LOGARITHMIC KLEIN–GORDON EQUATION

Przemysław Górka

Department of Mathematics and Information Sciences
Warsaw University of Technology
Pl. Politechniki 1, 00-661 Warsaw, Poland
pgorka@mini.pw.edu.pl

(Received August 28, 2008; revised version received November 6, 2008)

We study weak solutions to the Klein–Gordon equation with the logarithmic nonlinearity on interval. Such kinds of nonlinearities appear in inflation cosmology and in supersymmetric field theories. Moreover, this framework is applied in nuclear physics, optics, and geophysics. We obtained the existence of weak solutions. For this purpose the Galerkin method, logarithmic Sobolev inequality and compactness theorem are applied.

PACS numbers: 03.65.Pm, 02.30.Jr, 02.30.Sa

1. Introduction

In this paper we shall deal with the initial-boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= -u + \varepsilon u \log |u|^2 , \qquad x \in \mathcal{O}, t \in (0,T) , \\ u(x,0) &= u_0(x) , \qquad u_t(x,0) = u_1(x) , \qquad x \in \mathcal{O} , \\ u(x,t) &= 0 , \qquad x \in \partial\mathcal{O}, t \in (0,T) , \end{aligned} \tag{1}$$

where $\mathcal{O}$ is a finite interval $\mathcal{O} = [a,b]$, and parameter $\varepsilon \in [0,1]$ is fixed.

The problem (1) is a relativistic version of logarithmic quantum mechanics introduced by Białynicki-Birula and Mycielski (see [1,2]). The parameter ε measures the force of the nonlinear interactions. It has been shown experimentally (see [3–5]), that the nonlinear effects in quantum mechanics are very small. The upper bound for the parameter ε has been estimated, namely $|\varepsilon| \leq 3.3\, 10^{-15}$. Still, this framework is applied in many branches of physics, *e.g.* nuclear physics, optics, geophysics (see *e.g.* [6–9]).

This nonlinearity is selected by assuming the separability of noninteracting subsystems property (see [1,2]). It means that a solution of the nonlinear equation for the whole system can be constructed, as in the linear theory, by taking the product of two arbitrary solutions of the nonlinear equations for the subsystems. In other words, for noninteracting subsystems no correlations are introduced. Moreover, this is unique nonlinear theory which poses that property. Its most attractive feature are: existence of the lower energy bound and validity of Planck's formula $E = \hbar\omega$. The Born interpretation of the wave function is consistent with logarithmic nonlinearity.

The Klein–Gordon equation with logarithmic potential has been also introduced in the quantum field theory by Rosen [10]. Such kinds nonlinearity appear naturally in inflation cosmology and in supersymmetric field theories (see [11–13]).

The logarithmic quantum mechanics posses some special analytic solutions (see [14–16]). For example, this model has a large set of oscillating localized solutions. In the paper [15] the authors studied the so-called Gaussons. Gaussons represent solutions of the Gaussian shape. Moreover, the interaction of Gaussons was studied [17]. Using the Bohr–Sommerfeld quantization of localized solutions the mass spectrum of the localized particle-like collective excitations have been found [18].

Let us finally comment on the relevant mathematical literature. The evolution problem (1) in three dimensions was treated mathematically by Cazenave and Haraux (see [19]). Assuming high regularity of the initial data, they have shown global existence in time of the distributional solutions. Let us mention, that the logarithmic Schrödinger equation was also discussed in the mathematical literature (see [19–22]).

The goal of the present paper is to establish the existence of a weak solution to the problem (1). We show this in a few steps. First of all, we write this system in a weak form. Next, we shall construct approximate solutions. Subsequently, using compactness tools we shall show the convergence of the sequence to the solution of the problem. Let us stress that the Gross logarithmic Sobolev inequality and logarithmic Gronwall inequality, will be fundamental here. Namely, it will play a crucial role in *a priori* estimates. Let us also mention that this theorem was announced in the paper [23].

Notation. Throughout the paper, we shall try to use the standard notation. Moreover, we use the following convention: whenever we see the inequality $\mathbb{A} \leq c\mathbb{B}$ we tacitly understand, that it holds with some positive constant c independent of $\mathbb{A}$ and $\mathbb{B}$.

2. Main result

The process of multiplication of equation (1) by a test function ϕ and integration by parts leads us to the following definition:

Definition 1. *We shall say that u is a weak solution to the problem (1) if and only if*

$$\begin{aligned} u &\in L^\infty\left(0,T;H_0^1(\mathcal{O})\right), \\ u_t &\in L^\infty\left(0,T;L^2(\mathcal{O})\right), \\ u_{tt} &\in L^\infty\left(0,T;H^{-1}(\mathcal{O})\right) \end{aligned}$$

and the following identity holds

$$\langle u_{tt},\phi\rangle + \int\limits_{\mathcal{O}} \nabla u \nabla \phi dx = -\int\limits_{\mathcal{O}} u\phi dx + \varepsilon \int\limits_{\mathcal{O}} u \log|u|^2 \phi dx\,,$$

for each $\phi \in H_0^1(\mathcal{O})$ and for almost each $t \in (0,T]$ and u satisfies the initial conditions, i.e.

$$u(0) = u_0\,, \quad u_t(0) = u_1\,.$$

Where $\langle .,.\rangle$ is a pairing between $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$.

Remark 1. *If u is a weak solution to the problem (1), then $u \in C\left(0,T;L^2(\mathcal{O})\right)$ and $u_t \in C\left(0,T;H^{-1}(\mathcal{O})\right)$.*

Now, we can formulate the main result of the paper.

Theorem 1. *Suppose that $u_0 \in H_0^1(\mathcal{O})$ and $u_1 \in L^2(\mathcal{O})$. Then there exists a weak solution to the problem (1).*

Proof. **1. Approximate problem.**
It will be done by the Galekin method. Let $\{w_i\}_{i=1}^\infty$ be a basis of $H_0^1(\mathcal{O})$ and V_n the subspace spanned by n first vectors $w_1, w_2, \ldots, w_n$. Let u_m be the function

$$u_m(t) = \sum_{j=1}^{m} g_m^j(t) w_j\,,$$

defined by the solution of the system

$$\begin{aligned} &\left(u_m'', w_j\right)_{L^2(\mathcal{O})} + \left(\nabla u_m, \nabla w_j\right)_{L^2(\mathcal{O})} = \\ &-\left(u_m, w_j\right)_{L^2(\mathcal{O})} + \varepsilon\left(u_m \log|u_m|^2, w_j\right)_{L^2(\mathcal{O})}\,, \qquad \forall j = 1,\ldots,m \\ &u_m(0) = \sum_{j=1}^{m} \left(u_0, w_j\right) w_j\,, \\ &u_m'(0) = \sum_{j=1}^{m} \left(u_1, w_j\right) w_j\,. \end{aligned} \tag{2}$$

The system (2) has a solution u_m defined in $[0, t_m)$, where $0 < t_m \le T$. The *a priori* estimates to be obtained in the following step, in particular show, that $t_m = T$.

2. A priori estimates.

By multiplying both sides of (2) by ${g_m^j}'$, and adding from $j = 1$ to m we obtain:

$$\frac{d}{dt}\left(\|u_m'\|^2_{L^2(\mathcal{O})} + \|\nabla u_m\|^2_{L^2(\mathcal{O})} + (1+\varepsilon)\|u_m\|^2_{L^2(\mathcal{O})} \right.$$
$$\left. -2\varepsilon \int_{\mathcal{O}} |u_m|^2 \log|u_m|\,dx \right) = 0\,.$$

Hence

$$\begin{aligned}\|u_m'(t)\|^2_{L^2(\mathcal{O})} \quad &+\|\nabla u_m(t)\|^2_{L^2(\mathcal{O})} + (1+\varepsilon)\|u_m(t)\|^2_{L^2(\mathcal{O})}\\ &-2\varepsilon \int_{\mathcal{O}} |u_m(t)|^2 \log|u_m(t)|dx = \|u_m'(0)\|^2_{L^2(\mathcal{O})}\\ &+\|\nabla u_m(0)\|^2_{L^2(\mathcal{O})} + (1+\varepsilon)\|u_m(0)\|^2_{L^2(\mathcal{O})}\\ &-2\varepsilon \int_{\mathcal{O}} |u_m(0)|^2 \log|u_m(0)|dx \le C\left(\|u_1\|^2_{L^2(\mathcal{O})} + \|u_0\|^2_{H_0^1(\mathcal{O})}\right)\\ &-2\varepsilon \int_{\mathcal{O}} |u_m(0)|^2 \log|u_m(0)|dx\,.\end{aligned}$$

Since $u_m(0) \to u_0$ in the space $H_0^1(\mathcal{O})$, we obtain, that the sequence $u_m(0)$ is bounded in $H_0^1(\mathcal{O})$. Next, the Sobolev embedding $H_0^1(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ yields

$$\left| \int_{\mathcal{O}} |u_m(0)|^2 \log|u_m(0)|\,dx \right| \le c\,.$$

Subsequently, we can write

$$\begin{aligned}&\|u_m'(t)\|^2_{L^2(\mathcal{O})} + \|\nabla u_m(t)\|^2_{L^2(\mathcal{O})} + (1+\varepsilon)\|u_m(t)\|^2_{L^2(\mathcal{O})}\\ &\le C + 2\varepsilon \int_{\mathcal{O}} |u_m(t)|^2 \log|u_m(t)|\,dx\,,\end{aligned} \tag{3}$$

where C depends on the initial data, namely $C = C(\|u_0\|_{H_0^1(\mathcal{O})}, \|u_1\|_{L^2(\mathcal{O})})$. In order to estimate the right hand side of the inequality (3) we need local version of the Gross logarithmic Sobolev inequality. Namely,

Lemma 1. *If $h \in H_0^1(\mathcal{O})$, then for each $a > 0$ the following estimate holds:*

$$\int_{\mathcal{O}} h^2 \log|h|\,dx \;\le\; \tfrac{1}{2}\log\|h\|^2_{L^2(\mathcal{O})}\,\|h\|^2_{L^2(\mathcal{O})} + a\|\nabla h\|^2_{L^2(\mathcal{O})} - \tfrac{1}{2}\left(1+\tfrac{1}{2}\log 2\pi a\right)\|h\|^2_{L^2(\mathcal{O})}. \tag{4}$$

Proof. First of all, we recall the logarithmic Sobolev inequality (see [24] and [25]). For each $g \in H^1(\mathbb{R})$ and for each $a > 0$, the following estimate holds:

$$\int_{\mathbb{R}} g^2 \log|g|\,dx \;\le\; \tfrac{1}{2}\log\|g\|^2_{L^2(\mathbb{R})}\,\|g\|^2_{L^2(\mathbb{R})} + a\|\nabla g\|^2_{L^2(\mathbb{R})} - \tfrac{1}{2}\left(1+\tfrac{1}{2}\log 2\pi a\right)\|g\|^2_{L^2(\mathbb{R})}\,. \tag{5}$$

Next, let us define the map g as follows:

$$g(x) = \begin{cases} h(x) & \text{if} \quad x \in \mathcal{O} \\ 0 & \text{if} \quad x \in \mathbb{R}\setminus\mathcal{O}. \end{cases}$$

Since $g \in H^1(\mathbb{R})$, we can apply inequality (5):

$$\begin{aligned}
&\int_{\mathcal{O}} h^2 \log|h|\,dx = \int_{\mathbb{R}} g^2 \log|g|\,dx \\
&\le \tfrac{1}{2}\log\|g\|^2_{L^2(\mathbb{R})}\,\|g\|^2_{L^2(\mathbb{R})} + a\|\nabla g\|^2_{L^2(\mathbb{R})} - \tfrac{1}{2}\left(1+\tfrac{1}{2}\log 2\pi a\right)\|g\|^2_{L^2(\mathbb{R})} \\
&= \tfrac{1}{2}\log\|h\|^2_{L^2(\mathcal{O})}\,\|h\|^2_{L^2(\mathcal{O})} + a\|\nabla h\|^2_{L^2(\mathcal{O})} - \tfrac{1}{2}\left(1+\tfrac{1}{2}\log 2\pi a\right)\|h\|^2_{L^2(\mathcal{O})}\,.
\end{aligned}$$

From this the proof of the lemma follows. □

Now, we can back to the proof of the main result. Namely, from lemma 1 we can estimate the right hand side of the inequality (3):

$$\begin{aligned}
&\|u'_m(t)\|^2_{L^2(\mathcal{O})} + (1-2a\varepsilon)\|\nabla u_m(t)\|^2_{L^2(\mathcal{O})} \\
&+ \left(1+\varepsilon\left(2+\tfrac{1}{2}\log 2\pi a\right)\right)\|u_m(t)\|^2_{L^2(\mathcal{O})} \\
&\le \varepsilon \log\|u_m\|^2_{L^2(\mathcal{O})}\,\|u_m\|^2_{L^2(\mathcal{O})}\,.
\end{aligned}$$

Taking $a = \frac{1}{4\varepsilon}$ we obtain:

$$\begin{aligned}
&\|u'_m(t)\|^2_{L^2(\mathcal{O})} + \|\nabla u_m(t)\|^2_{L^2(\mathcal{O})} + \|u_m(t)\|^2_{L^2(\mathcal{O})} \\
&\le C(\varepsilon)\log\|u_m\|^2_{L^2(\mathcal{O})}\,\|u_m\|^2_{L^2(\mathcal{O})}\,.
\end{aligned} \tag{6}$$

Subsequently, let us note:

$$u_m(t) = u_m(0) + \int_0^t \frac{\partial u_m}{\partial t}(s)\, ds\,.$$

Therefore,

$$\|u_m(t)\|^2_{L^2(\mathcal{O})} \leq 2\|u_0\|^2_{L^2(\mathcal{O})} + \max\{2T,1\}\left(\frac{1+C(\varepsilon)}{C(\varepsilon)}\right)\int_0^t \|u'_m(t)\|^2_{L^2(\mathcal{O})} ds\,.$$

Next, by inequality (6) we have:

$$\begin{aligned}\|u_m(t)\|^2_{L^2(\mathcal{O})} \ &\leq\ 2\|u_0\|^2_{L^2(\mathcal{O})} + \max\{2T,1\}\,(1+C(\varepsilon)) \\ &\quad\times \int_0^t \log \|u_m\|^2_{L^2(\mathcal{O})}\, \|u_m\|^2_{L^2(\mathcal{O})}\, ds\,.\end{aligned}$$

If we put $\tilde{C} = \max\{2T,1\}\,(1+C(\varepsilon))$, we obtain the inequality:

$$\|u_m(t)\|^2_{L^2(\mathcal{O})} \leq 2\|u_0\|^2_{L^2(\mathcal{O})} + \tilde{C}\int_0^t \log\left(\tilde{C} + \|u_m\|^2_{L^2(\mathcal{O})}\right) \|u_m\|^2_{L^2(\mathcal{O})}\, ds\,.$$

Since $\tilde{C} \geq 1$, we obtain, by logarithmic Gronwall inequality (see [19]), the estimate:

$$\|u_m(t)\|^2_{L^2(\mathcal{O})} \leq \left(\tilde{C} + 2\|u_0\|^2_{L^2(\mathcal{O})}\right) e^{\tilde{C}t} \leq C\,.$$

Hence, from inequality (6) follows:

$$\|u'_m\|^2_{L^2(\mathcal{O})} + \|\nabla u_m(t)\|^2_{L^2(\mathcal{O})} + \|u_m(t)\|^2_{L^2(\mathcal{O})} \leq C\,.$$

The *a priori* estimates shows, that $t_m = T$. Therefore,

$$\|u'_m\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} + \|u_m(t)\|^2_{L^\infty\left(0,T;H^1_0(\mathcal{O})\right)} \leq C\,. \tag{7}$$

Next, in the standard way, we obtain the estimate for u''_m:

$$\|u''_m\|^2_{L^\infty(0,T;H^{-1}(\mathcal{O}))} \leq C\,. \tag{8}$$

3 Passing to the limit.

From inequalities (7) and (8) follows the existence of a subsequence (still denoted by u_m), such that:

$$\begin{array}{lll}
u_m \rightharpoonup u & \text{weakly in} & L^2\left(0,T;H_0^1(\mathcal{O})\right), \\
u'_m \rightharpoonup u' & \text{weakly in} & L^2\left(0,T;L^2(\mathcal{O})\right), \\
u''_m \rightharpoonup u'' & \text{weakly in} & L^2\left(0,T;H^{-1}(\mathcal{O})\right), \\
u_m \rightharpoonup u & \text{weakly} - * \text{ in} & L^\infty\left(0,T;H_0^1(\mathcal{O})\right), \\
u'_m \rightharpoonup u' & \text{weakly} - * \text{ in} & L^\infty\left(0,T;L^2(\mathcal{O})\right), \\
u''_m \rightharpoonup u'' & \text{weakly in} & L^\infty\left(0,T;H^{-1}(\mathcal{O})\right), \\
u_m \to u & \text{strongly in} & L^2\left(0,T;L^2(\mathcal{O})\right),
\end{array}$$

where the last arrow follows from the Aubin–Lions lemma. As a consequence we obtain:

$$u_m \to u \quad \text{a.e. in } (0,T)\times\mathcal{O}\,.$$

Since the map $x\log|x|^2$ is continuous, we have the convergence:

$$\left|u_m\log|u_m|^2 - u\log|u|^2\right|^2 \to 0 \quad \text{a.e. in } (0,T)\times\mathcal{O}\,.$$

By the Sobolev embedding theorem we obtain that $\left|u_m\log|u_m|^2 - u\log|u|^2\right|^2$ is bounded in $L^\infty((0,T)\times\mathcal{O})$. Next, taking into account Lebesgue dominated convergence theorem, we obtain:

$$u_m\log|u_m|^2 \to u\log|u|^2 \quad \text{strongly in } L^2\left(0,T;L^2(\mathcal{O})\right)\,.$$

Finally, we can pass to the limits in a standard way. Next, one can easily check that the initial conditions are fulfilled. This finishes the proof of the theorem. □

The author wish to thank to the anonymous referee for invaluable suggestions and comments.

REFERENCES

[1] I. Białynicki-Birula, J. Mycielski, *Bull. Acad. Pol. Sc.* **23**, 461 (1975).

[2] I. Białynicki-Birula, J. Mycielski, *Ann. Phys.* **100**, 62 (1976).

[3] R. Gähler, A.G. Klein, A. Zeilinger, *Phys. Rev.* **A23**, 1611 (1981).

[4] A. Shimony, *Phys. Rev.* **A20**, 394 (1979).

[5] C.G. Shull, D.K. Atwood, J. Arthur, M.A. Horne, *Phys. Rev. Lett.* **44**, 765 (1980).

[6] H. Buljan, A. Šiber, M. Soljačic, T. Schwartz, M. Segev, D.N. Christodoulides, *Phys. Rev.* **E68**, 036607 (2003).

[7] S. De Martino, M. Falanga, C. Godano, G. Lauro, *Europhys. Lett.* **63**, 472 (2003).

[8] E.F. Hefter, *Phys. Rev.* **A32**, 1201 (1985).

[9] W. Królikowski, D. Edmundson, O. Bang, *Phys. Rev.* **E61**, 3122 (2000).

[10] G. Rosen, *Phys. Rev.* **183**, 1186 (1969).

[11] A. Linde, *Phys. Lett.* **B284**, 215 (1992).

[12] K. Enqvist, J. McDonald, *Phys. Lett.* **B425**, 309 (1998).

[13] J.D. Barrow, P. Persons, *Phys. Rev.* **D52**, 5576 (1995).

[14] E.M. Maslov, *Phys. Lett.* **A151**, 47 (1990).

[15] I. Białynicki-Birula, J. Mycielski, *Phys. Scri.* **20**, 539 (1979).

[16] V.A. Koutvitsky, E.M. Maslov, *J. Math. Phys.* **47**, 022302 (2006).

[17] V.G. Makhankov, I.L. Bogolubsky, G. Kummer, *Phys. Scri.* **23**, 767 (1981).

[18] I.L. Bogolubsky, *Zh. Eksp. Teor. Fiz.* **76**, 422 (1979).

[19] T. Cazenave, A. Haraux, *Ann. Fac. Sci. Toulouse Math.* **2**, 21 (1980).

[20] T. Cazenave, A. Haraux, *C.R. Acad. Sci. Paris Sér. A-B* **288**, 253 (1979).

[21] P. Górka, *Found. Phys. Lett.* **19**, 591 (2006).

[22] P. Górka, *Lett. Math. Phys.* **81**, 253 (2007).

[23] K. Bartkowski, P. Górka, *J. Phys. A* **41**, 355201 (2008).

[24] L. Gross, *Amer. J. Math.* **97**, 1061 (1975).

[25] E. Carlen, *J. Func. Anal.* **101**, 194 (1991.