LETTERS TO THE EDITOR

ON THE CONNECTION BETWEEN THE BEL-ROBINSON TENSOR AND GRAVITATIONAL RADIATION

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We present here a new method which allows to connect the Bel-Robinson tensor with the gravitational radiation. The method is based on the Einstein canonical pseudotensor.

1. In our previous paper [1] we have connected the components of the Bel-Robinson tensor [2–5] with the differences of the energy and momentum of the free gravitation. These differences were evaluated in a fixed normal coordinate system. Here we want to show a new method connecting the components of the Bel-Robinson tensor with the gravitational radiation. This approach is based on the canonical pseudotensor ${}_{E}t_{\alpha}^{\beta}$ ($\alpha, \beta, \gamma, \delta, \mu, \nu = 0, 1, 2, 3$; $k, l = 1, 2, 3$).

2. Pirani [6] has constructed the approximate mean tensor, ${}_{P}t_{\alpha}^{\beta}$, of the energy-momentum for the vacuum gravitational field. This tensor is constructed as an average of the canonical pseudotensor over a small 2-sphere S_2; the average is taken in a fixed normal coordinate system. The sphere S_2 is located on the hypersurface $y^0 = \text{const}$ of the normal coordinate system, its center being in the origin P of that system. The resulting expression from which we can get the approximate mean tensor of Pirani, ${}_{P}t_{\alpha}^{\beta}$, has the form

$$ {}_{P}t_{\alpha}^{\beta} = \tfrac{1}{6}\sum_{k=1}^{3} {}_{E}\overset{0}{t}{}_{\alpha}^{\beta}{}_{,kk}. \tag{1}$$

Here ${}_{E}\overset{0}{t}{}_{\alpha}^{\beta}{}_{,\gamma\delta}$ denotes the first generalized normal tensor belonging to the canonical pseudotensor ${}_{E}t_{\alpha}^{\beta}$ and to the point P [7]. One can show that

$$ {}_{E}\overset{0}{t}{}_{\alpha}^{\beta}{}_{,\gamma\delta} = \frac{c^4}{16\pi G}\left[\tfrac{2}{9}\overset{0}{T}{}^{\beta}_{\alpha\gamma\delta} + \tfrac{2}{9}\overset{0}{T}{}^{\beta}_{.\alpha\gamma\delta} - \tfrac{1}{9}\delta_{\alpha}^{\beta}\overset{0}{R}{}^{\mu\nu\varrho}_{\ldots\delta}(\overset{0}{R}_{\mu\nu\varrho\gamma} + \overset{0}{R}_{\mu\varrho\nu\gamma})\right]. \tag{2}$$

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Index "0" above a quantity denotes its value in the point P. In the formula (2) $T^{\beta}{}_{\alpha\gamma\delta}$ is the Bel-Robinson tensor

$$T^{\beta}{}_{\alpha\gamma\delta} = R^{\beta\mu\nu}{}_{\gamma}R_{\alpha\mu\nu\delta}+R^{\beta\mu\nu}{}_{\delta}R_{\alpha\mu\nu\gamma}-\tfrac{1}{2}\delta_{\alpha}{}^{\beta}R^{\varrho\mu\nu}{}_{\gamma}R_{\varrho\mu\nu\delta}, \tag{3}$$

and $\overline{T}^{\beta}{}_{\alpha\gamma\delta}$ is the tensor

$$\overline{T}^{\beta}{}_{\alpha\gamma\delta} = R^{\beta\mu\nu}{}_{\gamma}R_{\alpha\nu\mu\delta}+R^{\beta\mu\nu}{}_{\delta}R_{\alpha\nu\mu\gamma}-\tfrac{1}{2}\delta_{\alpha}{}^{\beta}R^{\varrho\mu\nu}{}_{\gamma}R_{\varrho\nu\mu\delta}. \tag{4}$$

Pirani has shown that the off-diagonal terms ${}_{P}t_{0}{}^{k}$ of the mean tensor ${}_{P}t_{\alpha}{}^{\beta}$ are different from zero only for the vacuum metric of Type II or of Type III in the Petrov algebraic classification [8]. According to Pirani's criterion, in the empty space-times of Type II and III the gravitational radiation is present. Substituting (2) into (1) one obtains for the components ${}_{P}t_{0}{}^{k}$ of the tensor ${}_{P}t_{\alpha}{}^{\beta}$:

$${}_{P}t_{0}{}^{k} = \frac{1}{27}\frac{c^4}{8\pi G}\overset{0}{T}{}^{k}_{\cdot 000}, \tag{5}$$

because of $T^{\alpha\cdot\mu\cdot}_{\cdot\beta\cdot\mu} = 0$, $\overline{T}^{\alpha\cdot\mu\cdot}_{\cdot\beta\cdot\mu} = 0$ and $\overset{0}{T}{}^{k}_{000} = \overset{0}{\overline{T}}{}^{k}_{\cdot 000}$. Consequently, if in an empty space-time the gravitational radiation is present (according to Pirani's criterion) then, in every normal coordinate system, the components $\overset{0}{T}{}^{k}_{\cdot 000}$ of the Bel-Robinson tensor are different from zero and *vice versa*. Therefore the equality (5) shows that the criteria of Bel [5] and of Pirani are equivalent.

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