Vol. 40 (2009) *ACTA PHYSICA POLONICA B* No 1

A NOTE ON PROPER AFFINE VECTOR FIELDS IN NON-STATIC SPHERICALLY SYMMETRIC SPACE-TIMES

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(Received April 22, 2008; revised version received September 23, 2008; final version received October 24)

A study of non-static spherically symmetric space-times according to their proper affine vector fields is given by using holonomy and decomposability, the rank of the 6×6 Riemann matrix and direct integration techniques. It is shown that when the above space-times admit proper affine vector fields, they turn out to be static and spherically symmetric. In the non-static cases affine vector fields are just Killing vector fields.

PACS numbers: 04.20.–q, 04.20.Jb

1. Introduction

In this paper we study the proper affine vector fields in non-static spherically symmetric space-times by using holonomy and decomposability, the rank of the 6×6 Rieman matrix and direct integration techniques. The affine vector field which preserves the geodesic structure and affine parameter of a space-time carries significant information and rises interest in the Einstein's theory of general relativity. It is, therefore, important to study this symmetry. Let (M, g) be a space-time with M a smooth connected Hausdorff four dimensional manifold and g a smooth metric of Lorentz signature $(-,+,+,+)$ on M. The curvature tensor associated with g, through Levi–Civita connection Γ, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time M will be assumed non-flat in the sense that the Riemann tensor does not vanish over any non-empty open subset of M.

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A vector field X on M is called an affine vector field if it satisfies

$$X_{a;bc} = R_{abcd}X^d\,, \tag{1}$$

where $R_{abcd} = g_{af}R^f{}_{bcd} = g_{af}(\Gamma^f_{bd,c} - \Gamma^f_{bc,d} + \Gamma^f_{ce}\Gamma^e_{bd} - \Gamma^f_{ed}\Gamma^e_{bc})$. If one decomposes $X_{a;b}$ on M into its symmetric and skew-symmetric parts

$$X_{a;b} = \tfrac{1}{2}h_{ab} + G_{ab}\,, \qquad \Big(h_{ab}(\equiv X_{a;b} + X_{b;a}) = h_{ba}\,, \quad G_{ab} = -G_{ba}\Big) \tag{2}$$

then equation (1) is equivalent to

$$(i)\ h_{ab;c} = 0\,, \qquad (ii)\ G_{ab;c} = R_{abcd}X^d\,, \qquad (iii)\ G_{ab;c}X^c = 0\,. \tag{3}$$

Now, we are interested in proving equation (3) using equation (1). To prove $(3(i))$ consider h_{ab} and take its covariant derivative as $h_{ab;c} = X_{a;bc} + X_{b;ac}$ and using equation (1) we get $h_{ab;c} = (R_{abcd} + R_{bacd})X^d$, now using the fact $R_{abcd} = -R_{bacd} \Rightarrow h_{ab;c} = 0$. To prove $(3(ii))$ consider equation (2) and take its covariant derivative. Using equation (1) we get

$$R_{abcd}X^d = \tfrac{1}{2}h_{ab;c} + G_{ab;c}\,. \tag{4}$$

Comparing symmetric and skew-symmetric parts of the above equation (4) we get $h_{ab;c} = 0$ and $G_{ab;c} = R_{abcd}X^d$. For proving $(3(iii))$ contract equation (4) with X^c. Using the fact $R_{abcd}X^dX^c = 0$ and comparing the symmetric and skew-symmetric parts we get $G_{ab;c}X^c = 0$.

Now, we are interested in proving equation (1) using equation $(3(i))$. Consider equation (2) and take its covariant derivative. Using $h_{ab;c} = 0$ we get $X_{a;bc} = G_{ab;c}$. From the Ricci identity $(X_{a;bc} - X_{a;cb} = R_{abcd}X^d)$ we get

$$G_{ab;c} - G_{ac;b} = R_{abcd}X^d\,, \tag{5}$$

similarly permuting indices a, b and c in (5) gives

$$G_{ba;c} - G_{bc;a} = R_{bacd}X^d\,, \tag{6}$$

$$G_{cb;a} - G_{ca;b} = R_{cbad}X^d\,. \tag{7}$$

Adding equations (5), (6) and (7) and using the property of Riemann tensor $(R_{[abc]d} = 0)$ gives $X_{a;bc} = R_{abcd}X^d$. Next we will prove equation (1) using equation $(3(ii))$. Consider equation (2) and take its covariant derivative; using $G_{ab;c} = R_{abcd}X^d$ and $h_{ab;c} = 0$ we get $X_{a;bc} = R_{abcd}X^d$. Equation $(3(iii))$ follows from equation $(3(ii))$ by contracting X^c and using the fact that $R_{abcd}X^dX^c = 0$. One can find the above proofs in [7].

If $h_{ab} = 2cg_{ab}$, $c \in R$ then the vector field X is called *homothetic* (and *Killing* if $c = 0$). The vector field X is said to be proper affine if it is not homothetic vector field and also X is said to be proper homothetic vector field if it is not Killing vector field on M [2]. Let us define the subspace Z_p of the tangent space T_pM to M at p as those $k \in T_pM$ satisfying

$$R_{abcd}k^d = 0\,. \tag{8}$$

It is important to note that the time-like vector field $t_a \equiv t_{,a}$, where

$$t_{,a} = \frac{\partial t}{\partial x^a} \equiv \frac{\partial x^0}{\partial x^a}$$

satisfying $t_a t^a = -1$ is covariantly constant, that is $t_{a;b} = 0$, if and only if the components of the Christoffel symbol Γ^a_{bc} are zero whenever any of a, b or c takes the value zero. Now, we are interested in proving the above result. It follows from the definition that $t_0 = 1$ and $t_d = 0$ whenever d takes the value 1, 2 or 3. Consider $t_{a;b} = 0 \Rightarrow t_{a,b} - \Gamma^c_{ab}t_c = 0$. Using the fact $t_{a,b} = 0$, we get $\Gamma^c_{ab}t_c = 0$. Since we are following the Einstein's summation convention hence c can take the values 0, 1, 2 or 3. First, consider c taking the value 1, 2 or 3. The equation $\Gamma^c_{ab}t_c = 0$ is satisfied identically (here we are using the fact $t_c = 0$ whenever c takes the values 1, 2 or 3). Now consider c equals zero; then the above equation ($\Gamma^c_{ab}t_c = 0$) gives $\Gamma^0_{ab} = 0$ (here we are using the fact $t_0 = 1$). Now we are interested to prove the converse. Consider $\Gamma^0_{ab} = 0$ or $-\Gamma^c_{ab}t_c = 0$ (since we have $t_0 = 1$ and $t_c = 0$ for $c = 1$, 2 or 3) which can be written as $t_{a,b} - \Gamma^c_{ab}t_c = 0$ (because $t_{a,b} = 0$) which implies $t_{a;b} = 0$.

2. Affine vector fields

Suppose that M is a simple connected space-time. Then the holonomy group of M is a connected Lie subgroup of the identity component of the Lorentz group and is thus characterized by its subalgebra in the Lorentz algebra. These have been labeled into fifteen types R_1–R_{15} [1]. It follows from [2] that the only such space-times which could admit proper affine vector fields are those which admit nowhere zero covariantly constant second order symmetric tensor field h_{ab}. This forces the holonomy type to be either R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} [2]. A study of the affine vector fields for the above holonomy type can be found in [2]. It follows from [3] that the rank of the 6×6 Riemann matrix of the above space-times which have holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} is at most three. Hence for studying affine vector fields we are interested in those cases when the rank of the 6×6 Riemann matrix is less than, or equal to three.

3. Main results

Consider a non-static spherically symmetric space-time in the usual coordinate system (t, r, θ, ϕ) (labeled by (x^0, x^1, x^2, x^3)) with line element [4]

$$ds^2 = -e^{A(t,r)}dt^2 + e^{B(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{9}$$

The above space-time admits three linearly independent Killing vector fields which are

$$\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\,, \qquad \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\,, \qquad \frac{\partial}{\partial\phi}\,. \tag{10}$$

The non-zero independent components of the Riemann tensor are

$$\begin{aligned}
R_{0101} &= \frac{1}{4}\Big[e^{A(t,r)}(A_r^2(t,r) + 2A_{rr}(t,r)) - e^{B(t,r)}(B_t^2(t,r) + 2B_{tt}(t,r) \\
&\quad - A_t(t,r)B_t(t,r)) - e^{A(t,r)}A_r(t,r)B_r(t,r)\Big] \equiv \alpha_1\,,\\
R_{0202} &= \frac{r}{2}e^{A(t,r)-B(t,r)}A_r(t,r) \equiv \alpha_2\,,\\
R_{0303} &= \frac{r}{2}e^{A(t,r)-B(t,r)}A_r(t,r)\sin^2\theta \equiv \alpha_3\,,\\
R_{1212} &= \frac{r}{2}B_r(t,r) \equiv \alpha_4\,,\\
R_{1313} &= \frac{r}{2}B_r(t,r)\sin^2\theta \equiv \alpha_5\,,\\
R_{2323} &= r^2\sin^2\theta(1 - e^{-B(t,r)}) \equiv \alpha_6\,,\\
R_{0212} &= \frac{r}{2}B_t(t,r) \equiv \alpha_7\,,\\
R_{0313} &= \frac{r}{2}B_t(t,r)\sin^2\theta \equiv \alpha_8\,.
\end{aligned}$$

Writing the curvature tensor with components R_{abcd} at p as a 6×6 symmetric matrix as [5]

$$R_{abcd} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha_2 & 0 & \alpha_7 & 0 & 0\\ 0 & 0 & \alpha_3 & 0 & \alpha_8 & 0\\ 0 & \alpha_7 & 0 & \alpha_4 & 0 & 0\\ 0 & 0 & \alpha_8 & 0 & \alpha_5 & 0\\ 0 & 0 & 0 & 0 & 0 & \alpha_6 \end{pmatrix}. \tag{11}$$

As mentioned in Section 2, the space-times which admit proper affine vector fields have holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} and the rank of the 6×6 Riemann matrix is at most three. Therefore, we are only interested in those cases when the rank of the 6×6 Riemann matrix is less than, or equal to three. In general, for any 6×6 symmetric matrix there exist total forty one possibilities when the rank of the 6×6 symmetric matrix is less or equal to three, that is, twenty possibilities for rank three, fifteen possibilities for rank two and six possibilities for rank one. Suppose the rank of the 6×6 Riemann matrix is one. Then there is only one non-zero row or column in (11). If we set five rows or columns identically zero in (11) then there exist six possibilities when the rank of the 6×6 Riemann matrix is one. In these six possibilities five give the contradiction and only one will arise which is given in case (G). For example, consider the case when the rank of the 6×6 Riemann matrix is one *i.e.* $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$ and $\alpha_1 \neq 0$. The constraints $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0 \Rightarrow B(t,r) = 0$ and $A_r(t,r) = 0$. Subsituting back in (11) one has $\alpha_1 = 0$ which gives contradiction (here we assume that $\alpha_1 \neq 0$). So this case is not possible. Now consider another case when the rank of the 6×6 Riemann matrix is one *i.e.* $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$ and $\alpha_2 \neq 0$. The constraints $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$ implay $B(t,r) = 0$ and $A_r(t,r) = 0$. Subsituting back in (11) one has $\alpha_2 = 0$ which gives contradiction (here we assume that $\alpha_2 \neq 0$). Hence again this case is not possible. By similar analysis we come to the conclusion that there are all together five possibilities when the rank of the 6×6 Riemann matrix is three or less which are:

(C) Rank = 3, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0, \alpha_4 \neq 0, \alpha_5 \neq 0$ and $\alpha_6 \neq 0$.

(D) Rank = 3, $\alpha_1 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0, \alpha_2 \neq 0, \alpha_3 \neq 0$ and $\alpha_6 \neq 0$.

(E) Rank = 3, $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0, \alpha_1 \neq 0, \alpha_2 \neq 0$ and $\alpha_3 \neq 0$.

(F) Rank = 2, $\alpha_1 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0, \alpha_2 \neq 0$ and $\alpha_3 \neq 0$.

(G) Rank = 1, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0$ and $\alpha_6 \neq 0$.

We will consider each case in turn.

Case C

In this case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0, \alpha_4 \neq 0, \alpha_5 \neq 0, \alpha_6 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exists a unique (up to a multiple) number where zero time-like vector field $t_a = t_{,a}$ satisfies $t_{a;b} = 0$. From the Ricci identity $R^a{}_{bcd}t_a = 0$. From the above constraints we have $A_r(t,r) = 0$ and $B_t(t,r) = 0 \Rightarrow A(t,r) = \alpha(t)$ and $B(t,r) = \beta(r)$. The line element can, after a rescaling of t, be written in the form

$$ds^2 = -dt^2 + e^{\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \qquad (12)$$

The above space-time is clearly $1+3$ decomposable. Affine vector fields in this case [2] are

$$X = (c_7 t + c_8)\frac{\partial}{\partial t} + X', \qquad (13)$$

where $c_7, c_8 \in R$ and $X^{'}$ is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant t. The completion of case C requires finding an homothetic vector fields in the induced geometry of the submanifolds of constant t. The induced metric $g_{\alpha\beta}$ (where $\alpha, \beta = 1, 2, 3$) with non-zero components is given by

$$g_{11} = e^{\beta(r)}\,, \qquad g_{22} = r^2\,, \qquad g_{33} = r^2 \sin^2\theta\,. \tag{14}$$

A vector field $X^{'}$ is a homothetic vector field if it satisfies

$$L_{X'} g_{\alpha\beta} = 2c g_{\alpha\beta}\,, \qquad c \in R\,. \tag{15}$$

One can expand (15) and using (14) gets

$$\begin{aligned}
\beta^{'} X^1 + 2X^1{}_{,1} &= 2c\,, & (16)\\
e^{\beta} X^1{}_{,2} + r^2 X^2{}_{,1} &= 0\,, & (17)\\
e^{\beta} X^1{}_{,3} + r^2 \sin^2\theta X^3{}_{,1} &= 0\,, & (18)\\
\frac{1}{r} X^1 + X^2{}_{,2} &= c\,, & (19)\\
X^2{}_{,3} + \sin^2\theta X^3{}_{,2} &= 0\,, & (20)\\
\frac{1}{r} X^1 + \cot\theta X^2 + X^3{}_{,3} &= c\,. & (21)
\end{aligned}$$

Equation (16) gives

$$X^1 = c e^{-\frac{\beta}{2}} \int e^{\frac{\beta}{2}} dr + e^{-\frac{\beta}{2}} A^1(\theta, \phi)\,, \tag{22}$$

where $A^1(\theta, \phi)$ is a function of integration. Substituting the value of X^1 in (17) and (18) gives

$$\begin{aligned}
X^2 &= -A^1_{\theta}(\theta, \phi) \int \frac{1}{r^2} e^{\frac{\beta}{2}} dr + A^2(\theta, \phi)\,,\\
X^3 &= \frac{A^1_{\phi}(\theta, \phi)}{\sin^2\theta} \int \frac{1}{r^2} e^{\frac{\beta}{2}} dr + A^3(\theta, \phi)\,, \qquad (23)
\end{aligned}$$

where $A^2(\theta, \phi)$ and $A^3(\theta, \phi)$ are functions of integration. Considering equation (20) differentiating with respect to r one finds

$$A^1(\theta, \phi) = \sin\theta B^1(\phi) + B^2(\theta)\,,$$

where $B^1(\phi)$ and $B^2(\theta)$ are functions of integration. Substituting back into (22) and (23) gives

$$\begin{aligned} X^1 &= ce^{-\frac{\beta}{2}}\int e^{\frac{\beta}{2}}dr + e^{-\frac{\beta}{2}}(\sin\theta B^1(\phi) + B^2(\theta))\,, \\ X^2 &= -(\cos\theta B^1(\phi) + B^2_\theta(\theta))\int \frac{1}{r^2}e^{\frac{\beta}{2}}dr + A^2(\theta,\phi), \\ X^3 &= \frac{B^1_\phi(\phi)}{\sin\theta}\int \frac{1}{r^2}e^{\frac{\beta}{2}}dr + A^3(\theta,\phi)\,. \end{aligned} \tag{24}$$

Now consider equation (19) and differentiate with respect to ϕ to get

$$\sin\theta B^1_\phi(\phi)\left(\frac{e^{-\frac{\beta}{2}}}{r} + \int \frac{1}{r^2}e^{\frac{\beta}{2}}dr\right) + A^2_{\theta\phi}(\theta,\phi) = 0\,.$$

Differentiating with respect to r we get $B^1_\phi(\phi)((e^{-\frac{\beta}{2}}/r)' + e^{\frac{\beta}{2}}/r^2) = 0$ and there exist two possible cases:

$$\begin{aligned} (1)\quad B^1_\phi(\phi) &= 0\,, \quad \left(\left(\frac{e^{-\frac{\beta}{2}}}{r}\right)' + \frac{1}{r^2}e^{\frac{\beta}{2}}\right) \neq 0\,, \\ (2)\quad B^1_\phi(\phi) &\neq 0\,, \quad \left(\left(\frac{e^{-\frac{\beta}{2}}}{r}\right)' + \frac{1}{r^2}e^{\frac{\beta}{2}}\right) = 0\,. \end{aligned}$$

(**1**) In this subcase we have $B^1_\phi(\phi) = 0$ and $((e^{-\frac{\beta}{2}}/r)' + e^{\frac{\beta}{2}}r^2) \neq 0$. Equation $B^1_\phi(\phi) = 0 \Rightarrow B^1(\phi) = c_1$, where $c_1 \in R$ thus we have (from (24))

$$\begin{aligned} X^1 &= ce^{-\frac{\beta}{2}}\int e^{\frac{\beta}{2}}dr + e^{-\frac{\beta}{2}}(c_1\sin\theta + B^2(\theta))\,, \\ X^2 &= -(c_1\cos\theta + B^2_\theta(\theta))\int \frac{1}{r^2}e^{\frac{\beta}{2}}dr + A^2(\theta,\phi)\,, \\ X^3 &= A^3(\theta,\phi)\,. \end{aligned}$$

A straightforward calculation shows that a homothetic vector field exists if and only if

$$\frac{1}{r}e^{-\frac{\beta}{2}}\int e^{\frac{\beta}{2}}dr = 1\,, \qquad \left(\Rightarrow \int e^{\frac{\beta}{2}}dr = re^{\frac{\beta}{2}}\right),$$

which upon differentiation with respect to r gives $\beta =$const. One then easily see from equation (11) that the rank of 6×6 Riemann matrix is reduced to 1 or zero, giving us a contradiction (since we are assuming that the rank of 6×6 Riemann matrix is 3). Thus no proper homothetic vector field exists in the induced geometry. Therefore, it admits Killing vector fields which are

$$\begin{aligned} X^1 &= 0\,, \\ X^2 &= (c_1 \sin\phi + c_2\cos\phi)\,, \\ X^3 &= \cot\theta(c_1\cos\phi - c_2\sin\phi) + c_3\,, \end{aligned} \tag{25}$$

where $c_1, c_2, c_3 \in R$. Thus, from (25) and (13) affine vector fields in this case are

$$\begin{aligned} X^0 &= (c_7 t + c_8)\,, \quad & X^2 &= (c_1\sin\phi + c_2\cos\phi)\,, \\ X^1 &= 0\,, & X^3 &= \cot\theta(c_1\cos\phi - c_2\sin\phi) + c_3\,. \end{aligned} \tag{26}$$

One can write the above equation after subtracting Killing vector fields as

$$X = (t, 0, 0, 0)\,. \tag{27}$$

Clearly, in this case the above space-times (12) admit proper affine vector fields.
(**2**) In this subcase we have $((e^{-\beta/2}/r)' + e^{\beta/2}/r^2) = 0 \Rightarrow e^{\beta} = a^2/(a^2 - r^2)$, where $a \in R - \{0\}$. The line element can, after a rescaling of t, be written in the form

$$ds^2 = -dt^2 + \frac{a^2}{a^2 - r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{28}$$

The above space-time is the well known Einstein static space-time. Affine vector fields in this case are [6]

$$\begin{aligned} X^0 &= (c_7 t + c_8)\,, \\ X^1 &= \sqrt{\left(1 - \frac{r^2}{a^2}\right)}(\sin\theta(c_2\cos\phi + c_3\sin\phi) + c_1\cos\theta)\,, \\ X^2 &= \frac{1}{r}\sqrt{\left(1 - \frac{r^2}{a^2}\right)}(\cos\theta(c_2\cos\phi + c_3\sin\phi) - c_1\sin\theta) + (c_4\cos\phi + c_5\sin\phi)\,, \\ X^3 &= \frac{1}{r\sin\theta}\sqrt{\left(1 - \frac{r^2}{a^2}\right)}(-c_2\sin\phi + c_3\cos\phi) + \cot\theta(-c_4\sin\phi + c_5\cos\phi) + c_6\,, \end{aligned} \tag{29}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \in R$. One can write the above equation (29) after subtracting Killing vector fields as in (27). This completes the case (C).

Case D

In this case $\alpha_1 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0, \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_6 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exists a unique (up to a multiple) number with zero vector field $r_a = r_{,a}$ solution of equation (8) and $r_{a;b} \neq 0$. From the above constraints we have $A_r(t,r) \neq 0$, $B_r(t,r) = 0$, $B_t(t,r) = 0$ and $A_r^2(t,r) + 2A_{rr}(t,r) = 0$. Equations $B_r(t,r) = 0$, $B_t(t,r) = 0$ and $A_r^2(t,r) + 2A_{rr}(t,r) = 0$ implay $B(t,r) = \lambda(\neq 0, 1) \in R$ and $A = \ln(rU_1(t) + U_2(t))^2$, where $U_1(t)$ and $U_2(t)$ are numbers where functions of integration vanish. The subcase when $U_2(t) = 0$ will be considered later. The line element can be written in the form

$$ds^2 = -\left(rU_1(t) + U_2(t)\right)^2 dt^2 + e^{\lambda} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{30}$$

Substituting the above information into the affine equations and after some calculation one finds that affine vector fields in this case are

$$\begin{aligned} X^0 &= 0, \qquad X^2 = (c_1 \sin\phi + c_2 \cos\phi), \\ X^1 &= 0, \qquad X^3 = \cot\theta(c_1\cos\phi - c_2\sin\phi) + c_3, \end{aligned} \tag{31}$$

where $c_1, c_2, c_3 \in R$. Affine vector fields in this case are Killing vector fields.

Now consider the special case when $U_2(t) = 0$. The line element can, after a rescaling of t, be written in the form

$$ds^2 = -r^2dt^2 + e^{\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{32}$$

Affine vector fields in this case are

$$\begin{aligned} X^0 &= c_4, \qquad X^2 = (c_1 \sin\phi + c_2 \cos\phi), \\ X^1 &= c_5 r + c_6, \qquad X^3 = \cot\theta(c_1\cos\phi - c_2\sin\phi) + c_3, \end{aligned} \tag{33}$$

where c_1, c_2, c_3, c_4, c_5, $c_6 \in R$. One can write the above equation (33) after subtracting Killing vector fields as

$$X = (0, c_5 r + c_6, 0, 0). \tag{34}$$

Clearly, in this case the above space-times (32) admit proper affine vector fields.

Case E

In this case $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0, \alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exists no non-trivial solution of equation (8). From the above constraints we get $B(t,r) = 0$, $A_r(t,r) \neq 0$ and $A_r^2(t,r) + 2A_{rr}(t,r) \neq 0$. The line element can be written in the form

$$ds^2 = -e^{A(t,r)}dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{35}$$

Substituting the above information into the affine equations and after some calculation one finds that affine vector fields in this case are Killing vector fields which are given in equation (31).

Case F

In this case we have $\alpha_1 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0, \alpha_2 \neq 0, \alpha_3 \neq 0$, the rank of the 6×6 Riemann matrix is 2 and there exists a unique (up to a multiple) number with zero vector field $r_a = r_{,a}$ solution of equation (8) and $r_{a;b} \neq 0$. From the above constraints we have $A_r(t,r) \neq 0$, $B(t,r) = 0$ and $A_r^2(t,r) + 2A_{rr}(t,r) = 0$. Equation $A_r^2(t,r) + 2A_{rr}(t,r) = 0$ implay $A = \ln(rP_1(t) + P_2(t))^2$, where $P_1(t)$ and $P_2(t)$ are numbers where functions of integration vanish. The subcase when $P_2(t) = 0$ will be consider later. The line element can be written in the form

$$ds^2 = -(rP_1(t) + P_2(t))^2 dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{36}$$

Substituting the above information into the affine equations and after some lengthy calculation one finds that affine vector fields in this case are given in equation (31).

Now consider the special case when $P_2(t) = 0$. The line element can, after a rescaling of t, be written in the form

$$ds^2 = -r^2 dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{37}$$

Using the above information into the affine equations one finds that affine vector fields in this case are given in equation (33).

Case G

In this case we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0, \alpha_6 \neq 0$ and the rank of the 6×6 Riemann matrix is 1. From the above constraints we have $A_r(t,r) = 0$, $B_r(t,r) = 0$ and $B_t(t,r) = 0$. Equations $B_r(t,r) = 0$ and $B_t(t,r) = 0 \Rightarrow B(t,r) = \eta(\neq 0,1) \in R$. Here, there exist two linear independent solutions $t_a = t_{,a}$ and $r_a = r_{,a}$ of equation (8). The vector field t_a is covariantly constant, whereas r_a is not covariantly constant. The line element can, after a rescaling of t, be written in the form

$$ds^2 = -dt^2 + e^{\eta} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{38}$$

Affine vector fields in this case are

$$\begin{aligned} X^0 &= c_7 t + c_8 r + c_9 , \qquad X^2 = (c_1 \sin\phi + c_2 \cos\phi) , \\ X^1 &= c_4 t + c_5 r + c_6 , \qquad X^3 = \cot\theta(c_1 \cos\phi - c_2 \sin\phi) + c_3 , \end{aligned} \tag{39}$$

where c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, $c_9 \in R$. One can write the above equation (39) after subtracting Killing vector fields as

$$X = (c_7 t + c_8 r + c_9, c_4 t + c_5 r + c_6, 0, 0) . \tag{40}$$

Clearly, in this case the above space-time (38) admits proper affine vector fields.

4. Summary

In this paper a study of non-static spherically symmetric space-times according to their proper affine vector fields is given. A different approach is adopted to study proper affine vector fields in the above space-times by using holonomy and decomposability, the rank of the 6×6 Riemann matrix and direct integration techniques. From the above we obtain the following results:

(i) The case when the rank of the 6×6 Riemann matrix is three and there exists a unique number with zero independent time-like vector field which is a solution of equation (8) and is covariantly constant. This is the space-time (12) and (28) and it admits proper affine vector fields (see case C).

(ii) The case when the rank of the 6×6 Riemann matrix is three or two and there exists a unique number with zero independent vector field which is a solution of equation (8) and is not covariantly constant. This is the space-time (30) and (36) and it admits affine vector fields which are Killing vector fields (for details see cases D and F).

(iii) The case when the rank of the 6×6 Riemann matrix is three or two and there exists a number with zero independent vector field which is the solution of equation (8) and is not covariantly constant. These are the space-times (32) and (37) and they admit proper affine vector fields (see equation (33)).

(iv) The case when the rank of the 6×6 Riemann matrix is one and there exist two numbers with zero independent solution of equation (8) but only one independent covariantly constant vector field. This is the space-time (38) and it admits proper affine vector fields (see case G).

(v) The case when the rank of the 6×6 Riemann matrix is three and there exist a number of non-trivial solution of equation (8). This is the space-time (35) and it admits affine vector fields which are Killing vector fields (for details see case E).

REFERENCES

[1] J.F. Schell, *J. Math. Phys.* **2**, 202 (1961).
[2] G.S. Hall, D.J. Low, J.R. Pulham, *J. Math. Phys.* **35**, 5930 (1994).
[3] G.S. Hall, W. Kay, *J. Math. Phys.* **29**, 420 (1988).
[4] H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselears, E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2003.
[5] G. Shabbir, *Class. Quantum Grav.* **21**, 339 (2004).
[6] G. Shabbir, *Differential Geometry-Dynamical Systems* **8**, 244 (2006).
[7] G.S. Hall, *World Scientific* 2004.