EQUATIONS OF MOTION IN A CYLINDRICAL FIELD

BY A. H. KLOTZ AND G. K. RUSSELL

Department of Applied Mathematics, University of Sydney*

(Received September 25, 1972; Revised paper received March 1, 1973)

The equations of motion of a test particle are derived from the field equations of Einstein's unifield field theory in the case when there is a cylindrically symmetric source. An exact form of the equations of motion, corresponding to a particular solution of the field equations is obtained.

1. *Introduction*

We considered in a recent article [1] derivation of the equations of motion of an electrically charged test particle from the field equations of the unified field theories of Einstein and of Einstein and Straus [2, 3]. We have shown that electromagnetic interaction terms (a quasi-Lorentz force) can be obtained from the field equations in the desired order of approximation, providing the electromagnetic field is suitably interpreted, or, rather, identified within the non-symmetric geometry. In fact, we take

$$f_{\mu\nu} = {}^*g^{\alpha\beta} g_{\underset{+-}{\check{\mu\nu}};\alpha\beta}, \tag{1}$$

where $f_{\mu\nu}$ stands for the electromagnetic field tensor [4], Greek indices go from 1 to 4, $g_{\mu\nu}$ is the fundamental tensor, $g_{\check{\mu\nu}}$, its skew symmetric part and $^*g^{\mu\nu}$, its tensorial inverse. The case considered in the above work (Ref. [1]) corresponds to two structureless "point" charges so that, both the test particle and the field tested are assumed to be spherically symmetric. The equations of motion of the former are derived by an application of the Einstein-Infeld-Hoffman (EIH) method (see *e. g.* [5]).

In the present article we shall extend the EIH method to the case when the source field is cylindrically symmetric. Thus, it may be due in principle either to an electrically charged wire, or to a magnetized one. We shall find that in the former case the resulting equations of motion appear to deviate from what might be expected in the classical theory.

A form of the equations of motion was suggested previously (Ref. [6]) on the basis

* Address: The University of Sydney, Sydney NSW 2006, Australia.

that they should reduce to the special relativistic, Lorentz equations in what may be regarded as a non-relativistic approximation. We obtained then a radial force proportional to $(1+\varrho)^{-1}$ where $\varrho = r+m$, m is a constant, and r is the radial distance from the z-axis of symmetry of the field. Although interpretation of the coordinates in relation to the field is somewhat arbitrary, such a force bears no resemblance to its classical form. We can now correct this result by utilising a more general solution of the field equations given in Ref. [7].

We adopt throughout the coordinate scheme

$$(x^{\mu}) \equiv (r, z, \theta, t), \tag{2}$$

and rely freely on the notation and concepts introduced in Ref. [1].

2. *The electromagnetic force*

It has been shown in Ref. [5] that the electromagnetic force term obtained from the field equations is given by (Latin indices going from 1 to 3)

$$-\frac{1}{2\pi}\int S_{mk}n^{k}dS, \tag{3}$$

where

$$S_{mn} = \tfrac{1}{4}\,\delta_{mn}\varphi^{2} - \tfrac{1}{2}\,\varphi_{,m}\Phi_{,n} + \tfrac{1}{2}\,\varphi_{,mn}\Phi, \tag{4}$$

and the function Φ is defined, up to the addition of an arbitrary, harmonic function, by

$$\Phi_{,ss} = \varphi. \tag{5}$$

We take for φ, as previously, the classical solution of a given potential problem.

In the case of the motion of an electron in the field of line charge, the situation is complicated (in comparison with a two-point charge problem) by the fact that the field of the former is spherically symmetric while that of the latter is cylindrical. Let us assume, nevertheless, that the two can be considered concurrently and that the functions Φ corresponding to both, are additive. This seems to be warranted by the linearity of the equation (5).

Let ϱ_1 be the radial distance of a field point from the moving charge and ϱ_2, its distance from the line source. We solve separately the spherically symmetric equation

$$\Phi_{1,ss} = \frac{k_1}{\varrho_1}$$

and the cylindrically symmetric $\Phi_{2,ss} = k_2 \ln \varrho_2$, where k_1 and k_2 are constants. The complete solution is

$$\Phi = -\frac{a_1}{\varrho_1} + \frac{k_1}{2}\varrho_1 + b_1 + \frac{k_2}{4}\left[(\varrho_2^2+a)\ln\varrho_2 - (\varrho_2^2+b)\right], \tag{6}$$

where a_1, b_1, a and b are constants of integration. As in Ref. [5] we must add to Φ an harmonic function

$$^*\Phi(\varrho_1, \varrho_2).$$

Let us suppose that $^*\Phi$ is separately harmonic in ϱ_1 and in ϱ_2. Then

$$^*\Phi = -\frac{L}{\varrho_1}\ln\frac{\varrho_2}{\varrho_0} + M\ln\frac{\varrho_2}{\varrho_0}, \tag{7}$$

again with L, M and ϱ_0 constant. Actually, we can put $M = 0$ because this term already appears in Φ. A straightforward calculation now shows that the only surviving terms in (3) are

$$\frac{k_1 k_2}{6\pi}\left[2\varrho_2 \ln \varrho_2 + \left(1+\frac{a}{\varrho_2^2}\right)\varrho_2 - 2\varrho_2\right] - \frac{2k_2}{3\varrho_2}\left[a_1 + L\ln\frac{\varrho_2}{\varrho_0}\right].$$

For the equation of motion of a test particle, we take $k_1 = 0$, so that the force term aquires the form

$$-\frac{2k_2}{3\varrho_2}\left[a_1 + L\,\text{'n}\frac{\varrho}{\varrho_0}\right]. \tag{8}$$

3. *The exact equations of motion*

The general, static, cylindrically symmetric solution of the field equations is

$$g_{11} = g_{22} = -g_{44} = -1,\ g_{33} = -\varrho^2(l-(m_3^2-m_4^2)\,\varrho^2),$$
$$g_{23} = -g_{32} = m_3\varrho^2, g_{34} = -g_{43} = -m_4\varrho^2, \tag{9}$$

where m_3 and m_4 are constants of integration. According to (1), the field components corresponding to this solution are

$$f_{23} = -6m_3 \text{ and } f_{34} = 6m_4. \tag{10}$$

These give the electric and magnetic line charge distribution fields

$$\boldsymbol{E} = -\frac{6m_3}{r}\boldsymbol{r} \quad \text{and} \quad \boldsymbol{H} = \frac{6m_4}{r}\boldsymbol{\theta}, \tag{11}$$

respectively, $\boldsymbol{r}$ being the radial and $\boldsymbol{\theta}$, the transverse unit vectors. It follows that when $m_4 = 0$ we are dealing with a purely electrostatic field and when $m_3 = 0$, a purely magnetostatic one. However, neither of these reduces to the solution used in Ref. [1], which must, therefore, be regarded as a particular solution only.

We investigated in the above article the exact equations of motion in the form

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\underline{\mu\nu}}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{1}{2}\frac{e}{m}\varepsilon^{\alpha\lambda\mu\nu}(-g)^{-1/2}g_{\lambda\kappa}f_{\underline{\mu\nu}}\frac{dx^\kappa}{ds}, \tag{12}$$

where $\Gamma^{\alpha}_{\underline{\mu\nu}}$ is the symmetric part of the affine connection, $g_{\underline{\mu\nu}}$, the symmetric part of the fundamental tensor whose determinant is g, e is the charge and m, the rest mass of the test particle; s is the parameter of the path which may be identified with proper time. Previously, the reason for studying equations (12) was heuristic — with the solution of Ref. [1], they reduced to classical form, at least in what seemed to be a non-relativistic approximation. They bore no relation to the field equations. When we employ solution (9) instead, we find that in the electrostatic case, $m_4 = 0$, we obtain exactly the equations predicted by the EIH method. Indeed, if dots denote differentiation with respect to s, equations (12) become

$$\ddot{\varrho} - \varrho\dot{\theta}^2 = -\frac{6em_3/\mu}{\varrho}\dot{t} = -\frac{k}{\varrho}\dot{t}, \text{ say},$$

$$\ddot{z} = 0, \quad \ddot{\theta} + \frac{2}{\varrho}\dot{\varrho}\dot{\theta} = 0, \quad \ddot{t} = -\frac{k}{\varrho}\dot{\varrho}, \tag{13}$$

e being the charge and μ the mass of the test particle. Hence the orbit is given, in general, by

$$\ddot{\varrho} - \varrho\dot{\theta}^2 = -\frac{k^2}{\varrho}\ln\frac{\varrho_0}{\varrho}, \quad \varrho^2\dot{\theta} = h, \tag{14}$$

where $\varrho_0 > 0$ and h are constants. When the electron is projected parallel to the linear field source, we have $\dot{\theta} = 0$, throughout its motion, and the equations reduce to

$$\ddot{\varrho} = -\frac{k^2}{\varrho}\ln\frac{\varrho_0}{\varrho}. \tag{15}$$

The classical equation is obtained if we let $k^2 \to 0$ but $k^2 \ln \varrho_0 \to q^2/2$ say, (for the line distribution and the test particle being of opposite sign) and put $m = 0$. It is impossible to assign theoretically a physical meaning to the constant ϱ_0 except that it is presumably very large (or very small, depending on relative sign of the charges involved) in comparison with ϱ.

It is of considerable interest to note that the right-hand side of equation (15), the force, is exactly what we obtained by an EIH calculation in the expression (8).

4. *An approximate solution*

Neither the classical nor the above, unified field equation of motion can be integrated in terms of elementary functions. However, some idea of the deviation from the classical case can be obtained by considering the initial stages of the motion.

Let us suppose that $\dot{\varrho} = 0$ when $\varrho = \varrho_1$ and $s = 0$ and that both s remains small and $\varrho \sim \varrho_1$ ($\varrho < \varrho_1$ but close to it). Then the classical case gives $\sqrt{\ln \varrho_1/\varrho} = \tan \omega_0 s \simeq \omega_0 s$ or

$$\varrho_c = \varrho_1 e^{-\omega_0^2 s^2}, \text{ say}, \tag{16}$$

where $\omega_0 = q/2\varrho_1$.

Similarly, the unified field case gives $\sqrt{\ln \varrho_1/\varrho} \approx \omega_0 s(1-\varepsilon)$, where

$$\varepsilon \simeq \frac{\ln \varrho_1}{\ln \varrho_0} - \frac{k^2}{16\varrho_1^2\omega_0^2} \tag{17}$$

so that, if we denote this ϱ by ϱ_u, $\varrho_u \approx \varrho_c \exp(-2\varepsilon\omega_0^2 s^2)$, or

$$\frac{\varrho_c - \varrho_u}{\varrho_c} \simeq 2\varepsilon\omega_0^2 s^2. \tag{18}$$

5. *Conclusions*

We should note that in addition to the EIH approximation used in obtaining the electromagnetic force term from the field equations (implicit in our calculation) a test particle is characterised by putting $k_1 = 0$ (or, more exactly $k_1 \ll k_2$). The second assumption is clearly responsible for the result being the same as that derived from the "exact" equations (12). We cannot say therefore, that the latter have been rigorously derived. The result must be regarded merely as confirming the general form of the exact equations when the exact solution of the field equations (solution (9)) is employed. Furthermore, equations (12) give the classical result when an electric line charge distribution is replaced by a uniformly magnetised wire. The reason for that is that in this case, the fourth of the equations (13) becomes $\dot{t} = 0$. We mention in Ref. [6] the possibility of constructing an empirical test of the unified field theory. The results obtained herein may go some way towards finding one. This is why we have attempted to compare in the last Section the conclusions to be drawn from the present theory with the classical case. While the exact nature and the numerical values of the quantities appearing there remain uncertain, the "test" must be regarded as purely speculative.

We should like to express our appreciation of a constructive criticism due to Dr A. Staruszkiewicz of an earlier draft of this paper.

REFERENCES

[1] A. H. Klotz, G. K. Russell, *Acta Phys. Polon.*, **B3**, 649 (1972).
[2] A. Einstein, *Ann. Math.*, **46**, 578 (1945).
[3] A. Einstein, E. G. Straus, *Ann. Math.*, **47**, 731 (1946).
[4] A. H. Klotz, G. K. Russell, *Acta Phys. Polon.*, **B3**, 407 (1972).
[5] L. Infeld, *Acta Phys. Polon.*, **10**, 284 (1950).
[6] A. H. Klotz, G. K. Russell, *Acta Phys. Polon.*, **B2**, 637 (1971).
[7] A. H. Klotz, *Acta Phys. Polon.*, **B3**, 341 (1972).