MANIFESTLY COVARIANT HAMILTONIAN FORM OF THE WAVE EQUATION FOR THE DIRAC PARTICLE IN THE EXTERNAL ELECTROMAGNETIC FIELD

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The inhomogeneous Dirac equation (depending on the potential of the external electromagnetic field) is transformed into a covariant Hamiltonian form similar to that obtained earlier for the free Dirac particle. The same procedure is extended to the case of the "amplified Dirac equation" containing additional tensor terms.

1. Introduction

The manifestly covariant, quantum-mechanical Hamiltonian formalism previously proposed by one of us (Hanus [1], [2]) has been applied, as yet, merely to problems of free quantum particles (described by homogenous first-order wave equations). In particular, the problem of the free Dirac particle has been systematically investigated (Hanus, Słomiński [3], Hanus [4]) including questions concerning the physical interpretation of the displayed formalism. Now, the problem of extending this formalism to descriptions more general than that of a free particle may be raised. It is well known that no strictly covariant description of interacting particles can be given on the "first-quantization" level. However, the simplified model of a quantum particle in a given external field can be expressed by a formally exact equation, suitable for solving a wide class of important problems — the well-known case of the charged Dirac particle in the Coulomb field being the most representative example. This first exact solution of the inhomogeneous Dirac equation has been followed soon by that for a constant magnetic field (Rabi [5]) and, subsequently, by two other (Sauter [6], Volkov [7]). After about thirty years, the new interest in problems of quantum-mechanical description of relativistic particles in external electromagnetic fields has manifested itself by a series of papers on this subject[1], mainly owing

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[1] See *e.g.* Redmond [8], Stanciu [9], Sen Gupta [10], Canuto, Ciuderi [11], Holz [12] and Lam [13]–[16], where several new exact solutions in specified particular cases of external stationary fields (and also in the presence of the light beams) have been found for the Dirac particle, using the Dirac, or the Feynman and Gell-Mann equations [17]. Similar investigations for the boson particles have also been initiated in [16].

to its close connection with contemporary astrophysics and physics of plasma, as well as with other problems. Our proposed approach, although essentially different from that represented in Refs [8]–[16], concerns the same actual problem. The inhomogeneous Dirac equation will be investigated in Chapter 2. There is still another possibility — to follow the idea of Pauli [18] and to start from the "amplified Dirac equation" containing higher order tensor terms (which allow, in particular, to take account of the anomalous magnetic moment of the particle). The special form of this equation (see Bethe, Salpeter [19], p. 136 and 175) suitable for an approximate phenomenological description of all radiative corrections has been investigated by the present autors (Hanus, Mrugała [20]) who have transformed this equation from its standard covariant form into the non-covariant Hamiltonian one, reducing, subsequently, the so obtained quantum-mechanical wave equation to the subspace of positive energy states. Now, the possibility of extending into this equation the previously displayed manifestly covariant Hamiltonian formalism will be discussed in Chapter 3.

2. *The inhomogeneous Dirac equation*

In accordance with the idea of the discussed fromalism and of its further physical interpretation[2], the homogeneous Dirac equation

$$\left(\gamma_\nu \frac{\partial}{\partial x_\nu} + m\right)\psi^0 = 0 \tag{1}$$

has been transformed (in the sense of a "weak" relation, *i. e.* of a subsidiary condition defining the subspace of "quantum-mechanical states" $\Psi_0(\hat{\mathscr{X}}_{\kappa\lambda}, x_N)$) into the covariant Hamiltonian form

$$-d_N\Psi^0 = \mathscr{H}_N^0\Psi^0, \quad d_N = -in_\mu \frac{\partial}{\partial x_\mu}, \tag{2}$$

$$\mathscr{H}_N^0 = \varrho_{\text{III}} m + \varrho_{\text{I}}\theta_N^0, \tag{3}$$

where n_μ denotes a unit timelike vector, interpreted as the time arrow in a "laboratory frame" (arbitrary, in principle but specified for measuring observables of the described particle): $x_N = -in_\mu x_\mu$ (the *c*-number timelike variable) represents the time in this frame and d_N the respective time derivative. The following equalities also hold

$$\varrho_{\text{III}} = \gamma_N = -in_\mu\gamma_\mu, \quad \varrho_{\text{I}} = -\gamma_5, \tag{4}$$

$$\theta_N^0 = \frac{1}{2}\hat{\Sigma}_{\kappa\lambda}\hat{P}_{\kappa\lambda} = \frac{1}{2i}\gamma_\kappa\gamma_\lambda\hat{P}_{\kappa\lambda}, \tag{5}$$

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[2] See [4], in particular Chapter 2, where a short recapitulation of the formalism may be found. The notation used in our present paper is the same as in [2]–[4], with the only difference that for the free particle $\mathscr{H}_N, \theta_N, \psi$ and Ψ (see [1]–[5]) have been replaced by $\mathscr{H}_N^0$, θ_N^0, ψ^0 and Ψ^0, respectively, the former symbols being reserved for the formulae holding in the presence of the external field.

for simple bivectors of spacelike character[3] $\tilde{\mathscr{X}}_{\kappa\lambda}$, $\hat{P}_{\kappa\lambda}$ and $\hat{\Sigma}_{\kappa\lambda}$ being covariant generalizations of $\boldsymbol{x}$, $\boldsymbol{p}$ and $\boldsymbol{\sigma}$, respectively.

It will be shown that this way of obtaining the covariant Hamiltonian wave equation (2) can be generalized to the case of the inhomogeneous Dirac equation

$$\left[\gamma_\nu\left(\frac{\partial}{\partial x_\nu} - iea_\nu\right) + m\right]\psi = 0, \tag{6}$$

$a_\nu(x)$ denoting the four-potential of the electromagnetic field. Its components will appear in our further calculations combined into $\tilde{A}_{\kappa\lambda}(\tilde{\mathscr{X}}_{\mu\nu}, x_N)$ and $a_N(\hat{\mathscr{X}}_{\mu\nu}, x_N)$ where

$$\hat{A}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\lambda\kappa\mu\nu}A_{\mu\nu}, \qquad \hat{A}_{\mu\nu} = n_\mu a_\nu - n_\nu a_\mu, \tag{7}$$

$$a_N = -in_\mu a_\mu, \tag{8}$$

(these being, obviously, covariant generalizations of $\boldsymbol{a}$ $(\boldsymbol{r}, t)$ and $a_4(\boldsymbol{r}, t)$ respectively). It can also be verified that the Lorenz condition expressed in terms of $\hat{A}_{\kappa\lambda}$, a_N assumes the form

$$\tfrac{1}{2}\,\hat{D}_{\kappa\lambda}\hat{A}_{\kappa\lambda} + d_N a_N = 0, \tag{9}$$

where

$$\hat{D}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\kappa\lambda\mu\nu}D_{\mu\nu}, \qquad D_{\mu\nu} = n_\mu\frac{\partial}{\partial x_\nu} - n_\nu\frac{\partial}{\partial x_\mu}. \tag{10}$$

The symbolic bivector $\hat{D}_{\kappa\lambda}$ has the meaning of a "spacelike derivative". Its presence in our formulae denotes the use of the "covariant Schrödinger representation". On the other hand the obvious relation[4]

$$\hat{P}_{\kappa\lambda} = -i\hat{D}_{\kappa\lambda} \tag{11}$$

allows to express the covariant Hamiltonian in a more general form, independent of the representation used.

The explicit calculations transforming (6) into the covariant Hamiltonian form are as follows: multiplying (6) on the left by $\gamma_N = -in_\mu\gamma_\mu$ and putting $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\nu)$ we get

$$\left\{in_\mu\left(\frac{\partial}{\partial x_\mu} - iea_\mu\right) + \frac{i}{2}\,\gamma_\mu\gamma_\nu\left[n_\mu\left(\frac{\partial}{\partial x_\nu} - iea_\nu\right) - \right.\right.$$

$$\left.\left. -n_\nu\left(\frac{\partial}{\partial x_\mu} - iea_\mu\right)\right] - m\gamma_N\right\}\psi = 0. \tag{12}$$

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[3] Defined according to the general rule $\hat{V}_{\kappa\lambda} = \frac{i}{2}\,\varepsilon_{\kappa\lambda\mu\nu}V_{\mu\nu}$, $V_{\mu\nu} = n_\mu v_\nu - n_\nu v_\mu$ (for details see [4]).

[4] Immediately resulting from $p_\mu = -i\dfrac{\partial}{\partial x_\mu}$; (for the role of the latter relation in the formalism see [2]).

Passing now over to the "weak" formulation (*i. e.* introducing the timelike derivative into the first term of (12)), interpreting, simultaneously, the remaining terms as the Hamiltonian operating in the Hilbert space of wave functions $\Psi(\hat{\mathscr{X}}_{\kappa\lambda}, x_N)$, we obtain, using (7), (8) and (10)

$$-d_N\Psi = \left\{m\gamma_N + \frac{1}{2i}\gamma_\mu\gamma_\nu(D_{\mu\nu} - ieA_{\mu\nu}) - iea_N\right\}\Psi. \tag{13}$$

Taking into account the relations occuring between the bivectors $D_{\mu\nu}$, $A_{\mu\nu}$ and their dual counterparts and using the known formula

$$\varepsilon_{\mu\nu\kappa\lambda}\gamma_\mu\gamma_\nu = -\gamma_5(\gamma_\kappa\gamma_\lambda - \gamma_\lambda\gamma_\kappa) \tag{14}$$

we have

$$\frac{1}{2i}\gamma_\mu\gamma_\nu(D_{\mu\nu} - ieA_{\mu\nu}) = \frac{1}{2i}\gamma_\mu\gamma_\nu\left(-\frac{i}{2}\right)\varepsilon_{\mu\nu\kappa\lambda}(\hat{D}_{\kappa\lambda} - ie\hat{A}_{\mu\lambda}) =$$

$$= -\frac{1}{2i}\gamma_5\gamma_\mu\gamma_\lambda(-i\hat{D}_{\kappa\lambda} - e\hat{A}_{\mu\lambda}). \tag{15}$$

Hence, owing to (4), (5) and (11) we can express our final result in the form

$$-d_N\Psi = \mathscr{H}_N\Psi, \tag{16}$$

$$\mathscr{H}_N = m\varrho_{\mathrm{III}} + \varrho_{\mathrm{I}}\theta_N - iea_N, \tag{17}$$

$$\theta_N = \frac{1}{2i}\gamma_\kappa\gamma_\lambda(\hat{P}_{\kappa\lambda} - e\hat{A}_{\kappa\lambda}) = \frac{1}{2}\hat{\Sigma}_{\kappa\lambda}(\hat{P}_{\mu\lambda} - e\hat{A}_{\kappa\lambda}). \tag{18}$$

(The latter formula for θ_N, containing $\tilde{\Sigma}_{\kappa\lambda}$ can easily be verified by using the explicit expression for $\hat{\Sigma}_{\kappa\lambda}$. The structure of (17) requires still some explanation: $\hat{\mathscr{H}}_N$ ought to be expressed in terms of operators which unlike, a_N, do not have the geometrical character of "timelike scalars". The controversy is apparent only, since $a_N(\hat{\mathscr{X}}_{\mu\nu}, x_N)$ as well as $\hat{A}_{\kappa\lambda}$ $(\hat{\mathscr{X}}_{\mu\nu}, x_N)$ represent in $\mathscr{H}_N$ functions of the spacelike operators $\hat{\mathscr{X}}_{\mu\nu}$ and of the c-number variable x_N. The role of the mass term containing $\varrho_{\mathrm{III}} = \gamma_N$ has already been explained in [4] by the fact that γ_μ is not a Lorentz four-vector so that the timelike character of γ_N is rather illusory.

3. The amplified Dirac equation

The Dirac equation (6) supplemented (according to [19]) by phenomenological terms describing radiative corrections assumes the form[5]

$$\left\{\Gamma + g_1\frac{ie}{4m}\gamma_\mu\gamma_\nu F_{\mu\nu} - g_2\frac{ie}{m^2}\gamma_\mu\Box a_\mu\right\}\psi = 0, \tag{19}$$

[5] The notation has been somewhat changed, as compared to that used in [20].

where

$$\Gamma = \gamma_\mu \left(\frac{\partial}{\partial x_\mu} - iea_\mu \right) + m \tag{20}$$

stands for the operator of the equation (6), while

$$F_{\mu\nu} = \frac{\partial}{\partial x_\mu} a_\nu - \frac{\partial}{\partial x_\nu} a_\mu, \tag{21}$$

$$\Box a_\mu = -4\pi j_\mu, \tag{22}$$

$_\mu$ denoting the current-charge four-vector related to the source of the given external field. The symbols g_1, g_2 represent dimensionless constants whose numerical values are (according to the results of quantum electrodynamics) of the order of the fine structure constant. The possibility of transforming (19) to the respective covariant Hamiltonian form, in analogy to the procedure applied to (6) in the preceding Chapter, must now be examined. The main difficulty follows from the fact that $F_{\mu\nu}$ appears explicitly in (19). $F_{\mu\lambda}$ is not a simple bivector but, according to the known property of antisymmetrical tensors of second rank, it can be decomposed into a sum of two such bivectors. Namely, we define two quantities:

$$f_\kappa = -\frac{i}{2} \varepsilon_{\kappa\mu\nu\omega} F_{\mu\nu} n_\omega, \qquad g_\kappa = -\frac{i}{2} \varepsilon_{\kappa\mu\nu\omega} \hat{F}_{\mu\nu} n_\omega, \tag{23}$$

and subsequently

$$f_{\kappa\tau} = n_\kappa f_\tau - n_\tau f_\kappa, \qquad g_{\kappa\tau} = n_\kappa g_\tau - n_\tau g_\kappa. \tag{24}$$

From the definitions (23) it follows that

$$f_N = -in_\mu f_\mu = 0, \qquad g_N = -in_\kappa f_\kappa = 0, \tag{25}$$

and also

$$g_\kappa = F_{\kappa\omega} n_\omega. \tag{26}$$

After applying (23) and (26) to relation (24) we get

$$\hat{f}_{\kappa\tau} = \frac{i}{2} \varepsilon_{\kappa\tau\mu\nu} f_{\mu\nu} = F_{\kappa\tau} - n_\mu n_\tau F_{\mu\kappa} - n_\mu n_\kappa F_{\tau\mu}, \tag{27}$$

$$g_{\kappa\tau} = n_\mu n_\kappa F_{\tau\mu} + n_\mu n_\tau F_{\mu\kappa}. \tag{28}$$

Adding these two last quantities we obtain the desired decomposition of the antisymmetric field tensor $F_{\mu\nu}$ into the sum of two simple bivectors

$$\hat{f}_{\kappa\tau} + g_{\kappa\tau} = F_{\kappa\tau}. \tag{29}$$

In a special Lorentz frame $n_\mu = (0, 0, 0, i)$ we have

$$\hat{f}_{kl} = B_m, \quad \hat{f}_{k4} = 0, \quad g_{kl} = 0, \quad g_{4k} = iE_k, \tag{30}$$

where k, l, m stands for a cyclic permutation of 1. 2. 3; therefore we introduce the new symbols

$$\hat{f}_{\mu\nu} = \hat{B}_{\mu\nu}, \qquad g_{\mu\nu} = E_{\mu\nu} \tag{31}$$

and correspondingly we may write

$$F_{\mu\nu} = \hat{B}_{\mu\nu} + E_{\mu\nu}. \tag{32}$$

The bivectors $\hat{B}_{\mu\nu}$ and $E_{\mu\nu}$ have of course different characters. Only the first one is spacelike. $E_{\mu\nu}$ has a timelike character — this is quite natural because $F_{\mu\nu}$ is not a plane tensor. However, there is no difficulty since we can always write instead of $E_{\mu\nu}$ a dual tensor.

The equation (19) shall be transformed into the covariant Hamiltonian form in the same way as the homogeneous and inhomogeneous Dirac equations. Namely we shall multiply it on the left by $\gamma_N = -in_\mu\gamma_\mu$. For shortness we shall transform the terms proportional to $\gamma_\mu\gamma_\nu F_{\mu\nu}$ and $\gamma_\mu \Box a_\mu$ separately, neglecting temporarily in calculations the constant factors

$$\gamma_N\gamma_\mu\gamma_\nu F_{\mu\nu} = \gamma_N\gamma_\mu\gamma_\nu(\hat{B}_{\mu\nu} + E_{\mu\nu}) = \gamma_N\gamma_\mu\gamma_\nu\left(\hat{B}_{\mu\nu} - \frac{i}{2}\varepsilon_{\mu\nu\tau\varrho}\hat{E}_{\tau\varrho}\right) =$$
$$= \gamma_N\gamma_\mu\gamma_\nu\hat{B}_{\mu\nu} + i\gamma_N\gamma_5\gamma_\tau\gamma_\varrho\hat{E}_{\tau\varrho} = i\varrho_{\rm III}\hat{\Sigma}_{\mu\nu}\hat{B}_{\mu\nu} + i\varrho_{\rm II}\hat{\Sigma}_{\mu\nu}\hat{E}_{\mu\nu}. \tag{33}$$

We have used here the relations (14), $\hat{B}_{\mu\nu}n_\nu = 0$, $\hat{E}_{\mu\nu}n_\nu = 0$ and $\varrho_{\rm III}\gamma_{\rm I} = i\varrho_{\rm II}$. The last term with $\gamma_\mu \Box a_\mu$ is transformed according to (22)

$$\gamma_N\gamma_\mu\Box a_\mu = 4\pi i n_\nu\gamma_\nu\gamma_\mu j_\mu = 4\pi\left[in_\mu j_\mu + \frac{i}{2}\gamma_\mu\gamma_\nu(n_\mu j_\nu - n_\nu j_\mu)\right] =$$
$$= -4\pi j_N + 2\pi i\gamma_\mu\gamma_\nu J_{\mu\nu} = -4\pi j_N + \pi\gamma_\mu\gamma_\nu\varepsilon_{\mu\nu\tau\varrho}\hat{J}_{\tau\varrho} =$$
$$= -4\pi j_N - 2\pi\gamma_5\gamma_\tau\gamma_\varrho\hat{J}_{\tau\varrho} = -4\pi j_N + 2\pi i\varrho_{\rm I}\hat{\Sigma}_{\tau\varrho}\hat{J}_{\tau\varrho}. \tag{34}$$

The way of generalizing the current-charge four-vector j_μ consists in extending it to the plane antisymmetric pseudotensor

$$J_{\mu\nu} = n_\mu j_\nu - n_\nu j_\mu \tag{35}$$

and the scalar variable $j_N = -in_\mu j_\mu$. It is easy to test that

$$\tfrac{1}{2}\hat{J}_{\mu\nu}\hat{J}_{\mu\nu} + j_N^2 = j_\mu j_\mu \tag{36}$$

and in a special Lorentz frame $n_\mu = (0, 0, 0, i)$

$$j_N = j_4 = i\varrho, \qquad \hat{J}_{kl} = j_m, \qquad \hat{J}_{k4} = 0. \tag{37}$$

Finally, taking into account the relations (17), (33) and (34), we arrive at the "amplified

Dirac equation" in the following form

$$-d_N\Psi = \mathscr{H}_N\Psi, \tag{38}$$

$$\mathscr{H}_N = m\varrho_{\text{III}} + \tfrac{1}{2}\varrho_{\text{I}}\hat{\Sigma}_{\mu\nu}(\hat{P}_{\mu\nu} - e\hat{A}_{\mu\nu}) - iea_N -$$

$$-g_1\frac{e}{4m}\hat{\Sigma}_{\mu\nu}(\varrho_{\text{III}}\hat{B}_{\mu\nu} + \varrho_{\text{II}}\hat{E}_{\mu\nu}) + g_2 i\frac{2\pi e}{m^2}j_N + g_2\frac{2\pi e}{m^2}\varrho_{\text{I}}\hat{\Sigma}_{\mu\nu}\hat{J}_{\mu\nu}. \tag{39}$$

The scalar term j_N introduces no difficulties, since $j_N(\hat{\mathscr{X}}_{\mu\nu}, x_N)$ as well as $\hat{J}_{\kappa\lambda}(\hat{\mathscr{X}}_{\mu\nu}, x_N)$ are functions of the spacelike operators $\hat{\mathscr{X}}_{\mu\nu}$ and of the c-number variable x_N.

4. *Concluding remarks*

Our considerations have shown that the covariant Hamiltonian formalism displayed and applied in [1]–[4] for the problem of the free Dirac particle can be generalized in a consistent way to the case of an external electromagnetic field. The Dirac equation (6) as well as its modification (19) can be used, alternatively, as starting points of the considerations. Both these equations are formally treated as exact ones. Similarly the way of transforming them to the covariant Hamiltonian form contains no approximations. The approximate character is, however, inherent in the very formulation of the problem (of a quantum particle in a given classical field). This character does not diminish the large practical usefulness of such equations, as it has been mentioned in Section 1. Among special cases, the most important are those with stationary fields. The assumption of independence of $\hat{P}_{\kappa\lambda}$ and a_N on x_N means that in a specified laboratory frame they do not depend on t, the condition is, however, expressed in a covariant way. The physical interpretation of the formalism (discussed in detail in [4]) remains unchanged. No new difficulties appear beside those existing for the free particle. This result extends considerably the domain of applicability of the investigated formalism to actual relativistic problems of quantum mechanics.

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