THE STRUCTURE OF NON SYMMETRIC UNIFIED FIELD THEORIES

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The structure of a unified field theory is investigated in which the non symmetric "Ricci" tensor is expressed in terms of the most general variables with respect to which it is automatically Transposition Invariant in the sense of Einstein and Kaufman. It is shown that all such theories reduce to unified field theory proposed by the latter authors (E-K). Moreover, it is shown that this theory itself is fully equivalent to the theory of Einstein and Straus. Thus, unless one is prepared to complicate the Lagrangian unduly and arbitrarily, the latter appears as the only non symmetric, four dimensional extension of General Relativity.

Newly discovered symmetries of the theory give rise to a conservation law enabling an identification of the current vector density to be made.

1. Introduction

It is only very recently [1] that some indications have been found of how a class of unified field theories of gravitation and electromagnetism, the four dimensional, non symmetric theories, might be tested empirically. Without an experimental verification (or otherwise) the only criterion by means of which it would have been possible to judge their relevance to physics was the principle of simplicity. In the case of unified field theories, however, the principle was particularly difficult to apply, owing partly to their intrinsic complexity and partly to the uncertainty as to the direction in which relativity should develop.

In these circumstances, the investigation of the mathematical and logical structure of the theories proposed acquires special significance. Einstein himself regarded the theory which he outlined with Strauss [2] and later modified in collaboration with Kaufman [3] as the natural (that is, presumably logically necessary, Ref. [4]) extension of General Relativity. (We shall refer to these two theories in the sequel as E−S and E−K, respectively.) More important is the fact that unlike most other proposals for a comprehensive theory of macrophysics, Einstein's theories begin with a hypothesis which has a direct

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physical meaning. This is the principle of Hermitian symmetry or Transposition Invariance interpreted [5] as charge conjugation invariance of the field equations. We should recall also that Einstein argued the E−K theory to be stronger than E−S (and therefore to be preferred for lack of other evidence, Ref. [3]). We shall find below that this claim cannot be maintained.

The field equations of E−S are

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} \equiv g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0$$

$$R_{\underline{\mu\nu}} = 0$$

$$R_{\check{\mu}\check{\nu},\lambda} \equiv R_{\check{\mu}\check{\nu},\lambda} + R_{\check{\nu}\check{\lambda},\mu} + R_{\check{\lambda}\check{\mu},\nu} = 0$$

and

$$\Gamma_{\mu} \equiv \tfrac{1}{2}(\Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\sigma}_{\sigma\mu}) = 0.$$

Here $R_{\mu\nu}$ denotes as usual, the contracted curvature (Ricci) tensor $R^{\alpha}_{\mu\nu\alpha}$ formed from the non symmetric affine connection (we use Einstein's summation convention over repeated indices, Greek indices go from 1 to 4 and Latin from 1 to 3) $\Gamma^{\lambda}_{\mu\nu}$, a line under two indices denotes the symmetric, and a hook, the skew-symmetric part of a given quantity.

In the E−K theory, the field equations are derived from the variational principle

$$\delta \int \mathfrak{g}^{\mu\nu} R_{\mu\nu} d\tau = \int (R_{\mu\nu} \delta \mathfrak{g}^{\mu\nu} + \mathscr{M}^{\mu\nu}_{\lambda} \delta U^{\lambda}_{\mu\nu}) d\tau = 0$$

where $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$, the contravariant tensor $g^{\mu\nu}$ is defined by

$$g_{\mu\alpha} g^{\nu\alpha} = g_{\alpha\mu} g^{\alpha\nu} = \delta^{\nu}_{\mu},$$

and g is the determinant of $g_{\mu\nu}$. The non symmetric Ricci tensor is expressed in terms of the pseudo connection $U^{\lambda}_{\mu\nu}$ defined by

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} - \tfrac{1}{3} U^{\sigma}_{\mu\sigma} \delta^{\lambda}_{\nu}.$$

$R_{\mu\nu}$ is automatically Transposition Invariant with respect to $U^{\lambda}_{\mu\nu}$, without the need to assume the last of the E−S questions (the vanishing of Γ_{μ}).

It has been found by one of us [6] that the above expression of the affine connection in terms of $U^{\lambda}_{\mu\nu}$ is not unique. If we let

$$\Gamma^{\lambda}_{\mu\nu} = V^{\lambda}_{\mu\nu} - \tfrac{1}{3} V^{\sigma}_{\mu\sigma} \delta^{\lambda}_{\nu} - \tfrac{1}{3} V_{\nu} \delta^{\lambda}_{\mu}$$

where

$$V_{\nu} = \tfrac{1}{2}(V^{\sigma}_{\nu\sigma} - V^{\sigma}_{\sigma\nu}),$$

the tensor $R_{\mu\nu}$ will remain Transposition Invariant with respect to the new symbols $V^{\lambda}_{\mu\nu}$. It seems, therefore, that we can have two (or indeed infinitely many, since below we shall exhibit a general form of the substitution which serves the same purpose) theories of the E−K type.

We shall show, however, that this is not the case. Not only do all the above theories of the E−K class have the same domain of solutions, but they overlap fully also with E−S. The result is due to a new kind of invariance arising from the desire to make $R_{\mu\nu}$ Transposition Invariant.

One other point ought to be stressed before we can proceed. The affine connection $\Gamma^{\lambda}_{\mu\nu}$ or the quantities $U^{\lambda}_{\mu\nu}$, $V^{\lambda}_{\mu\nu}$ *etc.* are invariably defined by equations of the form

$$\mathcal{M}^{\mu\nu}_{\lambda} = 0$$

which differ from the remaining field equations by being purely algebraic (Mme Tonnelat solved them completely in the E−S case). Consequently, the theories (or theory) contain only sixteen variables to which physical meaning can be attached. These are the sixteen components of the fundamental tensor $g_{\mu\nu}$. The other variables Γ, U or V are merely auxiliary. When one of these is such that $R_{\mu\nu}$ is automatically Transposition Invariant with respect to it, we shall call the variable "Hermitian".

2. *The general Hermitian substitution*

We begin, for the sake of precision, with a general definition of Transposition Invariance. Suppose that a two index quantity $H_{\mu\nu}$ is a function of the fundamental tensor $g_{\mu\nu}$, of a quantity $U^{\lambda}_{\mu\nu}$, and of their derivatives with respect to the coordinates x^{μ} (denoted as usual by a comma):

$$H_{\mu\nu} = H_{\mu\nu}(g_{\alpha\beta}, U^{\tau}_{\varrho\sigma}, \ldots).$$

We say that $H_{\mu\nu}$ is Transposition Invariant ("Hermitian") with respect to $g_{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$ if

$$H_{\mu\nu} = H_{\nu\mu}(g_{\beta\alpha}, U^{\tau}_{\sigma\varrho}, \ldots).$$

In terms of the affine connection $\Gamma^{\lambda}_{\mu\nu}$, the ("Ricci") tensor $R_{\mu\nu}$ is [7]

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma}. \tag{1}$$

Let now $U^{\lambda}_{\mu\nu}$ be an Hermitian variable so that if

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}(U^{\alpha}_{\beta\gamma}) \tag{2}$$

is substituted into (1), $R_{\mu\nu}$ is Transposition Invariant with respect to $U^{\lambda}_{\mu\nu}$.

In addition, we require (as in Ref. [8]) that the equation (2) should be invertible, so that

$$U^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu}(\Gamma^{\alpha}_{\beta\gamma}). \tag{3}$$

For example, this condition excludes Schrödinger's substitution

$$\Gamma^{*\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tfrac{2}{3}\Gamma_{\nu}\delta^{\lambda}_{\mu},$$

for which $\Gamma^{*}_{\mu} \equiv 0$.

Moreover, it virtually forces us to write (2) as a linear expression

$$\Gamma^\lambda_{\mu\nu} = U^\lambda_{\mu\nu} + \alpha_1 U^\sigma_{\mu\sigma}\delta^\lambda_\nu + \alpha_2 U^\sigma_{\nu\sigma}\delta^\lambda_\mu + \beta_1 U^\sigma_{\sigma\mu}\delta^\lambda_\nu + \beta_2 U^\sigma_{\sigma\nu}\delta^\lambda_\mu \tag{4}$$

where α_1, α_2, β_1 and β_2 are numerical constants.

Then the two conditions, namely that $U^\lambda_{\mu\nu}$ should be an Hermitian variable and that the equation (4) should be invertible, require that

$$\Gamma^\lambda_{\mu\nu} = U^\lambda_{\mu\nu} + (2\alpha_1 + \tfrac{1}{3})\delta^\lambda_\nu U^\sigma_{\underline{\mu\sigma}} - \tfrac{1}{3}\,\delta^\lambda_\nu\, U_\mu - (3\alpha_1+1)\delta^\lambda_\mu U^\sigma_{\underline{\sigma\nu}} + \\ + (3\alpha_1 + 2\alpha_2 + 1)\delta^\lambda_\mu U_\nu. \tag{5}$$

We readily find that

$$U^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \mathscr{D}[(B-A)\delta^\lambda_\nu \Gamma^\sigma_{\underline{\mu\sigma}} - (A+B)\delta^\lambda_\nu \Gamma_\mu + \\ + (C-D)\delta^\lambda_\mu \Gamma^\sigma_{\underline{\nu\sigma}} + (C+D)\delta^\lambda_\mu \Gamma_\nu], \tag{6}$$

where

$$A = 12\alpha_1^2 + 8\alpha_1\alpha_2 + 3\alpha_1 + \tfrac{4}{3}\alpha_2,$$

$$B = 3\alpha_1^2 + 2\alpha_1\alpha_2 + 2\alpha_1 + \tfrac{1}{3}\alpha_2 + \tfrac{1}{3},$$

$$C = 3\alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_1 + \tfrac{4}{3}\alpha_2,$$

$$D = 12\alpha_1^2 + 8\alpha_1\alpha_2 + 7\alpha_1 + \tfrac{7}{3}\alpha_2 + 1,$$

and

$$\mathscr{D}^{-1} = (5\alpha_1 + \tfrac{4}{3})\,(9\alpha_1 + 6\alpha_2 + 2). \tag{7}$$

Thus $U^\lambda_{\mu\nu}$ given by (6) is the most general Hermitian variable with arbitrary α_1 and α_2 providing

$$\mathscr{D}^{-1} \neq 0.$$

The excluded values of α_1 and α_2 are therefore

$$\alpha_1 = -4/15 \text{ and } 9\alpha_1 + 6\alpha_2 + 2 = 0. \tag{8}$$

In particular, the theory of Ref. [6] is obtained if we put

$$\alpha_1 = -\tfrac{1}{3} \text{ and } \alpha_2 = -\tfrac{1}{6}.$$

and E−K if

$$\alpha_1 = -\tfrac{1}{3},\ \alpha_2 = 0.$$

In terms of $U^\lambda_{\mu\nu}$, the Ricci tensor becomes

$$R_{\mu\nu} = -U^\sigma_{\mu\nu,\sigma} + (1 + 3\alpha_1 + \alpha_2)\,(U^\sigma_{\mu\sigma,\nu} + U^\sigma_{\sigma\nu,\mu}) - \\ - \alpha_2(U^\sigma_{\sigma\mu,\nu} + U^\sigma_{\nu\sigma,\mu}) + U^\sigma_{\mu\varrho}U^\varrho_{\sigma\nu} - 2(1+3\alpha_1)U^\sigma_{\mu\nu}U^\varrho_{\underline{\sigma\varrho}} - \\ - 3\alpha_1^2 U^\sigma_{\mu\sigma}U^\varrho_{\varrho\nu} - \alpha_1(1+3\alpha_1)\,(U^\sigma_{\mu\sigma}U^\varrho_{\nu\varrho} + U^\sigma_{\sigma\mu}U^\varrho_{\varrho\nu}) - \\ - 3(\tfrac{1}{3} + \alpha_1)^2 U^\sigma_{\sigma\mu}U^\varrho_{\nu\varrho}, \tag{9}$$

and is easily seen to be Transposition Invariant as required.

3. *The field equations*

We derive the field equations from the variational principle

$$\delta \int_V \mathfrak{g}^{\mu\nu} R_{\mu\nu} d\tau = 0 \tag{10}$$

with $\mathfrak{g}^{\mu\nu}$ and $U^\lambda_{\mu\nu}$ as the variational parameters. Variation in $\mathfrak{g}^{\mu\nu}$ gives immediately the sixteen equations

$$R_{\mu\nu} = 0, \tag{11}$$

$R_{\mu\nu}$ being expressed either by (9), that is in terms of $U^\lambda_{\mu\nu}$, or by (1) in terms of the affine connection $\Gamma^\lambda_{\mu\nu}$ related to the former variables by the equations (5) or (6).

With the usual assumption that all integrated parts (that is 3-integrals over the hypersurface bounding the four volume V) vanish on the boundary, variation in $U^\lambda_{\mu\nu}$ gives the 64 algebraic equations determining $U^\lambda_{\mu\nu}$ itself

$$\begin{aligned} &\mathfrak{g}^{\mu\nu}{}_{,\lambda} - (1+3\alpha_1+\alpha_2)(\mathfrak{g}^{\mu\sigma}{}_{,\sigma}\delta^\nu_\lambda + \mathfrak{g}^{\sigma\nu}{}_{,\sigma}\delta^\mu_\lambda) + \alpha_2(\mathfrak{g}^{\nu\sigma}{}_{,\sigma}\delta^\mu_\lambda + \mathfrak{g}^{\sigma\mu}{}_{,\sigma}\delta^\nu_\lambda) + \\ &+ \mathfrak{g}^{\mu\sigma}U^\nu_{\lambda\sigma} + \mathfrak{g}^{\sigma\nu}U^\mu_{\sigma\lambda} - 2(1+3\alpha_1)\mathfrak{g}^{\mu\nu}U^\sigma_{\underline{\lambda\sigma}} - (1+3\alpha_1)\mathfrak{g}^{\varrho\sigma}(U^\mu_{\varrho\sigma}\delta^\nu_\lambda + U^\nu_{\varrho\sigma}\delta^\mu_\lambda) - \\ &- 3\alpha_1^2(\mathfrak{g}^{\mu\sigma}U^\varrho_{\varrho\sigma}\delta^\nu_\lambda + \mathfrak{g}^{\sigma\nu}U^\varrho_{\sigma\varrho}\delta^\mu_\lambda) - 3(\tfrac{1}{3}+\alpha_1)^2(\mathfrak{g}^{\nu\sigma}U^\varrho_{\sigma\varrho}\delta^\mu_\lambda + \mathfrak{g}^{\sigma\mu}U^\varrho_{\varrho\sigma}\delta^\nu_\lambda) - \\ &- \alpha_1(1+3\alpha_1)(\mathfrak{g}^{\mu\sigma}U^\varrho_{\sigma\varrho}\delta^\nu_\lambda + \mathfrak{g}^{\sigma\mu}U^\varrho_{\sigma\varrho}\delta^\nu_\lambda + \mathfrak{g}^{\sigma\nu}U^\varrho_{\varrho\sigma}\delta^\mu_\lambda + \mathfrak{g}^{\nu\sigma}U^\varrho_{\varrho\sigma}\delta^\mu_\lambda) = 0. \end{aligned} \tag{12}$$

Equations (12) have some very interesting consequences. A direct calculation shows that four differential identities on $\mathfrak{g}^{\mu\nu}$ are implied. In fact, skew symmetrising equations (12) with respect to μ and ν and contracting over ν and λ, we obtain

$$-(2+9\alpha_1+6\alpha_2)\mathfrak{g}^{\underset{\vee}{\mu\lambda}}{}_{,\lambda} = 0. \tag{13}$$

Since the numerical factor is not allowed to vanish by the conditions on $\mathscr{D}^{-1}$ of the previous section

$$\mathfrak{g}^{\underset{\vee}{\mu\lambda}}{}_{,\lambda} = 0, \tag{14}$$

identically. Substituting these identities into (12) we see that the algebraic equations are independent of α_2.

We now show that they are also independent of α_1. Indeed, contracting equations (12) over ν and λ (but without previous skew symmetrisation), we find

$$\mathfrak{g}^{\mu\sigma}{}_{,\sigma} + \mathfrak{g}^{\varrho\sigma}U^\mu_{\varrho\sigma} = -\tfrac{2}{3}(1+6\alpha_1)\mathfrak{g}^{\underline{\mu\sigma}}U^\varrho_{\underline{\sigma\varrho}} - \tfrac{2}{3}\mathfrak{g}^{\mu\sigma}U_\sigma \tag{15}$$

the factor $(4+15\alpha_1)$ cancelling out. Collecting the cofactors of δ^ν_λ and δ^μ_λ respectively from the last three brackets of the equations (12) and eliminating $(1+6\alpha_1)\, g^{\underline{\mu\sigma}}U^\varrho_{\underline{\sigma\varrho}}$ with the help of (15) we can write the former equations in the form

$$\begin{aligned} &\mathfrak{g}^{\mu\nu}{}_{,\lambda} + \mathfrak{g}^{\mu\alpha}U^\nu_{\lambda\alpha} + \mathfrak{g}^{\alpha\nu}U^\mu_{\alpha\lambda} - 2(1+3\alpha_1)\mathfrak{g}^{\mu\nu}U^\alpha_{\underline{\lambda\alpha}} - \\ &- (\mathfrak{g}^{\mu\alpha}{}_{,\alpha} + \mathfrak{g}^{\alpha\beta}U^\mu_{\alpha\beta})\delta^\nu_\lambda - (\mathfrak{g}^{\alpha\nu}{}_{,\alpha} + \mathfrak{g}^{\alpha\beta}U^\nu_{\alpha\beta})\delta^\mu_\lambda - \\ &- \tfrac{1}{3}(1+6\alpha_1)U^\beta_{\underline{\alpha\beta}}(\mathfrak{g}^{\alpha\mu}\delta^\nu_\lambda + \mathfrak{g}^{\nu\alpha}\delta^\mu_\lambda) - \tfrac{1}{3}U_\alpha(\mathfrak{g}^{\nu\alpha}\delta^\mu_\lambda - \mathfrak{g}^{\alpha\mu}\delta^\nu_\lambda) = 0. \end{aligned}$$

Our aim is to convert these equations into a covariant tensor form. The usual method is to multiply them by, say, $g_{\mu\sigma}g_{\varrho\nu}$ (and later to contract in order to eliminate $\frac{1}{2}\, g^{\alpha\beta}g_{\alpha\beta,\lambda}$). But they are not in the right form for this because

$$g_{\mu\sigma}g^{\alpha\mu} \neq \delta^{\alpha}_{\mu}.$$

We observe first that because of the equations (14)

$$\mathfrak{g}^{\alpha\nu}{}_{,\alpha}+\mathfrak{g}^{\alpha\beta}U^{\nu}_{\alpha\beta} = \mathfrak{g}^{\nu\alpha}{}_{,\alpha}+\mathfrak{g}^{\alpha\beta}U^{\nu}_{\alpha\beta}.$$

If we now substitute for the latter expression from (15) (it occurs twice with μ and ν interchanged) we find that the equivalent form is

$$\mathfrak{g}^{\mu\nu}{}_{,\lambda}+\mathfrak{g}^{\mu\alpha}U^{\nu}_{\lambda\alpha}+\mathfrak{g}^{\alpha\nu}U^{\mu}_{\alpha\lambda}-2(1+3\alpha_1)\mathfrak{g}^{\mu\nu}U^{\alpha}_{\underline{\lambda\alpha}}+$$

$$+\tfrac{1}{3}\left[(1+6\alpha_1)\mathfrak{g}^{\mu\alpha}U^{\beta}_{\underline{\alpha\beta}}+\mathfrak{g}^{\mu\alpha}U_{\alpha}\right]\delta^{\nu}_{\lambda}+$$

$$+\tfrac{1}{3}\left[(1+6\alpha_1)\mathfrak{g}^{\alpha\nu}U^{\beta}_{\underline{\alpha\beta}}-\mathfrak{g}^{\alpha\nu}U_{\alpha}\right]\delta^{\mu}_{\lambda} = 0$$

and we can lower the indices in the usual way. The result is

$$g_{\varrho\sigma;\lambda}-U^{\alpha}_{\varrho\lambda}g_{\alpha\sigma}-U^{\alpha}_{\lambda\sigma}g_{\varrho\alpha}-\tfrac{1}{3}(1+6\alpha_1)\,(2g_{\varrho\sigma}U^{\alpha}_{\underline{\lambda\alpha}}+g_{\varrho\lambda}U^{\alpha}_{\underline{\sigma\alpha}}+$$

$$+g_{\lambda\sigma}U^{\alpha}_{\underline{\varrho\alpha}})-\tfrac{1}{3}(g_{\varrho\lambda}U_{\sigma}-g_{\lambda\sigma}U_{\varrho}) = 0. \tag{16}$$

If we substitute for $U^{\lambda}_{\mu\nu}$ from equation (6) in order to discover the equivalent equations from which the affine connection might be found, we get

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}-\tfrac{2}{3}\Gamma_{\lambda}g_{\mu\nu}-\tfrac{2}{3}\Gamma_{\nu}g_{\mu\lambda} = 0 \tag{17}$$

independently of α_1. These equations of course [9] are not Transposition Invariant with respect to the affine connection. However, the forms (16) and (17) are completely equivalent to each other so that regardless of which one we happen to select we must finish up with the same true field equations, namely, the differential equations from which the components of the fundamental tensor $g_{\mu\nu}$ are determined. It is only to these equations that physics, in the guise of suitable boundary conditions and symmetry restrictions, can apply.

4. *Equivalence of E–S and E–K*

Let us write $g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}(U)$ for the same expression as $g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}$ but with $U^{\lambda}_{\mu\nu}$ replacing $\Gamma^{\lambda}_{\mu\nu}$. Let also $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ denote the affine connection of Einstein and Straus. The field equations of E–S can be written in the form

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}(\tilde{\Gamma}) = 0, \quad R_{\underline{\mu\nu}}(\tilde{\Gamma}) = 0, \quad R_{\underset{\vee}{\mu\nu}} = A_{\mu,\nu}-A_{\nu,\mu}, \quad \tilde{\Gamma}_{\mu} = 0, \tag{18}$$

where A_{μ} is a vector.

Indeed, it is in this form that they are derived from a variational principle.

If we let

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \gamma^{\lambda}_{\mu\nu} + A_{\nu}\delta^{\lambda}_{\mu}$$

then $\gamma^{\lambda}_{\mu\nu}$ is still an affine connection but

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}(\gamma) = A_{\lambda}g_{\mu\nu} + A_{\nu}g_{\mu\lambda}, \quad R_{\underline{\mu\nu}}(\gamma) = 0, \quad R_{\mu\nu}(\gamma) = 0, \quad \gamma_{\mu} - \tfrac{3}{2}A_{\mu} = 0. \tag{19}$$

If we therefore choose

$$A_{\mu} = \tfrac{2}{3}\gamma_{\mu} \tag{20}$$

we obtain the equations of the E–K theory except for the addition of the equations (20). This could be called a hybrid theory H. Notice, however, that γ_{μ} has nothing to do with E–S theory so that all we are doing is requiring the arbitrary vector A_{μ} of E–S to be the contracted skew part of some affine connection. We obtain E–K if we select $\gamma^{\lambda}_{\mu\nu}$ to be the connection of that theory.

The reverse process is easily carried out. Starting with equations (17) we simply put

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tfrac{2}{3}\delta^{\lambda}_{\mu}\Gamma_{\nu},$$

and find immediately

$$R_{\underline{\mu\nu}}(\tilde{\Gamma}) = 0, \quad R_{\underset{\vee}{\mu\nu},\lambda}(\tilde{\Gamma}) = 0, \quad g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}(\tilde{\Gamma}) = 0, \quad \tilde{\Gamma}_{\mu} = 0 \tag{21}$$

i. e., the equation of E–**S**.

Equations (17), which are a consequence of the variational principle adopted, guarantee that

$$\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0. \tag{14}$$

The important point is that although in E–**S** these equations are equivalent to the vanishing of $\tilde{\Gamma}_{\mu}$, the affine connection of E–**S** and the affine connection of E–K (with respect to which the field equations of the latter are not Transposition Invariant) need not be the same at all. Indeed, as long as we regard $g_{\mu\nu}$ as the only fundamental quantity of the theories concerned, it does not matter what auxiliary quantities we may care to use. They are determined by equations of a very different nature from those which determine $g_{\mu\nu}$'s themselves, since they do not involve any boundary conditions in their selection. We are driven, therefore, to the conclusion that the theories of E–**S** and E–K (and, in view of the α_1 and α_2 invariance, any generalisations of the latter) are completely interchangeable in the sense that they possess the same domain of solutions.

5. Conclusions

The test of the non symmetric unified field theories mentioned in the Introduction [1] depends on the equations of motion of a test particle which, in accordance with the situation encountered in General Relativity, follow from the field equations. (We have shown elsewhere that the field equations of the E–**S** theory give rise to the "correct" equations of motion in the sense that a Lorentz-like term appears naturally in them providing the electromagnetic field is suitably identified, [9]. In view of the results obtained in this article, there is only one non symmetric four-dimensional theory that needs to be considered and that is the theory of Einstein and Straus. If the results obtained from it are experimentally verified we have no option but to accept it as (indeed, Einstein claimed it to be!) a next step in the development of a comprehensive theory of macrophysics.

There remains, however, the question of the strength of the field equations. Relying on the so-called λ-invariance of the E−K equations Einstein claimed the latter to be stronger than the E−S equations (and therefore preferable on the grounds of the Principle of Simplicity or Occam's razor). The fact is, however, that λ-invariance leads [3] to the equations (14) which are a consequence of our field equations anyway and which represent only a restriction on the choice of the fundamental tensor.

We can show easily that invariance corresponds to a case which is *a priori* excluded by the conditions of the theorem of section 2. If we put

$$V^{\lambda}_{\mu\nu} - \tfrac{1}{3} V^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} = U^{\lambda}_{\mu\nu} + \alpha_1 U^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} + (\alpha_1 + \tfrac{1}{3})U^{\sigma}_{\sigma\mu}\delta^{\lambda}_{\nu} + \\ + \alpha_2 U^{\sigma}_{\nu\sigma}\delta^{\lambda}_{\mu} - (3\alpha_1 + \alpha_2 + 1)U^{\sigma}_{\sigma\nu}\delta^{\lambda}_{\mu} \tag{22}$$

(so that $V^{\lambda}_{\mu\nu}$ is the pseudo-connection of E−K), then

$$V^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} - (3\alpha_1 + \alpha_2 + 1)U^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} + \alpha_2 \delta^{\lambda}_{\gamma} U^{\sigma}_{\sigma\mu} + \\ + \alpha_2 \delta^{\lambda}_{\mu} U^{\sigma}_{\nu\sigma} - (3\alpha_1 + \alpha_2 + 1)\delta^{\lambda}_{\mu} U^{\sigma}_{\sigma\nu}. \tag{23}$$

Hence, if we choose

$$\alpha_1 = -\tfrac{1}{3}, \quad \alpha_2 = \tfrac{1}{6}, \tag{24}$$

(excluded, since for these values $\mathscr{D}^{-1} = 0$)

$$V^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} - \tfrac{1}{3}\delta^{\lambda}_{\nu} U_{\mu} + \tfrac{1}{3}\delta^{\lambda}_{\mu} U_{\nu}, \tag{25}$$

(which cannot be inverted so that U_μ will be arbitrary). We obtain the E−K λ-invariance if there exists a scalar such that

$$V_{\mu} = \lambda_{,\mu}.$$

Since we have excluded this case, the λ-invariance does not affect the theorem considered in this article, which in any case have been shown to reduce to only one theory.

APPENDIX

Matter in the unified field theory

Einstein and Kaufman obtained in their article an expression for a quantity which could be identified as an energy momentum tensor. This followed directly from the invariants associated with the variational principle. The fact that their Lagrangian is λ-invariant gave rise only to the equations (14) which we have now seen to be an immediate consequence of the field equations (16). Hence λ-invariance cannot be said to contribute anything to the structure of the theory. The E−K identification of energy-momentum density, ($\mathfrak{g}^{\mu\nu}U^{\varrho}_{\mu\nu,\sigma} = \mathfrak{T}^{\varrho}_{\sigma}$) is made simply because this quantity is conserved in the sense that

$$\mathfrak{T}^{\varrho}_{\sigma,\varrho} = 0. \tag{26}$$

However, there is a serious drawback to this. $\mathfrak{T}^{\varrho}_{\sigma}$ is a tensor density only for linear transformations of coordinates reviving almost the distinction between Special and General Theories of Relativity.

In any case the problem of matter in the unified field theory remains formidable. How, for example, can one hope to obtain a model of a stratified stellar object with a large magnetic field? So far, we have good reasons for identifying [10] the electromagnetic field as

$$f_{\mu\nu} = g^{\alpha\beta} g_{\underset{+}{\mu}\,\underset{-}{\nu};\alpha\beta}. \tag{27}$$

The answer to the above question ought to be sought not so much from finding expressions for energy-momentum, but from attempting to extract from the theory a metric tensor. It is highly doubtful whether $g_{\underline{\mu\nu}}$ is an adequate description of the latter especially if $g_{\underset{\vee}{\mu\nu}}$ is not the electromagnetic field tensor. However, there does not seem to be at present any way in which a more general expression for the metric could be derived. Most probably discovery of such a way will have to wait until more solutions of the field equations are known.

There are on the other hand two alternative methods which may throw some light on the problem even in the present state of the theory. It has been pointed out by one of us [11] that according to the definition (27) the electromagnetic field $f_{\mu\nu}$ can vanish without simultaneous distinction of skew symmetry of the fundamental tensor. This is the case when $g_{\underset{\vee}{\mu\nu}}$ satisfies our analogue of the wave equations

$$g^{\alpha\beta} g_{\underset{+}{\mu}\,\underset{-}{\nu};\alpha\beta} = 0. \tag{28}$$

Solutions of this equation in the vicinity of the coordinate origin may represent distribution of matter, though of course, skewsymmetry precludes by Noether's theorem, $g_{\underset{\vee}{\mu\nu}}$ itself, from being regarded as the energy momentum tensor. Secondly, we can revert to the variational principle but consider the α-invariance discovered in this article.

A change of α_1 and α_2 corresponds to a "new" version of the theory (although not a new theory as we have seen above). Moreover, by equation (16), any solution for $U^{\lambda}_{\mu\nu}$ (but not for $\Gamma^{\lambda}_{\mu\nu}$) depends on $g_{\mu\nu}$, $g_{\mu\nu,\lambda}$ and α_1 (but not on α_2, by equation (14)). Consequently, variations in α_1 and $U^{\lambda}_{\mu\nu}$ are not independent but we can vary α_2. A glance at equation (9) (expression for $R_{\mu\nu}$ in terms of $U^{\lambda}_{\mu\nu}$ and the α's) shows that the result is

$$(\mathfrak{g}^{\underset{\vee}{\mu\nu}} U_{\mu})_{,\nu} = 0. \tag{29}$$

Unlike the E−K energy momentum, $\mathfrak{g}^{\underset{\vee}{\mu\nu}} U_{\mu}$ is a vector density. This can be verified since from equation (6)

$$(9\alpha_1 + 6\alpha_2 + 2)U_{\mu} = \tfrac{3}{2}\,\Gamma^{\lambda}_{\underline{\mu\lambda}} + \tfrac{1}{2}\,\Gamma_{\mu}, \tag{30}$$

and

$$\Gamma^{\lambda}_{\underline{\mu}\lambda,\nu} - \Gamma^{\lambda}_{\underline{\nu}\lambda,\mu}$$

is a tensor by the contracted equation (17) which gives

$$\Gamma^{\lambda}_{\underline{\mu\lambda}} + \tfrac{5}{3}\Gamma_{\mu} = \tfrac{1}{2}\, g^{\alpha\beta} g_{\alpha\beta,\mu} = (\ln \sqrt{-g})_{,\mu}. \tag{31}$$

In fact (29) reduces to

$$(\mathfrak{g}^{\mu\nu}\Gamma_{\mu})_{,\nu} = 0. \tag{32}$$

It is, therefore, tempting to identify current density as

$$J^{\nu} = \mathfrak{g}^{\mu\nu}_{\vee}\Gamma_{\mu}. \tag{33}$$

This does not conflict with (27) if we adopt Mie's point of view that the field tensors occurring in the first and the second set of Maxwell's equations should be regarded at least *a priori* as independent field entities.

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