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CONNECTIONS BETWEEN VON FOERSTER COALITION GROWTH MODEL AND TSALLIS q-EXPONENTIAL

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The paper will show the direct connections between q-Malthusian growth model, *i.e.* the classical exponential growth model generalized in Tsallis statistics, and the coalition growth model introduced by von Foerster in 1960. As it will turn out, the equations that have been taken into account when the coalition model was introduced, are closely related to the equations that describe the non-extensive generalization of entropy.

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1. Introduction

The problem of any population growth/fall can be expressed by many different equations. The development of population P, which lives in the environment with unlimited resources and when the population growth/fall rate w is always proportional to the current size of population, can be given with the use of the exponential law

$$P(t) = P(0)\exp(wt)\,, \tag{1}$$

where $P(0)$ denotes the initial value and t is time.

This model is sometimes called the Malthusian one and it is based on the classical definition of exponential, $\exp(\cdot)$. In a continuous form this simple exponential growth model is based on a constant rate of growth/fall w, which represents the difference between the fertility b and the mortality d *i.e.* $w = b - d = \text{const}$. But taking into account the Tsallis definition of the q-exponential [1] the equation (1) can be generalized by the definition of $\exp_q(x)$ and the q-generalized Malthusian growth model in Tsallis statistics[1]

[1] The notation $P_{(q)}$ means that this is a non-extensive q-generalization.

is given by [2]

$$P_{(q)}(t) = P_{(q)}(0) \exp_q(w_q t) \tag{2}$$

as the solution of equation

$$\frac{dP_{(q)}(t)}{dt} = w_q \left(P_{(q)}(t)\right)^q , \tag{3}$$

where in the simplest approach $w_q = \mathrm{const}$. The term $\exp_q(\cdot)$ mentioned above was proposed by Tsallis in [1] and is given by

$$\exp_q(x) = [1 + (1-q)\,x]^{1/(1-q)} \quad \text{for} \quad 1 + (1-q)x > 0, \quad q \in R \tag{4}$$

together with its inverse function, called q-logarithm defined as [1,3]

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q} \quad \text{for} \quad x \geq 0 \tag{5}$$

as the basis of Tsallis entropy definition.

2. Coalition growth model

In the study of the behavior of biological (but not only) populations the attempts made to estimate the number of its elements in the past and in the future are one of the most important problems. Usually the number of population elements under conditions, that are stable and unchanged during time, depends at least on two things: the *birth* b and the *death* d rates. In addition, when the population lives in the environment with the resources that are unlimited or their number is relatively high its development can be described by Malthusian growth model represented by (1). However, in such a situation a tacit assumption is made: all elements in population behave identically and undergo the same phenomena (*e.g. equivivant population* [4]). But in reality the situation can be different — two parameters, *i.e.* b and d describing the fertility and the mortality, may vary from element to element and can also depend on: the age of particular elements, hazards in the environment, competition between elements for a limited food supply, the abundance of predators or prey, communication between elements, *etc.* In Prigogine book [5] there is a very interesting example about the behavior of great spruce bark beetle (lat. *Dendroctonus micans*), whose larvas group in big clusters in some cases. This is due to the existence of long-range correlations between them and as a result one can see the larvas self-organization — they made local clusters. Prigogine states that in thermodynamics this situation can be related to the non-equilibrium, which is *the source of order*. A similar problem has appeared in the case of information handling capacity in computer systems. Moravec noted that it had been growing about ten million times faster than it did in nervous systems during our evolution [6]. The computer systems power doubled every 24 months in the 1950s, 1960s and 1970s, doubled every 18 months in the 1980s, and is now doubling each 12 months. The numerical data collected by Moravec and his cooperators showed that the growth of computer systems power is faster than the exponential one: in three decades the doubling time has fallen from two years to one year. This observation was also presented in [7,8] where the generalized version of Moore's Law, which studies the increase of computing speed amount[2] that can be bought at a fixed price, was considered.

In 1960 Heinz von Foester *et al.* supposed that the classical exponential growth in the case of human population should be changed because elements in the system (people) can form coalitions that for example can grow faster [4]. In such a situation the equation (1) becomes obsolete and nothing can be said about the long-term behavior of analyzed population. As an example let us consider data (Table I) given by the U.S. Census Bureau [9].

TABLE I

World human population.

Year	Population (millions)	Year	Population (millions)
1000	200	1955	2780
1650	545	1960	3005
1750	728	1965	3345
1800	906	1970	3707
1850	1171	1975	4086
1900	1608	1980	4454
1910	1750	1985	4850
1920	1834	1990	5263
1930	2070	1995	5674
1940	2295	2000	6070
1950	2517	2005	6453

Plotting data on the figure it can be seen that the real growth is faster than the exponential one (Fig. 1).

Von Foerster assumed that the coalition growth model should be based on equation [4]

$$\frac{dP}{dt} = kP^{1+r}\,, \tag{6}$$

[2] Expressed as Million Instructions Per Second — MIPS.

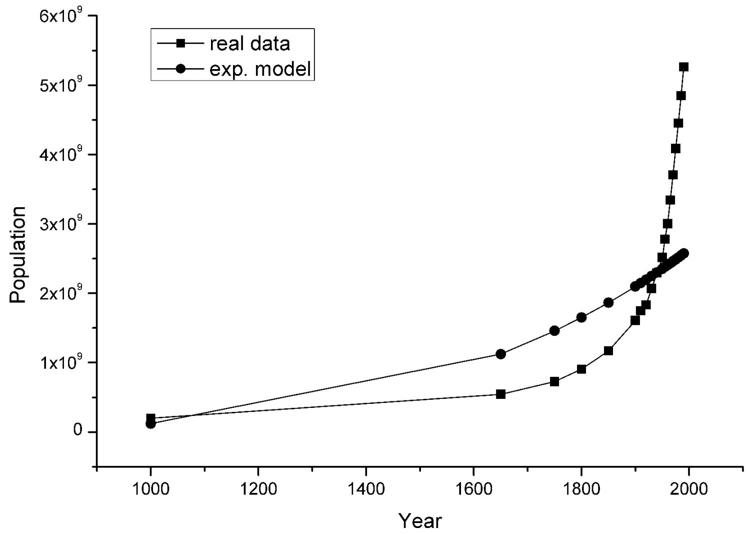

Fig. 1. World human population — the exponential model with $w = 0.0035286$ and $P(0) = 3 \times 10^6$ is represented by continues line with filled dots, while the real data are represented by continuous line with filled squares. As one can see the real data are not well approximated by the classical Malthusian model.

where k and r are positive constants. In this model the productivity (*i.e.* dP/dt) is not constant but is an increasing function of P (it depends on value of k — if $k < 0$ then it will be the decreasing function) and r is presumably small[3].

The solution of equation (6) can be written as

$$P(t) = \frac{1}{(rk\,(T-t))^{\frac{1}{r}}} \tag{7}$$

for some constant T that represents a finite time, at which the population P becomes infinite or would if the growth pattern continues follow the coalition model. von Foerster calculated T value, which appeared in the title of his article: 13 November A.D. 2023. This time was called *Doomsday* [4]. Basing on actualized data given in Table I and assuming, according to von Foerster calculations, that $T = 2026.8683$ the values of r and k for equation (7) were once again calculated. Now they are respectively: $r \cong 1.09$ and $k \cong 7.8 \times 10^{-13}$. In the original von Foerster's article these values were: $r = 0.99$ and $k = 5.5 \times 10^{-12}$, but they were calculated in 1960 and, as it can be seen, since then they have changed a little bit.

[3] For $r = 0$ model reduces to the Malthusian one.

The q-Malthusian growth model given by equation (2), which taking into account equation (4) and assuming that $w_q(t) = w = \text{const}$ can be written as:

$$P_{(q)}(t) = P_{(q)}(0)\left(1 + (1-q)\, wt\right)^{\frac{1}{1-q}} \tag{8}$$

or

$$P_{(q)}(t) = \left(\left(P_{(q)}(0)\right)^{1-q} + \left(P_{(q)}(0)\right)^{1-q}(1-q)\, wt\right)^{\frac{1}{1-q}}, \tag{9}$$

while equation (7) in the following form

$$P(t) = \left(rk\,(T-t)\right)^{-\frac{1}{r}} = (rkT - rkt)^{-\frac{1}{r}}. \tag{10}$$

Let us assume that $r = q - 1$, thus equation (10) will be

$$P(t) = ((q-1)kT + (1-q)kt)^{\frac{1}{1-q}} \tag{11}$$

and as it can be seen between equations (9) and (11) exists the direct relation.

Because r, k and T are given, the value of $P_{(q)}(0)$ (initial condition) in the equation (8) can be computed as:

$$P_{(q)}(0) = (rkT)^{-\frac{1}{r}} = ((q-1)kT)^{\frac{1}{1-q}} \tag{12}$$

and in analyzed case it is $\approx 11 \times 10^7$. This value can also be obtained from (7) when $t = 0$.

Having $P_{(q)}(0)$ it is possible to obtain w for equation (8) *i.e.* $w = \frac{k}{\left(P_{(q)}(0)\right)^{1-q}} = \frac{1}{rT}$ and in the analyzed case it is ≈ 0.000452635.

However, the relations presented above (especially equation (12)) raise one very important question: how one can make the interpretation of the relation (12) for an initial condition. Usually an initial condition is given at first regardless of environment of human population growth and presented relation might suggest that $P_{(q)}(0)$ determines von Foerster Doomsday T or T determines $P_{(q)}(0)$. The first interpretation of this doubt is rather natural, because the analysis presented above shows the existing connections between q-Malthusian model and von Foerster proposal and equation (12) shows how one can compute $P_{(q)}(0)$ having r, k and T to compare both models.

The second solution of this situation can be given as follows. To avoid the problem of initial condition interpretation let us see that the q-Malthusian growth model given by equation (2) is related to the differential equation (3), whose solution can be also given by (if $w_q = k$):

$$\ln_q P = kt + C\,, \tag{13}$$

where C is a constant determined by a given initial condition. Then the solution of the differential equation (3) taking into account equation (13) is

$$P = \exp_q(kt + C) = \exp_q(C)\exp_q\left(\frac{k}{\left(\exp_q(C)\right)^{1-q}}t\right). \tag{14}$$

Finally comparing the solution (14) with equation (2) when $w_q = k$ it can be seen that $P_q(0) = \exp_q(C)$ and the initial condition can be also given independently of any parameters of the population growth model proposed by von Foerster.

Reassuming, the von Foerster coalition growth model, which is given by the differential equation (6), compared to equation (3) shows that $r = q - 1$ if $w_q = k$. Thus the solution of equation (7) can be rewritten to the following form

$$\begin{aligned} P &= \frac{1}{\left(rk\left(T-t\right)\right)^{\frac{1}{r}}} \\ &= \left((1-q)k(t-T)\right)^{\frac{1}{1-q}} \\ &= \exp_q\left(\frac{(1-q)k(t-T)-1}{1-q}\right) \\ &= \exp_q\left(kt - \frac{\left(\exp_q(kT)\right)^{1-q}}{1-q}\right), \end{aligned} \tag{15}$$

and as it can be seen the existing relations between q-Malthusian model and the von Foerster growth model can be confirmed once again without the problem of $P_{(q)}(0)$ interpretation.

3. Conclusions

In the presented paper the non-extensive generalization of Mathusian growth model was given with the simple application to the von Foerster growth model obtained in 1960. The q-generalization of exponential, $\exp_q(x)$, given by Tsallis provides a possibility to build a simple model of growth that can be applied to describe the long-term behavior of human population (see Fig. 2).

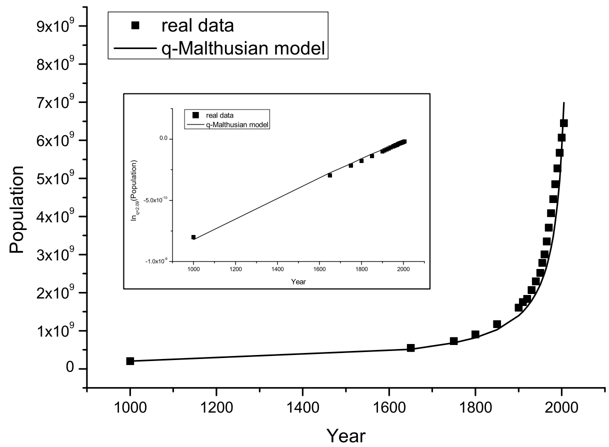

Fig. 2. Population growth expressed by q-Malthusian (von Foerster) model. The estimated value of $r \approx 1.09$ *i.e.* $q = 2.09$, $k \approx 7.8 \times 10^{-13}$ ($w = 0.000452635$) and the initial value $P_{(q)}(0) \approx 11 \times 10^7$. The inset shows the same data but in q-log-lin scale with $q = 2.09$.

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