REISSNER–NORDSTROM BLACK HOLE IN NONCOMMUTATIVE SPACES

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We investigate the behaviour of a non-commutative radiating Reissner–Nordstrom(Re–No)black hole. We find some interesting results: (a) the existence of a minimal non-zero mass to which the black hole can shrink, (b) a finite maximum temperature that the black hole can reach before cooling down to absolute zero, (c) compared to the neutral black holes the effect of charge is to increase the minimal non-zero mass and lower the maximum temperature, (d) the absence of any curvature singularity. We also derive some essential thermodynamic quantities from which we study the stability of the black hole. Finally we find an upper bound for the non-commutativity parameter θ.

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1. Introduction

It is generally believed that the picture of continuous space-time should break down at very short distances of the order of the Planck length. Field theories on noncommutative spaces may play an important role in unraveling the properties of nature at the Planck scale. It has been shown that the noncommutative geometry naturally appears in string theory with a non zero antisymmetric B-field.

Beside the string theory arguments the noncommutative field theories are very interesting on their own right. In a noncommutative space-time the coordinate operators satisfy the relation

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu}\,, \tag{1}$$

where $\hat{x}$ are the coordinate operators and $\theta^{\mu\nu}$ is an antisymmetric tensor of dimension $(\text{length})^2$. In general noncommutative version of a field theory is obtained by replacing the product of the fields appearing in the action by the star products

$$(f \star g)(x) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}\right) f(x)g(y)\mid_{x=y}\,, \tag{2}$$

where f and g are two arbitrary functions that we assume to be infinitely differentiable.

In recent years there have been a lot of work devoted to the study of noncommutative field theory and noncommutative quantum mechanics, and possible experimental consequences of extensions of the standard formalism (see the reviews [1] and references therein). Apart from this there has been also a growing interest in possible cosmological consequences of space non-commutativity. Here we focus on Reissner–Nordstrom black hole in noncommutative spaces.

In a recent work [2], the authors studied the Re–No black hole in noncommutative spaces. They argued that using commutation relations (1) and coordinate transformation $x_i = \hat{x}_i + \frac{1}{2}\theta_{ij}\hat{p}_j$, $p_i = \hat{p}_i$, where p_i and x_i satisfy the usual commutation relations of quantum mechanics, the Re–No black hole can be extended to noncommutative spaces. By a substitution of radial coordinate in terms of its noncommutative equivalent $r \rightarrow \hat{r} = \hat{x}_i\hat{x}_i$, the authors derived a line element for Re–No black hole in a noncommutative space and studied its thermodynamics. The main problem regarding their line element is: It does not seem to be solution of Einstein's equations. Then the question arises that what is the relevant equation, and what are the definition of energy and temperature for this new equation? There seems to be no modified Einstein's equations in this case, so the physical relevance of the resulting line element is obscure. Another unclear point is that once $\hat{r}$ is written in terms of the matrix θ^{ij} and the conventional position operators x_i and momenta p_i, ds^2 is far from what we mean by a line element.

Another important point is that the proposed line element (see Sec. 4 in [2]) exhibits, by the presence of the charge, a behaviour worse than $1/r^4$, with an inconsistent spherical symmetry breaking. And finally one more unconvincing result regarding this perturbative expansion (in θ parameter) approach is that curvature singularities continue to exist in spite of introducing a minimal length. Coordinate noncommutativity implies the existence of a finite minimal length $\sqrt{\theta}$, below which concept of "distance" becomes physically meaningless. This underlines the problem to define the line element, namely the infinitesimal distance between two nearby points in Einstein's gravity.

In Sec. 2 we study the Re–No black hole in noncommutative spaces that solve the above mentioned inconsistencies. We begin by a brief review of Re–No black hole in commutative spaces.

A spherically symmetric solution of the coupled Einstein's and Maxwell equations is that of Reissner and Nordstrom, which represents a black hole with mass M and charge Q.

The metric of the Re–No black hole is given by[1]:

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} - r^2 d\Omega^2 \,. \tag{3}$$

There are two apparent singularities at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}\,, \tag{4}$$

provided $M \geq Q$. Cosmic censorship dictates this inequality, and hence there is an external event horizon at r_+. The other horizon r_- is the internal Cauchy horizon. The limiting case when $Q = M$ and $r_+ = r_-$ is referred to as the extremal case.

2. Reissner–Nordstrom black hole in noncommutative spaces

To analyze black holes in the framework of noncommutative spaces one has to solve corresponding field equations. It is argued [3,4] that it is not necessary to change the Einstein's tensor part of the field equations, and that the noncommutative effects act only on the matter source. The underlying philosophy of this approach is to modify the distribution of point like sources in favor of smeared objects. This is in agreement with the conventional procedure for the regularization of UV divergences by introducing a cut off. Thus we conclude that in general relativity, the effect of noncommutativity can be taken into account by keeping the standard form of the Einstein's tensor in the left-hand side of the field equation and introducing a modified energy momentum tensor as a source in the right-hand side. This is exactly the gravitational analogue of the noncommutative modification of quantum field theory [5]. For the reasons mentioned in the previous section, we have developed an effective approach where noncommutativity is implemented only through a Gaussian de-localization of matter sources. In this way no problem arises in defining the line element and Einstein's equations are kept unchanged. We can summarize the approach as follows: (a) in noncommutative geometry there cannot be point-like object, because there is no physical distance smaller than a minimal position uncertainty of the order of $\sqrt{\theta}$, (b) this effect is implemented in space-time through matter de-localization, which by explicit calculations [5] turns out to be of

[1] We have employed Gaussian units along with natural units, and set Newton's constant to unity.

Gaussian form, (c) space-time geometry is determined through Einstein's equations with de-localized matter sources, (d) de-localization of matter results in a regular, *i.e.* curvature singularity free, metric. This is exactly what is expected from the existence of a minimal length.

The effect of smearing is mathematically implemented as a "substitution rule": position Dirac-delta function is replaced everywhere with a Gaussian distribution of minimal width $\sqrt{\theta}$. Inspired by this result, we choose the mass density of a static, spherically symmetric, smeared, particle-like gravitational source as [3,4]

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} \exp\left(\frac{-r^2}{4\theta}\right) . \tag{5}$$

A particle of mass M, instead of being perfectly localized at a point is diffused throughout a region of line size $\sqrt{\theta}$. This is due to the intrinsic uncertainty encoded in the coordinates commutator (1).

By solving the Einstein's equations with $\rho_\theta(r)$, as a matter source, we find the line element:

$$ds^2 = -g_{00}\, dt^2 + g_{00}^{-1}\, dr^2 + r^2\, d\Omega^2 , \tag{6}$$

where

$$g_{00} = 1 - \frac{4M}{\sqrt{\pi}r}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) + \frac{Q^2}{\pi r^2}\gamma^2\left(\frac{1}{2}, \frac{r^2}{4\theta}\right) - \frac{Q^2}{\pi r\sqrt{2\theta}}\gamma\left(\frac{1}{2}, \frac{r^2}{2\theta}\right) , \tag{7}$$

and

$$\gamma\left(\frac{a}{b}, x\right) \equiv \int_0^x \frac{du}{u}\, u^{a/b} e^{-u} , \tag{8}$$

is the lower incomplete Gamma function.

In the limit $r/\sqrt{\theta} \to \infty$, we get the classical Re–No metric *i.e.* the Re–No metric in commutative spaces. The line element (6) describes the geometry of a noncommutative Re–No black hole and gives us useful information about possible noncommutativity effects on the properties of this type of black hole. Using equation $g_{00}(r_{\rm H}) = 0$, one can find the event horizon(s):

$$\begin{aligned} r_\pm =& \frac{2M}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) + \frac{Q^2}{2\pi\sqrt{2\theta}}\gamma\left(\frac{1}{2}, \frac{r^2}{2\theta}\right) \\ &\pm\frac{1}{2}\left[\left(\frac{4M}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) + \frac{Q^2}{\pi\sqrt{2\theta}}\gamma\left(\frac{1}{2}, \frac{r^2}{2\theta}\right)\right)^2 + \frac{4Q^2}{\pi}\gamma^2\left(\frac{1}{2}, \frac{r^2}{4\theta}\right)\right]^{\frac{1}{2}} . \end{aligned} \tag{9}$$

It is convenient to invert Eq. (9) and consider the black hole mass M as a function of $r_{\rm H}$:

$$M = \frac{Q^2}{2\sqrt{2\pi\theta}} + \frac{1}{\gamma\left(\frac{1}{2}, \frac{r_{\rm H}^2}{4\theta}\right)} \left[\frac{\sqrt{\pi}}{4} r_{\rm H} + \frac{Q^2}{4\sqrt{\pi}\, r_{\rm H}} G(r_{\rm H})\right], \tag{10}$$

where

$$G(r) \equiv \gamma^2 \left(\frac{1}{2}, \frac{r^2}{4\theta}\right) - \frac{r}{\sqrt{2\theta}} \gamma \left(\frac{1}{2}, \frac{r^2}{2\theta}\right). \tag{11}$$

In the limit $r_{\rm H}/\sqrt{\theta} \ll 1$, where one expects significant changes due to space non-commutativity, Eq. (10) leads to

$$M \to M_0 \approx 0.5\sqrt{\pi\theta} + 0.2\frac{Q^2}{\sqrt{\pi\theta}}, \tag{12}$$

which is an interesting result. Noncommutativity implies a minimal non-zero mass that allows the existence of an event horizon. If the black hole has an initial mass $M > M_0$, it can radiate until the value M_0 is reached. At this point the horizon has totally evaporated leaving behind a massive relic. Since black holes with mass $M < M_0$ do not exist there are three possibilities:

1. For $M > M_0$ there is a black hole with regular metric at the origin.
2. For $M = M_0$ the event horizon shrinks to zero.
3. For $M < M_0$ there is no horizon.

The reason why it does not end-up into a naked singularity is due to the finiteness of the curvature at the origin. Compared to the neutral black hole ($Q = 0$) the effect of charge is to increase the minimal non-zero mass.

The physical nature of the mass M_0 remnant is clearer if we consider the black hole temperature as a function of $r_{\rm H}$. We have

$$\begin{aligned} T_{\rm H} &\equiv \left(\frac{1}{4\pi}\frac{dg_{00}}{dr}\right)_{r=r_{\rm H}} \\ &= \frac{1}{4\pi r_{\rm H}}\left[1 - N(\theta) - \frac{4Q^2}{\pi r_{\rm H}^3}\gamma^2\left(\frac{3}{2}, \frac{r_{\rm H}^2}{4\theta}\right) - \frac{Q^2}{\pi r_{\rm H}^3} N(\theta)\, G(r_{\rm H})\right], \end{aligned} \tag{13}$$

where

$$N(\theta) = \frac{r_{\rm H}^3 \exp(\frac{-r_{\rm H}^2}{4\theta})}{4\theta^{\frac{3}{2}}\gamma\left(\frac{3}{2}, \frac{r_{\rm H}^2}{4\theta}\right)}. \tag{14}$$

In the large radius limit, *i.e.* $r_{\rm H}/\sqrt{\theta} \gg 1$, one recovers the standard result for the Hawking temperature:

$$T_{\rm H} = \frac{1}{4\pi r_{\rm H}} - \frac{Q^2}{16\pi r_{\rm H}^3} = \frac{1}{4\pi r_{\rm H}}\left(1 - \frac{Q^2}{4r_{\rm H}^2}\right) . \tag{15}$$

On the other hand in the limit $r_{\rm H}/\sqrt{\theta} \to 0$, we have

$$T_{\rm H} \propto \frac{r_{\rm H}}{\pi\theta} , \qquad \frac{r_{\rm H}}{\sqrt{\theta}} \to 0 . \tag{16}$$

Eqs (15) and (16) are very interesting and have two important consequences. First, when the black hole completely evaporates it reaches zero temperature and there will be no horizon. Second, it reaches a maximum temperature while passing from the regime of large radius to the regime of small radius. This is the same behaviour encountered in the noncommutative neutral case [3]. The effect of charge is just to lower the maximum temperature, see Eq. (15).

3. Specific heat, free energy and thermodynamic stability

A black hole as a thermodynamic system is unstable if it has negative specific heat. We study the thermodynamic stability of noncommutative Re–No black hole by evaluating its specific heat and free energy.

We know that the entropy is proportional to the area of event horizon:

$$S = \frac{A}{4} = \pi r_{\rm H}^2 . \tag{17}$$

Using the first law of thermodynamics $dE = TdS + \Phi dq$, where Φ is the electrostatic potential, we obtain the following expression for the energy:

$$E = M_0 + 2\pi \int_{r_0}^{r_{\rm H}} r''_{\rm H} T(r''_{\rm H})\, dr''_{\rm H} + \int_{r_0}^{r_{\rm H}} \Phi(r''_{\rm H})\, dq(r''_{\rm H}) , \tag{18}$$

where M_0 is the minimal mass below which no black hole can be formed and r_0 is the minimal horizon, see Eq. (10).

In order to check the stability of the noncommutative Re–No black hole we evaluate the heat capacity:

$$C_v = \frac{\partial E(r_{\rm H})}{\partial T(r_{\rm H})} = \left(\frac{\partial E(r_{\rm H})}{\partial r_{\rm H}}\right)\left(\frac{1}{\frac{\partial T(r_{\rm H})}{\partial r_{\rm H}}}\right) . \tag{19}$$

E increases monotonically as r increases, but as mentioned earlier T has a maximum at $r = r_{\rm max}$. Below (above) $r_{\rm max}, T$ is a monotonically increasing (decreasing) function of r. The heat capacity is positive for $r_0 < r_{\rm H} < r_{\rm max}$ and negative for $r_{\rm H} > r_{\rm max}$. Thus the black hole is stable if $r_0 < r_{\rm H} < r_{\rm max}$, and is unstable if $r_{\rm H} > r_{\rm max}$.

The free energy of the noncommutative Re–No black hole is given by:

$$F = E(r_{\rm H}) - T(r_{\rm H})S(r_{\rm H})\,. \tag{20}$$

By evaluating F, and using the fact that the black hole is stable (unstable) when the free energy has a local minimum (maximum), we again see that for $r_{\rm H} < r_{\rm max}$ the black hole is stable while it is unstable if $r_{\rm H} > r_{\rm max}$.

4. The upper bound on the noncommutativity parameter $\boldsymbol{\theta}$

Using Eq. (12) and the extremal condition $M_{\rm ext} = Q$, one can find an upper bound on the noncommutativity parameter θ. We have:

$$0.5\sqrt{\pi\theta} + 0.2\,\frac{Q^2}{\sqrt{\pi\theta}} \geq M_{\rm ext} = Q\,, \tag{21}$$

which gives the following upper bound for the noncommutativity parameter θ:

$$\theta \leq 0.02\,Q^2\,. \tag{22}$$

It is also interesting to discuss our expectation about the lower bound for the parameter θ. As mentioned earlier passing from the regime of large radius to the regime of small radius, Eqs (15) and (16) imply the existence of a maximum temperature. The role of charge is to lower the maximum temperature.

In commutative case one expects relevant back-reaction effects during the terminal stage of evaporation because of huge increase of temperature. As it has been shown, the role of noncommutativity is to cool down the black hole in the final stage. As a consequence [4], there is a suppression of quantum back-reaction since the black hole emits less and less energy. But back-reaction may be important during the maximum temperature phase. In order to estimate its importance in this region, we consider the thermal energy $E = T_{\rm H}$ and the total mass M. In order to have significant back-reaction effect $T_{\rm H}^{\rm max.}$ should be of the same order of magnitude as M. For the neutral case *i.e.* $Q = 0$, from Eqs (10) and (13) we have $M \cong 2.4\sqrt{\theta}M_{\rm Pl}^2$ and $T_{\rm H}^{\rm max.} = 1.5\times 10^{-2}/\sqrt{\theta}$, so we shall obtain the following estimation:

$$\sqrt{\theta} \approx 10^{-1}\ell_{\rm Pl} = 10^{-34}{\rm cm}\,. \tag{23}$$

For the case of charged black holes, the role of charge is to lower the maximum temperature. Therefore, we obtain even smaller values for the noncommutativity parameter θ. Expected values of $\sqrt{\theta}$ are well above the Planck length $\ell_{\rm Pl}$, so (23) (and the smaller values for the case of charged black holes) indicate that back-reaction effects are suppressed if $\sqrt{\theta} \approx 10\ell_{\rm Pl}$ (or even $\sqrt{\theta} > 10\ell_{\rm Pl}$ for the case of charged black holes). For this reason we can safely use unmodified form of the metric (6) during all the evaporation process. So, we can safely consider $\sqrt{\theta} \geq 10^{-33}$ cm.

5. Discussion

In this section we discuss two important issues. First, since the concept of a black hole is inherently coordinate-independent, and since the restriction to space–space noncommutativity implies the choice of a specific space-time slicing, it is not obvious that the inferred modifications of black hole properties are coordinate independent features. Then the question is how we can justify the general covariance of the results. The modifications occurring at the level of energy momentum tensor (EMT) do not modify its tensorial properties. In other words, the noncommutativity provides a fluid type EMT instead of the conventional EMT generating the Schwarzschild solution, which is wrongly considered a vacuum solution [6,8]. We only need to solve the Einstein's equations plugging this new EMT in the same way as one considers a cosmological fluid in the Robertson–Walker space-time. Of course, these coordinates coincide with the Schwarzschild spherical coordinates as can be seen from the solution slightly away from the origin. Therefore, there is no problem with coordinate independence once the derivation is tensorially consistent.

Second, how we can implement noncommutativity by changing only the matter part of Einstein's equation and leaving the left hand side of the equation intact. One of the main differences between noncommutative and commutative theories stems from the fact that in a noncommutative space the coordinates operators have no common position eigenvectors due to Eq. (1). It has been known since the seminal work of Glauber in quantum optics [7], that there exist coherent states that are eigenstates of annihilation operator. As already stated, the reason behind use of coherent states is that there are no coordinate eigenstates for NC coordinates and no coordinate representation can be defined. Therefore, ordinary wave functions (in quantum mechanics) or fields defined over points (in Quantum field theory) can not be defined anymore. Coherent states are the closest to the sharp coordinate states that one can define for NC coordinates in the sense that they are minimal-uncertainty states and enable us to define mean values of coordinate operators. Coherent states, properly defined as eigenstates of ladder operators built from noncommutative coordinate operators only, are the closest to the sharp coordinate states, which we can define for non-commutative coordinates. This means that coordinate coherent states are the minimal uncertainty states over the noncommutative manifold and allow us to calculate the aforementioned mean values [8]. This implies that the matter field is also modified, since now it has to be written in terms of "mean coordinates", even though "formally" it is left unchanged.

6. Conclusions

In conclusion, we have studied the Re–No black hole in noncommutative spaces. We have found the Re–No metric and Hawking temperature in noncommutative spaces that reproduce exactly ordinary Re–No solution at large distances ($r/\sqrt{\theta} \to \infty$). We have shown that like the neutral case there is a minimal non-zero mass $M_0 \approx 0.5\sqrt{\pi\theta} + 0.2\,Q^2/\sqrt{\pi\theta}$ to which a black hole can decay through radiation. The effect of charge Q is to increase this minimal mass. The reason why it does not end-up into a naked singularity is due to the finiteness of the curvature at the origin. From thermodynamics point of view, the same kind of regularization takes place eliminating the divergent behaviour of Hawking temperature. As a consequence, there is a maximum temperature that the black hole can reach before cooling down to absolute zero. The effect of charge Q is to lower this maximum temperature.

We have also found an upper bound for the noncommutativity parameter θ.

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