BOUNDS ON K_{l3}-DECAY FORM FACTORS

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(Received August 29, 1972)

The hyperbolic metric principle is applied to study the bounds on K_{l3}-decay form factors. Some of the results obtained in previous works are rederived and improved.

1

Recently much theoretical effort has been devoted to the derivation of bounds on K_{l3}-decay form factors. Li and Pagels [1, 2], Okubo and co-workers [3, 4, 5] established rigorous bounds on these quantities using as input information the (zero momentum transfer) value of the propagator $\Delta(t)$ of the divergence of the strangeness-changing current.

Some of these results as well as new ones have been derived by Radescu [6] by a simpler method based on the maximum modulus theorem for holomorphic functions. Bourrely [7] derived bounds on the scalar form factor of K_{l3}-decay, valid in the physical region. For other papers on this subject see Acharya [8], Aubrecht *et al.* [9], Tanaka and Torgerson [10], Micu [11].

The purpose of this note is to rederive and improve some of the results quoted in the above papers. To this end we shall systematically use some results from the theory of analytic functions in the unit disc.

To make our note self-contained we shall review some well-known facts concerning K_{l3}-decay. Starting with the weak strangeness changing current $V_\mu^{(4-i5)}(x)$ one can define

$$\langle \pi^0(p)|V_\mu^{(4-i5)}(0)|K^+(k)\rangle = \tfrac{1}{2}\left[(k+p)_\mu f_+(t)+(k-p)_\mu f_-(t)\right], \tag{1}$$

where $f_\pm(t)$ are the K_{l3}-decay form factors, m_π and m_K are the pion and kaon masses respectively, and $t = (p-k)^2$.

The matrix element of the divergence of the current $V_\mu^{4-i5}(x)$ leads to

$$\langle \pi^0(p)|i\partial_\mu V_\mu^{(4-i5)}(0)|K^+(k)\rangle = \tfrac{1}{2}D(t) = \tfrac{1}{2}\left[(m_K^2-m_\pi^2)f_+(t)+tf_-(t)\right]. \tag{2}$$

It is known [1–5] that $D(t)$ is a real analytic function of t with a cut $t_0 \leqslant t \leqslant \infty$, $t_0 = (m_K+m_\pi)^2$. Moreover $D(t)$ is bounded on the cut by the spectral function of the

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propagator of the divergence of the strangeness changing current $V_\mu^{4-i5}(x)$, *i. e.* for $t > t_0$

$$\varrho(t) \geqslant \frac{3}{64\pi^2} t^{-1}[(t-t_0)(t-t_1)]^{1/2}|D(t)|^2, \tag{3}$$

where $t_1 = (m_K - m_\pi)^2$ and

$$\Delta(t) = \int d^4x e^{iqx} \langle 0 | T\partial_\mu V_\mu^{(4-i5)}(x), \partial_\nu V_\nu^{(4+i5)}(0) | 0 \rangle =$$

$$= \int_{t_0}^{\infty} \frac{\varrho(t')dt'}{t'-t}. \tag{4}$$

After this brief review we shall proceed now to derive our results.

2

Taking into account the analytical properties of $D(t)$ we can perform a standard conformal transformation which maps the cut t-plane onto the unit disc D in the z-plane, *i. e.*

$$z = \frac{1-(1-t/t_0)^{1/2}}{1+(1-t/t_0)^{1/2}}. \tag{5}$$

One can consider the unit disc D as a non-Euclidean (hyperbolic) domain and therefore one can apply the hyperbolic metric principle [12, 13]. First, we shall present a definition.

Let $w(z)$ be an arbitrary function which conformally map the domain A onto the unit disc $|w| < 1$. Then $\delta(z, A) = \dfrac{|w'(z)|}{1-|w(z)|^2}$ is uniquely defined in the domain A and does not depend on $w(z)$. The quantity $\delta(z, A)$ is called the hyperbolic metric of the domain A. It is invariant under conformal transformations.

Now we shall state the following theorem (the hyperbolic metric principle) [12, 13] which will play a central role in our derivation.

Let $f(z)$ be a regular function in the domain A whose values lie in the domain B. If A (B) does possess at least three boundary points then

$$|f'(z)|\,\delta(f(z), B) \leqslant \delta(z, A), \; z \in A, \tag{6}$$

where $\delta(z, A)$ and $\delta(f(z), B)$ are the hyperbolic metrics of A, B respectively.

We shall not give the proof of this theorem but instead we should like to make some observations.

i) Both A and B are connected domains.

ii) If B contains the point at infinity then naturally $f(z)$ will be considered as meromorphic in the domain A.

iii) The theorem is still valid if one considers $f(z)$ analytic in the domain A.

Now we shall present some applications of this theorem. We define the hyperbolic metric of the unit disc $|z| < 1$ as

$$\delta(z, D) = \frac{1}{1-|z|^2}, \quad |z| < 1. \tag{7}$$

A direct application of the relations (6) and (7) implies

$$\frac{|f'(z)|}{1-|f(z)|^2} \leqslant \frac{1}{1-|z|^2}, \quad |z| < 1, \tag{8}$$

where $f(z)$ is a regular or analytic in the unit disc $|z| < 1$.

Therefore

$$|f'(0)| \leqslant 1-|f(0)|^2. \tag{9}$$

This inequality permits one to obtain the best bound on $D'(0)$ where

$$D'(0) = f_+(0)\left(\xi + \frac{m_K^2 - m_\pi^2}{m_\pi^2}\lambda_+\right) \tag{10}$$

and

$$\xi = f_-(0)/f_+(0), \quad \lambda_+ = m_\pi^2 f'_+(0)/f_+(0) \tag{11}$$

if one considers $f(z)$ given by

$$f(z) = D(z)/\pi^{-1/2} \cdot I(z) \cdot |\Delta(z)|^{1/2}, \tag{12}$$

where $I(z)$ is the kinematical function

$$I(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln |w(\theta)|\right),$$

$$w(t) = \frac{8\pi}{\sqrt{3}} t^{1/2}[(t-t_0)(t-t_1)]^{-1/4}, \tag{13}$$

which in the above-quoted papers is considered as a known one. If one substitutes (12) and (13) into (9) one obtains the corresponding bound on $D'(0)$ (Eq. (41) in [6]).

We shall now present another application of the hyperbolic metric principle. First we shall define in the unit disc $|z| < 1$ (the conformal image of the holomorphic domain of $D(t)$) an outer function $F(z)$ in terms of $\varrho(z)$. Obviously one has

$$|D(t)| \leqslant |F(t)|, t > t_0. \tag{14}$$

In addition $F(z)$ will be assumed univalent (in an average sense) in the unit disc $|z| < 1$. Next we shall apply the above-stated theorem which implies the following inequality for univalent functions which do not possess zeros in $(z) < 1$

$$\left|\frac{F(z)}{F(0)}\right| \leqslant \left(\frac{1+|z|}{1-|z|}\right)^2. \tag{15}$$

Therefore one has

$$|D(z)| \leqslant \left(\frac{1+|z|}{1-|z|}\right)^2 \cdot |F(0)|. \tag{16}$$

This is the best bound one can obtain on $D(z)$ in the unit disc $|z| < 1$.

Let us now go further. On the cut $t_0 \leqslant t \leqslant \infty$ we have the inequality

$$|D(t)| \leqslant \frac{8\pi}{\sqrt{3}} \frac{\varrho^{1/2}(t)t^{1/2}}{[(t-t_0)(t-t_1)]^{1/4}} \equiv g(t), \quad t > t_0. \tag{3'}$$

We pass to the unit disc $|z| < 1$ and define the outer function

$$F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln|g(\theta)|\right) \tag{17}$$

which has the following properties:

a) $|g(z)| \leqslant |F(z)|$, $|z| < 1$

b) $|g(e^{i\theta})| = |F(e^{i\theta})|$ almost everywhere on the unit circle $|z| = 1$ *i. e.* on the cut $t_0 \leqslant t \leqslant \infty$

c) $F(z)$ does not possess zeros in the unit disc $|z| < 1$.

In addition we shall assume $F(z)$ is univalent in an average sense in $|z| < 1$. One can show [13] that this assumption is not too restrictive. Then the inequality (16) leads to

$$|D(z)| \leqslant \left(\frac{1+|z|}{1-|z|}\right)^2 \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \ln|g(\theta)|\right). \tag{18}$$

One can factorize the outher function (17) into two terms, one identical to the kinematical function $I(z)$, the other containing $\varrho(z)$. For the latter factor one can easily derive

$$|F_2(z)| \leqslant |\Delta(z)|^{1/2}/\pi^{1/2}m_K^2. \tag{19}$$

As it stands, the first term

$$F_1(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln\left|\frac{16\pi z^{1/2}}{\sqrt{3}\,[4t_0 z-(1+z)^2]^{1/4}[2z-z^2-1]^{1/4}}\right|\right) \tag{20}$$

can be calculated by means of the Jensen formula. Substituting all the results into Eq. (19) one obtains

$$|D(z)| \leqslant \frac{16\pi^{1/2}}{\sqrt{3}} \frac{|\Delta(0)|^{1/2}(m_K+m_\pi)^{1/2}}{(m_K^{1/2}+m_\pi^{1/2})} \left(\frac{1+|z|}{1-|z|}\right)^2. \tag{21}$$

This result generalizes the inequalities derived in references [2], [3], [6].

To conclude, we mention that the same theorem can lead to weaker bounds on the derivative of $D(t)$.

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