Vol. 40 (2009) ACTA PHYSICA POLONICA B No 1

A VACUUM SOLUTION FOR THE COSMOLOGICAL MODEL BIANCHI I IN THE CONFORMAL POINCARÉ-GAUGE THEORY OF GRAVITATION

V.V. Karbanovski[†], E.V. Martynova[‡], Y.N. Limonnikova
M.S. Bogdanov

Murmansk State Pedagogical University
Egorova-Street 15, 183720 Murmansk, Russia

(Received May 27, 2008)

The vacuum equations for Bianchi I cosmology in the conformal Poincaré -gauge theory of gravitation are considered. All possible cases are investigated. It is shown that do not exist the solutions which are different from the Kasner line element.

PACS numbers: 98.80.–k

In paper [1] the conformal Poincaré-gauge theory of gravitation has been developed. In the Einstein gauge and in the torsionless limit its equations are reduced to the form

$$R^{\mu}_{\nu} - \tfrac{1}{2}R\delta^{\mu}_{\nu} + \vartheta f C^{\mu\alpha}_{\nu\beta} R^{\beta}_{\alpha} \;=\; \vartheta T^{\mu}_{\nu}\,, \tag{1a}$$

$$f C^{\mu\alpha}_{\nu\beta;\mu} \;=\; 0\,, \tag{1b}$$

where $C^{\mu\alpha}_{\nu\beta}$ is the Weyl tensor, $\vartheta = 8\pi G/c^4$, f — an arbitrary parameter.

The asymptotical solutions of system (1) near the singular points for the perfect fluid configurations have been investigated in work [2]. In paper [3] the equations (1) have been considered for the cosmological model Bianchi I

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2\,. \tag{2}$$

They are reduced to the following system (a dot denotes the differentiation with respect to t)

† Karbanovski-V-V@mail.ru
‡ Martynova-E-V@mail.ru

$$
\begin{aligned}
-\vartheta\varepsilon &= G^0_0 + \vartheta f \left(A G^1_1 + B G^2_2 - (A+B) G^3_3 \right) , && \text{(3a)}\\
\vartheta P_1 &= G^1_1 + \vartheta f \left(A G^0_0 - (A+B) G^2_2 + B G^3_3 \right) , && \text{(3b)}\\
\vartheta P_2 &= G^2_2 + \vartheta f \left(B G^0_0 - (A+B) G^1_1 + A G^3_3 \right) , && \text{(3c)}\\
\vartheta P_3 &= G^3_3 + \vartheta f \left(-(A+B) G^0_0 + B G^1_1 + A G^2_2 \right) , && \text{(3d)}\\
& \dot{A} + (2A+B) h_2 + (A-B) h_3 = 0 \,, && \text{(3e)}\\
& \dot{B} + (A+2B) h_1 + (B-A) h_3 = 0 \,, && \text{(3f)}
\end{aligned}
$$

here $G^i_k = R^i_k - \frac{1}{2} R \delta^i_k$, $\varepsilon = -T^0_0$ is the energy density of matter, $P_1 = T^1_1$, $P_2 = T^2_2$, $P_3 = T^3_3$, — its pressures along axes x, y, z, respectively, $A = C^{01}_{01}$, $B = C^{02}_{02}$, $h_1 = \dot{a}/a, h_2 = \dot{b}/b, h_3 = \dot{c}/c$.

It was shown that the configurations of "usual" matter ($\varepsilon \geq 0$) can be realized only at $f > 0$.

In work [4] the system (3) has been considered for case of anisotropy on two directions ($h_1 = h_2$). It was found that the vacuum solution is

$$ds^2 = -dt^2 + \left(c_1 + \tfrac{3}{2} \sqrt{\gamma}\, t \right) (dx^2 + dy^2) + c_2 \left(c_1 + \tfrac{3}{2} \sqrt{\gamma}\, t \right)^{-1/3} dz^2 \,, \qquad (4)$$

which corresponds to "pancakes of Zel'dovich" (γ, c_1, c_2 are the integration constants). It was shown also that it is impossible to construct model with a matter satisfying standard physical requirement $P_3 \geq 0$.

Here we shall consider the system (3) for the general vacuum case $\varepsilon = P_1 = P_2 = P_3 = 0$, $h_1 \neq h_2 \neq h_3$.

The particular solution of this problem is the Kasner vacuum solution

$$ds^2 = -dt^2 + a_0 t^{q_1} dx^2 + b_0 t^{q_2} dy^2 + c_0 t^{q_3} dz^2 \qquad (5)$$

with additional constraints (3e), (3f). Using these conditions for expression (5) we find

$$
\begin{aligned}
& -4q_1 + 4q_1^2 + 2q_2 - 2q_2^2 + 2q_3 - 2q_3^2 + q_1 q_2 \\
& + q_1 q_3 - 2q_2 q_3 - 3q_1^2 q_2 + 3q_1 q_2^2 - 3q_1^2 q_3 + 3q_1 q_3^2 = 0 \,, && \text{(6a)}\\
& 2q_1 - 2q_1^2 - 4q_2 + 4q_2^2 + 2q_3 - 2q_3^2 + q_1 q_2 \\
& - 3q_1 q_2^2 + 3q_1^2 q_2 - 3q_2^2 q_3 + 3q_2 q_3^2 - 2q_1 q_3 + q_2 q_3 = 0 \,. && \text{(6b)}
\end{aligned}
$$

Substituting into (6) the known requirements for the Kasner solution

$$q_1 + q_2 + q_3 = q_1^2 + q_2^2 + q_3^2 = 1 \,, \qquad (7)$$

we shall obtain identity.

Thus, the metrics (5) is the particular solution for the vacuum equations (1). This can be interpreted as extension of Fairchild theorem.

Let us consider now the variants corresponding to $G^i_k \neq 0$. According to (3a)–(3d) it is possible for the condition:

$$\begin{vmatrix} 1 & V & W & -(V+W) \\ V & 1 & -(V+W) & W \\ W & -(V+W) & 1 & V \\ -(V+W) & W & V & 1 \end{vmatrix} = 0\,, \tag{8}$$

where $V = \vartheta f A$, $W = \vartheta f B$.

The requirement (8) can be fulfilled for $V = -\frac{1}{2}, W = -\frac{1}{2}$ or $V + W = \frac{1}{2}$. Obviously, these conditions are equivalent. Therefore, we shall consider only

$$V = -\tfrac{1}{2}\,. \tag{9}$$

Substituting (9) into (3a)–(3c) we obtain

$$G^0_0 - \tfrac{1}{2}G^1_1 + WG^2_2 + \left(\tfrac{1}{2} - W\right)G^3_3 \;=\; 0\,, \tag{10a}$$

$$-\tfrac{1}{2}G^0_0 + G^1_1 + \left(\tfrac{1}{2} - W\right)G^2_2 + WG^3_3 \;=\; 0\,, \tag{10b}$$

$$WG^0_0 + \left(\tfrac{1}{2} - W\right)G^1_1 + G^2_2 - \tfrac{1}{2}G^3_3 \;=\; 0\,. \tag{10c}$$

The requirements (10) lead to following conditions

$$W \;=\; 1\,, \qquad G^1_1 = -G^3_3\,, \qquad G^0_0 = -G^2_2\,, \tag{11a}$$

$$W \;=\; -\tfrac{1}{2}\,, \qquad G^1_1 = -G^2_2\,, \qquad G^0_0 = -G^3_3\,, \tag{11b}$$

$$G^0_0 \;=\; G^1_1 = -G^2_2 = -G^3_3\,. \tag{11c}$$

Using (3e)–(3f) for (11a)–(11b) we obtain $G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = 0$.

Thus, the variants (11a)–(11b) reduce to solution (5). Let us consider now the case (11c). It correspond to system

$$\begin{cases} \dot h_2 + h_2^2 + \dot h_3 + h_3^2 = h_1h_2 + h_1h_3\,, \\ \dot h_1 + h_1^2 + \dot h_3 + h_3^2 = -h_1h_2 - 2h_1h_2 - 2h_1h_3 - h_2h_3\,, \\ \dot h_1 + h_1^2 + \dot h_2 + h_2^2 = -2h_1h_2 - h_1h_3 - h_2h_3\,, \end{cases} \tag{12}$$

where $h_1 = \dot a/a,\ h_2 = \dot b/b,\ h_3 = \dot c/c$.

The equations (12) can be presented as

$$\dot h_1 \;=\; -h_1^2 - 2h_1h_2 - 2h_1h_3 - h_2h_2\,, \tag{13a}$$

$$\dot h_2 \;=\; -h_2^2 + h_1h_3\,, \tag{13b}$$

$$\dot h_3 \;=\; -h_3^2 + h_1h_2\,. \tag{13c}$$

From (3e) we find

$$W = \frac{h_2 + \frac{1}{2}h_3}{h_2 - h_3}\,. \tag{14}$$

Using expressions (13) and (14) in equation (3f) we obtain

$$(h_2 - h_3)(h_1 h_2 + h_1 h_3 + h_2 h_3) = 0\,. \tag{15}$$

A case $h_2 = h_3$ was considered in work [4]. Therefore, we research the equality

$$h_1 h_2 + h_1 h_3 + h_2 h_3 = 0\,. \tag{16}$$

Taking into account (16) we find from (13)

$$h_1 + h_2 + h_3 = (t + c_1)^{-1}\,, \tag{17}$$

where c_1 is a constant of integration.

From expressions (13b), (13c) and (16) we obtain

$$\frac{d}{dt}(h_2 + h_3) = -(h_2^2 + h_3^2) - h_2 h_3\,. \tag{18}$$

The equations (16), (17) give

$$-(h_2 + h_3)^2 + -\frac{h_2 + h_3}{t + c_1} + h_2 h_3 = 0\,. \tag{19}$$

Integrating (18) with help of (19) we find

$$h_2 + h_3 = \frac{c_2}{t + c_1}\,, \tag{20}$$

where c_2 is a constant of integration.

From (17) and (20) we obtain

$$h_1 = \frac{1 - c_2}{t + c_1}\,. \tag{21}$$

Using (20) and (21) in (13b) we find

$$\dot{h}_2 = -h_2^2 - \frac{1 - c_2}{t + c_1} h_2 + \frac{(1 - c_2)c_2}{(t + c_1)^2}\,. \tag{22}$$

The solution of equation (22) has a form

$$h_2 = \frac{c_2 - 2\alpha}{(t + c_1)\left[1 + c_3(t + c_1)^{2\alpha - c_2}\right]} + \frac{\alpha}{t + c_1}\,, \tag{23}$$

where c_3 is a constant of integration,

$$\alpha = \frac{c_2 \pm \sqrt{c_2(4 - 3c_2)}}{2} \,. \tag{24}$$

From (13b) and (23) we obtain

$$h_3 = \frac{c_2 - \alpha}{t + c_1} \left(1 - \frac{1}{1 + c_3(t + c_1)^{2\alpha - c_2}} \right) \,. \tag{25}$$

However, it is possible to check up that the expressions (21), (23) and (25) for equations (11c) and (3e), (3f) are incompatible. Therefore, we conclude the absence of solutions which are different from the vacuum GR solutions. This result can be interpreted as extension of conclusion [1] on vacuum equations for Bianchi I cosmology. At present we study the system (3) for non-zero functions ε and P_i $(i = 1, 2, 3)$.

REFERENCES

[1] B.N. Frolov, *Acta Phys. Pol. B* **9**, 823 (1978).

[2] B.N. Frolov, V.V. Karbanovski, *Phys. Lett.* **A169**, 1 (1992).

[3] V.V. Karbanovski *et al., Acta Phys. Pol. B* **36**, 2143 (2005).

[4] V.V. Karbanovski *et al., Acta Phys. Pol. B* **38**, 38 (2007).