alone. As is seen from Eq. (4.3) even the tensor $T_M^2$ cannot be found in this way. This is so because the coefficient $F(L)$ in Eq. (1.1) is in this case proportional to $3|F_1|^2-1$ and because $F_1$ is unknown.

In the method described above the system of equations (4.1)–(4.7) is overdetermined. For instance, $T_M^2$ can be found from Eqs (4.3), (4.4) or (4.6). This makes possible to use the least-squares method to reduce the errors.

In Appendix the formulae relating the statistical tensors to the non-vanishing moments are given for the following processes: $\frac{3}{2}^- \to 0^- + (\frac{3}{2}^+ \to \frac{1}{2} + 0)$, $2^- \to 0^- + (1^- \to 0+0)$, $1^+ \to 0^- + (2^+ \to 0+0)$ and $2^- \to 0^- + (2^+ \to 0+0)$. For the process $\frac{1}{2}^+ \to 0^- + (\frac{3}{2}^+ \to \frac{1}{2}+0)$ the only non-trivial moment is

$$H(2, 0; 0, 0) = \tfrac{1}{5}. \tag{4.12}$$

The method fails in this case since $H(2, 1; 1, 1) = 0$ because of the "accidental" vanishing of the Clebsch-Gordan coefficient $C(\frac{3}{2}, -\frac{1}{2}; 2, 1 \mid \frac{3}{2}, \frac{1}{2})$. However, Eq. (4.12) may be used to check the spin-parity assignments of the particles.

## 5. *Discussion*

We have shown how it is possible to measure the tensors for any $L$ and the decay coupling constants from the double moments in sequential decays. We stress again that the usual method of moments gives in these cases only the even $L$ tensors up to an unknown factor $F(L)$.

A possible difficulty in the practical application of the method presented here may be the correct spin-parity assignment of the particles and decay products, as the method relies on this assignment. However, in most cases the system of equations giving $T_M^L$ and $F_\lambda$ is overdetermined and the extra equations may be used to check the spin-parity assignments or to reduce the errors.

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## APPENDIX

### *Further examples of the double moments in sequential decays*

Only the non-vanishing moments are quoted below.

a. Decay $\frac{3}{2}^- \to \frac{3}{2}^+ + 0$ followed by $\frac{3}{2}^+ \to \frac{1}{2}+0$

$$H(2, 0; 0, 0) = \tfrac{1}{5}(1-4|F_3|^2), \tag{1a.1}$$

$$H(2, 1; 1, M) = -\frac{8}{5\sqrt{15}}\, i \,\mathrm{Im}\,(F_3F_1^*)\,(T_M^1)^*, \tag{1a.2}$$

$$H(0,0;2,M) = \tfrac{2}{5}(4|F_3|^2-1)(T^2_M)^*, \tag{1a.3}$$

$$H(2,0;2,M) = -\tfrac{2}{25}(T^2_M)^*, \tag{1a.4}$$

$$H(2,1;2,M) = H(2,2;2,M) = -\tfrac{8}{25}\,\mathrm{Re}\,(F_1F_3^*)(T^2_M)^*, \tag{1a.5}$$

$$H(2,1;3,M) = -\tfrac{8}{35}\sqrt{\tfrac{2}{5}}\,i\,\mathrm{Im}\,(F_3F_1^*)(T^3_M)^*, \tag{1a.6}$$

$$H(2,2;3,M) = \tfrac{8}{35}\,i\,\mathrm{Im}\,(F_3F_1^*)(T^3_M)^*. \tag{1a.7}$$

The coupling constants satisfy the relation

$$|F_1|^2+|F_3|^2 = \tfrac{1}{2} \tag{1a.8}$$

and can be expressed by the $S$ and $D$ wave decay constants

$$F_3 = \tfrac{1}{2}(F_S+F_D), \tag{1a.9}$$

$$F_1 = \tfrac{1}{2}(F_S-F_D). \tag{1a.10}$$

b. Decay $2^- \to 1^-+0^-$ followed by $1^- \to 0+0$

$$H(2,0;0,0) = \tfrac{2}{5}(1-3|F_1|^2), \tag{1b.1}$$

$$H(2,1;1,M) = -i\sqrt{\tfrac{2}{5}}\,\mathrm{Im}\,(F_1F_0^*)(T^1_M)^*, \tag{1b.2}$$

$$H(0,0;2,M) = -\sqrt{\tfrac{2}{7}}(1-|F_1|^2)(T^2_M)^*, \tag{1b.3}$$

$$H(2,0;2,M) = -\sqrt{\tfrac{2}{7}}(\tfrac{2}{5}-|F_1|^2)(T^2_M)^*, \tag{1b.4}$$

$$H(2,1;2,M) = -\tfrac{1}{5}\sqrt{\tfrac{6}{7}}\,\mathrm{Re}\,(F_1F_0^*)(T^2_M)^*, \tag{1b.5}$$

$$H(2,2;2,M) = -\tfrac{3}{5}\sqrt{\tfrac{2}{7}}\,|F_1|^2(T^2_M)^*, \tag{1b.6}$$

$$H(2,1;3,M) = -\tfrac{2}{7}\sqrt{\tfrac{3}{5}}\,i\,\mathrm{Im}\,(F_0F_1^*)(T^3_M)^*, \tag{1b.7}$$

$$H(0,0;4,M) = \tfrac{1}{3}\sqrt{\tfrac{10}{7}}(1-\tfrac{10}{3}|F_1|^2)(T^4_M)^*, \tag{1b.8}$$

$$H(2,0;4,M) = \tfrac{2}{3}\sqrt{\tfrac{2}{35}}(1-\tfrac{4}{3}|F_1|^2)(T^4_M)^*, \tag{1b.9}$$

$$H(2,1;4,M) = \frac{2}{3}\frac{1}{\sqrt{7}}\,\mathrm{Re}\,(F_1F_0^*)(T^4_M)^*, \tag{1b.10}$$

$$H(2,2;4,M) = \tfrac{2}{3}\sqrt{\tfrac{2}{21}}\,|F_1|^2(T^4_M)^*. \tag{1b.11}$$

The coupling constants satisfy the relation

$$2|F_1|^2+|F_0|^2 = 1 \tag{1b.12}$$

and can be related to the $P$ and $F$ wave coupling constants

$$F_1 = \sqrt{\frac{3}{10}}F_P+\frac{1}{\sqrt{5}}F_F, \tag{1b.13}$$

$$F_0 = \sqrt{\frac{2}{5}}F_P-\frac{3}{\sqrt{5}}F_F. \tag{1b.14}$$

c. Decay $1^+ \to 2^+ + 0^-$ followed by $2^+ \to 0+0$

$$H(2,0;0,0) = \tfrac{2}{7}(1-|F_1|^2), \tag{1c.1}$$

$$H(4,0;0,0) = \tfrac{2}{7}(1-\tfrac{10}{3}|F_1|^2), \tag{1c.2}$$

$$H(2,1;1,M) = -\frac{\sqrt{2}}{7}\, i\,\mathrm{Im}\,(F_1F_0^*)\,(T_M^1)^*, \tag{1c.3}$$

$$H(4,1;1,M) = -\tfrac{2}{7}\sqrt{\tfrac{5}{3}}\, i\,\mathrm{Im}\,(F_1F_0^*)\,(T_M^1)^*, \tag{1c.4}$$

$$H(0,0;2,M) = \frac{\sqrt{6}}{5}(3|F_1|^2-1)\,(T_M^2)^*, \tag{1c.5}$$

$$H(2,0;2,M) = \frac{\sqrt{6}}{35}(5|F_1|^2-2)\,(T_M^2)^*, \tag{1c.6}$$

$$H(2,1;2,M) = -\frac{3\sqrt{2}}{35}\,\mathrm{Re}\,(F_1F_0^*)\,(T_M^2)^*, \tag{1c.7}$$

$$H(2,2;2,M) = -\frac{3\sqrt{6}}{35}|F_1|^2(T_M^2)^*, \tag{1c.8}$$

$$H(4,0;2,M) = -\frac{2\sqrt{6}}{105}(3-4|F_1|^2)\,(T_M^2)^*, \tag{1c.9}$$

$$H(4,1;2,M) = -\tfrac{2}{7}\sqrt{\tfrac{3}{5}}\,\mathrm{Re}\,(F_1F_0^*)\,(T_M^2)^*, \tag{1c.10}$$

$$H(4,2;2,M) = -\tfrac{2}{7}\sqrt{\tfrac{2}{5}}|F_1|^2(T_M^2)^*. \tag{1c.11}$$

The normalization condition is

$$2|F_1|^2+|F_0|^2 = 1 \tag{1c.12}$$

and the relation with the $S$ and $D$ wave coupling constants are

$$F_1 = -\sqrt{\frac{3}{10}}F_S - \frac{1}{\sqrt{5}}F_D, \tag{1c.13}$$

$$F_0 = -\sqrt{\tfrac{2}{5}}F_S + \sqrt{\tfrac{3}{5}}F_D. \tag{1c.14}$$

d. Decay $2^- \to 2^+ + 0^-$ followed by $2^+ \to 0+0$

$$H(2,0;0,0) = \tfrac{2}{7}(1-4|F_2|^2-|F_1|^2), \tag{1d.1}$$

$$H(4,0;0,0) = \tfrac{2}{21}(3-5|F_2|^2-10|F_1|^2), \tag{1d.2}$$

$$H(2,1;1,M) = -\tfrac{1}{7}\sqrt{\tfrac{10}{3}}\, i\,\mathrm{Im}\,(F_1F_0^*+2F_2F_1^*)\,(T_M^1)^*, \tag{1d.3}$$

$$H(4,1;1,M) = \tfrac{10}{63}\, i\,\mathrm{Im}\,(F_2F_1^*+3F_0F_1^*)\,(T_M^1)^*, \tag{1d.4}$$

$$H(0,0;2,M) = \sqrt{\tfrac{2}{7}}(4|F_2|^2+|F_1|^2-1)\,(T_M^2)^*, \tag{1d.5}$$

$$H(2,0;2,M) = \tfrac{1}{7}\sqrt{\tfrac{2}{7}}(3|F_1|^2-2)\,(T_M^2)^*, \tag{1d.6}$$

$$H(2,1;2,M) = -\tfrac{1}{7}\sqrt{\tfrac{2}{7}}\,\mathrm{Re}\,(6F_2F_1^*+F_1F_0^*)\,(T_M^2)^*, \tag{1d.7}$$

$$H(2,2;2,M) = -\tfrac{1}{7}\sqrt{\tfrac{2}{7}}\,(4\,\mathrm{Re}\,(F_2F_0^*)+3|F_1|^2)\,(T_M^2)^*, \tag{1d.8}$$

$$H(4,0;2,M) = \tfrac{2}{21}\sqrt{\tfrac{2}{7}}\,(7|F_2|^2+8|F_1|^2-3)\,(T_M^2)^*, \tag{1d.9}$$

$$H(4,1;2,M) = \tfrac{2}{7}\sqrt{\tfrac{5}{21}}\,\mathrm{Re}\,(F_1F_2^*-F_0F_1^*)\,(T_M^2)^*, \tag{1d.10}$$

$$H(4,2;2,M) = \tfrac{2}{7}\sqrt{\tfrac{10}{21}}\,(\mathrm{Re}\,F_0F_2^*-|F_1^*|^2)\,(T_M^2)^*, \tag{1d.11}$$

$$H(2,1;3,M) = -\frac{2\sqrt{5}}{49}\,i\,\mathrm{Im}\,(3F_2F_1^*+F_0F_1^*)\,(T_M^3)^*. \tag{1d.12}$$

$$H(2,2;3,M) = -\tfrac{10}{49}\,i\,2\,\mathrm{Im}\,(F_2F_0^*)\,(T_M^3)^*, \tag{1d.13}$$

$$H(4,1;3,M) = \tfrac{5}{21}\sqrt{\tfrac{2}{21}}\,i\,\mathrm{Im}\,(F_2F_1^*+2F_1F_0^*)\,(T_M^3)^*, \tag{1d.14}$$

$$H(4,2;3,M) = \tfrac{5}{49}\sqrt{\tfrac{10}{3}}\,i\,\mathrm{Im}\,(F_2F_0^*)\,(T_M^3)^*, \tag{1d.15}$$

$$H(4,3;3,M) = \tfrac{5}{21}\sqrt{\tfrac{10}{7}}\,i\,\mathrm{Im}\,(F_2F_1)^*(T_M^3)^*, \tag{1d.16}$$

$$H(0,0;4,M) = -\tfrac{1}{9}\sqrt{\tfrac{10}{7}}\,(5|F_2|^2+10|F_1|^2-3)\,(T_M^4)^*, \tag{1d.17}$$

$$H(2,0;4,M) = -\tfrac{2}{63}\sqrt{\tfrac{10}{7}}\,(7|F_2|^2+8|F_1|^2-3)\,(T_M^4)^*, \tag{1d.18}$$

$$H(2,1;4,M) = -\tfrac{10}{21}\,\frac{1}{\sqrt{21}}\,\mathrm{Re}\,(F_2F_1^*-F_1F_0^*)\,(T_M^4)^*, \tag{1d.19}$$

$$H(2,2;4,M) = -\tfrac{10}{21}\sqrt{\tfrac{2}{21}}\,(\mathrm{Re}\,(F_2F_0^*)-|F_1|^2)\,(T_M^4)^*, \tag{1d.20}$$

$$H(4,0;4,M) = \tfrac{1}{189}\sqrt{\tfrac{10}{7}}\,(18-20|F_1|^2-35|F_2|^2)\,(T_M^4)^*, \tag{1d.21}$$

$$H(4,1;4,M) = \tfrac{5}{189}\sqrt{\tfrac{10}{7}}\,\mathrm{Re}\,(6F_1F_0^*+F_2F_1^*)\,(T_M^4)^*, \tag{1d.22}$$

$$H(4,2;4,M) = \tfrac{5}{189}\sqrt{\tfrac{10}{7}}\,(3\,\mathrm{Re}\,(F_2F_0^*)+4|F_1|^2)\,(T_M^4)^*, \tag{1d.23}$$

$$H(4,3;4,M) = \tfrac{5}{27}\sqrt{\tfrac{10}{7}}\,\mathrm{Re}\,(F_1F_2^*)\,(T_M^4)^*, \tag{1d.24}$$

$$H(4,4;4,M) = \tfrac{5}{27}\sqrt{\tfrac{10}{7}}\,|F_2|^2(T_M^4)^*. \tag{1d.25}$$

The normalization condition is

$$2|F_2|^2+2|F_1|^2+|F_0|^2 = 1, \tag{1d.26}$$

and the relations of the helicity coupling constants with the decay constants in the $S$, $D$ and $G$ waves are

$$F_2 = \frac{1}{\sqrt{5}}F_S+\sqrt{\frac{2}{7}}F_D+\frac{1}{\sqrt{70}}F_G, \tag{1d.27}$$

$$F_1 = \frac{1}{\sqrt{5}}F_S-\frac{1}{\sqrt{14}}F_D-\sqrt{\frac{8}{35}}F_G, \tag{1d.28}$$

$$F_0 = \frac{1}{\sqrt{5}}F_S-\sqrt{\frac{2}{7}}F_D+\sqrt{\frac{18}{35}}F_G. \tag{1d.29}$$

## REFERENCES

[1] A. Kotański, K. Zalewski, *Nuclear Phys.*, **B4**, 559 (1968); **B20**, 236 (1970) (E).
[2] A. Kotański, K. Zalewski, *Nuclear Phys.*, **B22**, 317 (1970).
[3] A. Kotański, K. Zalewski, *Many Body Decays and Statistical Tensors*, lectures at the XI Cracow School of Theoretical Physics, Zakopane, June 1971.
[4] N. Byers, S. Fenster, *Phys. Rev. Letters*, **11**, 52 (1963).
[5] A. Białas, A. Kotański, *Nuclear Phys.*, **B19**, 667 (1970).
[6] S. U. Chung, *Phys. Rev.*, **138B**, 1541 (1965); **169**, 1342 (1968).
[7] S. U. Chung, *Spin Formalisms*, Report CERN 71–8, Geneva 1971.
[8] K. Gottfried, J. D. Jackson, *Nuovo Cimento*, **33**, 309 (1964).