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MAXIMALLY SYMMETRIC SUPERSTRING VACUA

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Euclidean and Lorentzian quantum-cosmological methods for setting the cosmological constant to zero are discussed, with particular reference to the superstring theory.

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1. Introduction

The problem of the cosmological constant Λ, originally introduced into the field equations of the general theory of relativity by Einstein [1] to create a static universe, remains a subject of active research. Current astronomical observations limit the magnitude of Λ to at most an extremely small fraction of the Planck value,

$$|\Lambda| \approx 4\lambda \times 10^{-123} \Lambda_{\rm P}\,, \tag{1}$$

irrespective of acceleration, where

$$\rho_{\Lambda_{\rm P}} \equiv \frac{\Lambda_{\rm P}}{8\pi L_{\rm P}^2} = \frac{3M_{\rm P}}{4\pi L_{\rm P}^3}\,, \tag{2}$$

$M_{\rm P} \equiv 1.22 \times 10^{19}\,{\rm GeV}$ is the Planck mass, $G_{\rm N} \equiv M_{\rm P}^{-2} \equiv L_{\rm P}^2$ is the Newton gravitational constant, and

$$\lambda \equiv \frac{\rho_\Lambda}{\rho_0} \tag{3}$$

is the fraction of matter observed today in the form of cosmological constant, while $\rho_0 \equiv 1.88 \times 10^{-29} h^2\,{\rm g\,cm^{-3}}$ is the present-day energy-density for a flat three-space, so that $\lambda \lesssim 0.3$. Here we assume a Friedmann space-time

$$ds^2 = dt^2 - a^2(t) d\boldsymbol{x}^2\,, \tag{4}$$

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where t is comoving time, $a(t) \equiv a_0\, e^{\alpha(t)}$ is the radius function of the three-space $d\boldsymbol{x}^2$, whose curvature is k, and $d\alpha/dt \equiv 100\, h\, \mathrm{km\, s^{-1}}\ \mathrm{Mpc}^{-1}$ is the Hubble parameter, with $h \approx 1/2$.

The smallness of the ratio $|\Lambda|/\Lambda_{\mathrm{P}}$ leads one in the first instance to seek a mechanism which sets Λ equal to zero, and quantum cosmology seems naturally to provide such a method. In the covariant approach, the wave function of the Universe Ψ is expressed as a Feynman path integral of the form

$$\Psi = \sum_{\text{paths}} \exp\left(iS\right). \tag{5}$$

For certain topologies, including the space-time (4), the Lorentzian metric $g_{ij}(x^k)$ can be Euclideanized in such a way that all components of the metric remain real, via Wick rotation of the time coordinate,

$$t \to \pm i\tau\,. \tag{6}$$

The Euclidean action S_{E} is so defined that Euclidean, matter kinetic-energy terms occur with a positive sign. For the example of a scalar field ϕ, the Lorentzian Lagrangian $L \ni \dot{\phi}^2/2$ is chosen to transform into the Euclidean Lagrangian $L_{\mathrm{E}} \ni \phi'^2/2$, where $^\bullet \equiv d/dt$ and $' \equiv d/d\tau$. In Minkowski space-time, the purpose of Euclideanizing the action is to improve the convergence properties of the path integral (5), and positive semi-definiteness of L_{E} and S_{E} then necessitates replacing the exponent iS in expression (5) by $-S_{\mathrm{E}}$, which is achieved by choosing the minus sign in the rotation (6),

$$t \to -i\tau\,. \tag{7}$$

Thus, we have

$$iS = i\int dt d^3x\sqrt{-g}\left(\frac{\dot{\phi}^2}{2}+\ldots\right) \to -\int d\tau d^3x\sqrt{-g}\left(\frac{\phi'^2}{2}+\ldots\right) \equiv -S_{\mathrm{E}}\,. \tag{8}$$

Ignoring problems with the sign of the gravitational kinetic term in the action — which is the opposite from the sign of the matter kinetic terms if only the Einstein–Hilbert term $-R/2\kappa^2$ is taken into account — and assuming the conventional probabilistic interpretation of Ψ to hold good, in the sense that $\Psi\Psi^*$ can be regarded as a probability density, it then follows that the most probable configuration is obtained by maximizing the Euclidean exponent $(-S_{\mathrm{E}})$.

Following an earlier paper on quantum tunnelling probabilities by Vilenkin [2], this line of reasoning led Hawking [3, 4] to argue, from the existence of the de Sitter instanton, that the cosmological constant should vanish.

For assuming a spatially closed de Sitter metric, generated by a positive semi-definite cosmological constant Λ, it turns out that the Euclidean action of the instanton configuration is

$$S_{\mathrm{E}} = -\frac{3\pi}{G_{\mathrm{N}}\Lambda} \,. \tag{9}$$

Expression (9) implies that $-S_{\mathrm{E}} \to +\infty$ as $\Lambda \to 0_+$, and justifies the result, assuming the cosmological constant to be freely variable, that Λ should vanish.

A complication may arise if the cosmological constant is partly or wholly an effective one, produced by the non-vanishing, vacuum expectation value of an additional field, as discussed by Duff [5], with regard to the totally anti-symmetric four-index field H_{ijkl}, obtained from the three-index potential A_{ijk} [6, 7],

$$H_{ijkl} = A_{[ijk,l]} \,. \tag{10}$$

This field is defined by the Lagrangian

$$L_H = \frac{1}{48}\, H_{ijkl}H^{ijkl} \,, \tag{11}$$

the classical solution to the resulting field equations being

$$H_{ijkl} = K\varepsilon_{ijkl} \,, \tag{12}$$

where K is an arbitrary constant and $\varepsilon_{ijkl} = \sqrt{-g}\,\delta_{ijkl}$.

Eq. (12) results in a contribution to the cosmological constant

$$\Lambda_H = \tfrac{1}{2}\kappa^2 K^2 \tag{13}$$

in the Einstein equations. It was found in Ref. [5], however, that the field H_{ijkl} contributes differently to the net "cosmological constant" in the Lagrangian, actually reversing the sign of the action by comparison with the result for a genuine cosmological constant Λ. The underlying reason for this, of course, is the additional complexity due to the fact that expression (11) can be expanded as

$$H_{ijkl}H^{ijkl} \equiv H^{ijkl}H^{mnop}g_{im}g_{jn}g_{ko}g_{lp} \,, \tag{14}$$

which contains four factors of the metric tensor g_{ij}, whereas the cosmological constant *per se* contains none. As we shall see below, a similar effect occurs with other higher-derivative field contributions to the Lagrangian, although without necessarily reversing the sign of the coefficient of $\sqrt{-g}$ in $\mathcal{L}$.

This result raises the question of the sense of the Wick time-rotation (6). For if instead of (7) we choose the opposite rotation

$$t \to i\tau\,, \tag{15}$$

then $\Psi \sim \exp(S_{\mathrm{E}})$, and since $S_{\mathrm{E}} > 0$ in the case just discussed, then the argument of Refs [3,4] still applies, although now $L_{\mathrm{E}} \ni -\phi'^2/2$, which would lead to matter instabilities. The choice of sign in expression (6) reflects the boundary condition imposed upon the wave function — see Vilenkin [8] for a thorough discussion.

Recently, however, the situation has been clarified by Wu [9], who showed that addition of the surface term

$$\delta S = -\frac{1}{6}\int d\Sigma_i A_{jkl} H^{ijkl} \tag{16}$$

restores the equality between the net cosmological constant occurring in the Einstein equations and the coefficient of $-\kappa^{-2}\sqrt{-g}$ in the Lagrangian, which resolves the problem without need to change the sense of direction of the Wick rotation.

2. Further generalizations

The mechanism of Refs [3,4] can be straightforwardly generalized to induced gravity [10], for example, in which the gravitational constant is replaced by the vacuum expectation value of a scalar field, $(8\pi G_{\mathrm{N}})^{-1} \to \varepsilon\phi_0^2$, in a theory of the form

$$L = -\tfrac{1}{2}\varepsilon\phi^2 R + \tfrac{1}{2}(\nabla\phi)^2 - V(\phi)\,, \tag{17}$$

where ε is a positive constant, typically of order unity, and $V(\phi)$ is a suitable potential. It can also be applied to theories in dimensionality $D > 4$, via the corresponding higher-dimensional four-index field $\hat{H}_{ABCD}(A = 0, 1, \ldots D-1)$ [11].

Additionally, it is possible to envisage more complicated gravitational Lagrangians, which include higher-derivative terms $\hat{\mathcal{R}}^n$ as well as the Einstein–Hilbert term $-\hat{R}/2\hat{\kappa}^2$. To be specific, let us consider the D-dimensional theory containing terms of quadratic, cubic and quartic order, $n = 0, 1, 2, 3, 4$,

$$\begin{aligned}\hat{L} \;=\; & -\frac{(\hat{R} + 2\hat{\Lambda}_0)}{2\hat{\kappa}^2} + \hat{L}^{(2)}\left(\hat{R}, \hat{R}_{AB}, \hat{R}_{ABCD}\right) + \hat{L}^{(3)}\left(\hat{R}, \hat{R}_{AB}, \hat{R}_{ABCD}\right) \\ & + \hat{L}^{(4)}\left(\hat{R}, \hat{R}_{AB}, \hat{R}_{ABCD}\right)\,. \end{aligned} \tag{18}$$

Our conventions are those of Landau and Lifschitz [12], where the signature of the metric is $\mathrm{sgn}(\hat{g}_{AB}) = (+ - - - \ldots)$ and the Riemann–Christoffel and Ricci tensors are defined as

$$\hat{R}^{A}{}_{BCD} = \partial_C \hat{\Gamma}^{A}{}_{BD} - \partial_D \hat{\Gamma}^{A}{}_{BC} + \hat{\Gamma}^{A}{}_{CE}\hat{\Gamma}^{E}{}_{BD} - \hat{\Gamma}^{A}{}_{DE}\hat{\Gamma}^{E}{}_{BC} \tag{19}$$

and

$$\hat{R}_{AB} = \hat{R}_{ACB}{}^{C}\,, \tag{20}$$

respectively, so that the Einstein equations read

$$\hat{R}_{AB} - \tfrac{1}{2}\hat{R}\hat{g}_{AB} = \hat{\kappa}^2 \hat{T}_{AB}\,, \tag{21}$$

with a *positive* matter source $\hat{T}_{AB}$ on the right-hand side.

Thus, for a perfect fluid characterized by energy density $\hat{\rho}$ and pressure $\hat{p}$, we have

$$\hat{T}_{AB} = (\hat{\rho} + \hat{p})\hat{u}_A\hat{u}_B - \hat{p}\hat{g}_{AB}\,, \tag{22}$$

if the manifold admits a time-like unit vector $\hat{u}_A$ satisfying

$$\hat{u}_A\hat{u}^A = 1\,. \tag{23}$$

In this case, contraction of Eqs (21) yields the trace

$$-\tfrac{1}{2}\,(D-2)\,\hat{R} = \hat{\kappa}^2\,[\hat{\rho} - (D-1)\,\hat{p}]\,, \tag{24}$$

which for a cosmological constant $\hat{\rho} = -\hat{p} = \hat{\Lambda}/\hat{\kappa}^2$ yields

$$\hat{R} = -\left(\frac{2D}{D-2}\right)\hat{\Lambda}\,. \tag{25}$$

For the maximally symmetric space-time

$$\hat{R}_{ABCD} = \frac{2}{(D-1)(D-2)}\,\hat{\Lambda}\,(\hat{g}_{AD}\hat{g}_{BC} - \hat{g}_{AC}\hat{g}_{BD})\,, \tag{26}$$

we therefore obtain the field equations for the theory (18) in the form

$$\hat{R}_{AB} - \tfrac{1}{2}\hat{R}\hat{g}_{AB} = \left\{\hat{\Lambda}_0 + \frac{\hat{\kappa}^2}{D}\left[(4-D)\hat{L}^{(2)} + (6-D)\hat{L}^{(3)} + (8-D)\hat{L}^{(4)}\right]\right\}\hat{g}_{AB}\,, \tag{27}$$

the trace of which is

$$\begin{aligned}\hat{R} &= -\left(\frac{2D}{D-2}\right)\left\{\hat{\Lambda}_0 + \hat{\kappa}^2\left[\left(\frac{4-D}{D}\right)\hat{L}^{(2)} + \left(\frac{6-D}{D}\right)\hat{L}^{(3)} + \left(\frac{8-D}{D}\right)\hat{L}^{(4)}\right]\right\}\\ &\equiv -\left(\frac{2D}{D-2}\right)\hat{\Lambda}_{\text{eff}}\,,\end{aligned} \tag{28}$$

the effective cosmological constant being

$$\hat{\Lambda}_{\text{eff}} = \hat{\Lambda}_0 + \hat{\kappa}^2 \left[\left(\frac{4-D}{D} \right) \hat{L}^{(2)} + \left(\frac{6-D}{D} \right) \hat{L}^{(3)} + \left(\frac{8-D}{D} \right) \hat{L}^{(4)} \right]. \quad (29)$$

Substitution of Eq. (28) into Eq. (18) yields

$$\hat{L} = \left(\frac{2}{D-2} \right) \left(\frac{\hat{\Lambda}_0}{\hat{\kappa}^2} + \hat{L}^{(2)} + 2\hat{L}^{(3)} + 3\hat{L}^{(4)} \right). \quad (30)$$

Comparing Eqs (29) and (30), we see that the coefficient of $-\hat{\kappa}^{-2}\sqrt{-\hat{g}}$ in the Lagrangian density $\hat{\mathcal{L}}$ is a complicated multiple of $\hat{\Lambda}_{\text{eff}}$ in general, so that the application of probabilistic arguments to set $\hat{\Lambda}_{\text{eff}} = 0$ may require either of the Wick rotations (7) or (15), assuming that $\hat{\Lambda}_0$ is freely variable — clearly, we can envisage different Lagrangians with different $\hat{\Lambda}_0$'s.

3. The heterotic superstring theory

The heterotic superstring theory of Gross *et al.* [13-15] presents an interesting example. After reduction to four dimensions, the non-vanishing terms of relevance in the effective action are the Einstein–Hilbert, quadratic and quartic gravitational contributions, corresponding to $n = 1, 2$ and 4. From Eq. (28), setting $D = 4$ and $\Lambda_0 = \hat{L}^{(3)} = 0$, we have

$$R = -4\kappa^2 L^{(4)} \equiv -4\Lambda_{\text{eff}}, \quad (31)$$

while Eq. (30) yields

$$L = L^{(2)} + 3L^{(4)} = L^{(2)} + \frac{3\Lambda_{\text{eff}}}{\kappa^2}. \quad (32)$$

In this case, Λ_{eff} is not continuously variable, but can take the two discrete values given by Eq. (52) of Ref. [16], hereafter called paper I,

$$\Lambda_{\text{eff}} = 0, -\left[18/175\zeta(3)\right]^{1/3} A_{\text{r}}^{-1} \kappa^{-2}, \quad (33)$$

where $1/A_{\text{r}} \equiv g_0^2$ is the inverse modulus of the physical four-space, g_0^2 being the tree-level gauge coupling.

Note that the coefficient of $\sqrt{-g}$ in $\mathcal{L}$ is now arbitrarily variable *a priori*, since the contribution to the field equations derived from $L^{(2)}$ vanishes identically in a maximally symmetric space, while $L^{(2)}$ itself is non-vanishing in general. We have

$$L^{(2)} = B(R^2 - CR_{ij}R^{ij}) = 4B(4-C)\Lambda_{\text{eff}}^2, \quad (34)$$

showing that the magnitude and sign of $L^{(2)}$ can be freely adjusted by varying the coefficients B and C, unless $C = 4$, which causes $L^{(2)}$ to vanish. For the heterotic superstring theory, the constant B is given by the formula [17]

$$B = A_{\rm r} B_{\rm r}^{-2} \tilde{B}\,, \tag{35}$$

where $B_{\rm r}$ is the radius squared of the internal space $\bar{g}_{\mu\nu}$, $\tilde{B}$ is defined by the integral

$$\tilde{B} = \frac{1}{8}\zeta(3)\kappa^4 \int d^6y\sqrt{\bar{g}}\bar{R}_{\mu\nu\xi o}\bar{R}^{\mu\nu\xi o} \Big/ \int d^6y\sqrt{\bar{g}} \approx 4 \tag{36}$$

and $\zeta(3) \equiv 1.202$ is the Riemann zeta function, according to the numerical estimate [18], while $C = 1$. Therefore, the Lagrangian (32) is

$$L = 3(4B\Lambda_{\rm eff} + \kappa^{-2})\Lambda_{\rm eff}\,. \tag{37}$$

From Eqs (33), (35) and (36), we find that

$$4B\kappa^2\Lambda_{\rm eff} = -4\left[18/175\zeta(3)\right]^{1/3}\tilde{B}B_{\rm r}^{-2} = -0.783\left(\frac{\tilde{B}}{4}\right)\left(\frac{3}{B_{\rm r}}\right)^2\,, \tag{38}$$

in which $B_{\rm r}$ has been scaled approximately to the Hagedorn value $B_{\rm r}^{\rm (H)} = 2.914$.

Due to the uncertainty of the numerical estimate (38), the sign of expression (37) is unclear. It may even be that L is exactly zero for both solutions (33), if Eq. (38) is exactly equal to -1, but there seems no particular reason why this should be, especially since we have ignored the trace anomaly, which would then become significant.

We can now apply the above analysis [3, 4] to argue that the solution with vanishing $\Lambda_{\rm eff}$, that is Minkowski space, is preferred probabilistically over the anti-de Sitter space solution with non-vanishing $\Lambda_{\rm eff}$.

In reaching this result, we utilize the fact that a negative cosmological constant can be dealt with in the same way as a positive one. We can realize this by constructing the "anti-de Sitter instanton", obtained by Wick rotation, not of the time coordinate t, but rather of the radial spatial coordinate r, via Eq. (I90), referred to the Friedmann space-time (4),

$$r \to \pm i\tilde{\rho}\,. \tag{39}$$

The line element for a real open three-space is thereby converted into the corresponding imaginary closed one, expressed in coordinates $(t, \tilde{\rho}, \theta, \phi)$ as Eq. (I91),

$$ds^2 = dt^2 + a^2(t)\left[\frac{d\tilde{\rho}^2}{1-\tilde{\rho}^2} + \tilde{\rho}^2(d\theta^2 + \sin^2\theta d\phi^2)\right]\,, \tag{40}$$

where $a(t)$ is the solution to the Friedmann equation (I89), setting $\Lambda < 0$, $k < 0$,

$$\frac{\dot{a}^2}{a^2} = -\frac{|\Lambda|}{3} + \frac{|k|}{a^2} \,. \tag{41}$$

Up to an overall minus sign, expression (40) is precisely the line element that one would have obtained from the Wick rotation (6) applied to the time coordinate t in the spatially closed de Sitter space, for which the Euclideanized Friedmann equation (I74) reads, setting $\Lambda > 0, k > 0$,

$$\frac{a'^2}{a^2} = -\frac{\Lambda}{3} + \frac{k}{a^2} \,, \tag{42}$$

the two problems now being completely equivalent to one another. (Let us emphasize that the metric contains all the geometrical information about the space-time under consideration.) Finally, we can resolve the sign indeterminacy in Eq. (39) by choosing $r = i\tilde{\rho}$, so that

$$iS = -\tfrac{1}{2}\int d^4\tilde{x}\sqrt{\tilde{g}}\left[\dot{\phi}^2 + a^{-2}(t)(1-\tilde{\rho}^2)(\partial_{\tilde{\rho}}\phi)^2 + \ldots\right] \equiv -\tilde{S}_{\mathrm{E}} \,. \tag{43}$$

The Euclidean action of the anti-de-Sitter instanton is

$$\tilde{S}_{\mathrm{E}} = 288\pi^2 B + \frac{9\pi}{G_{\mathrm{N}}\Lambda_{\mathrm{eff}}} \,. \tag{44}$$

4. The Wheeler–DeWitt equation

Results obtained from the covariant approach to quantum cosmology should also generally be derivable from the canonical approach, in which the time coordinate is singled out for special treatment. Due to the high degree of symmetry of the maximal space-time (26), we restrict consideration to the mini-superspace, assuming a Lorentzian line element in the Friedmann form (4). Quantization of the resulting Hamiltonian constraint $H =$ yields the Wheeler–DeWitt equation [19,20] for the wave function of the Universe Ψ

$$\mathcal{H}\Psi = 0 \,, \tag{45}$$

where $\mathcal{H}$ is the operator corresponding to H. The four-dimensional theory (18), with $\Lambda = 0, k = 0$, is discussed from this viewpoint in Refs [17,21] (Ref. [21] is hereafter called paper II), the Lagrangian density being Eq. (23) of Ref. [22], hereafter called paper III. Including also a massless scalar field ϕ, we have

$$\begin{aligned} \mathcal{L} \;=\; a_0^3 e^{3\alpha}\Big(&\tfrac{1}{2}\dot{\phi}^2 - \kappa^{-2}\Lambda + A_0\dot{\alpha}^2 + B_0\dot{\alpha}^4 + B_2\ddot{\alpha}^2 + K_0\dot{\alpha}^6 + K_2\dot{\alpha}^2\ddot{\alpha}^2 \\ &+ K_3\ddot{\alpha}^3 + C_0\dot{\alpha}^8 + C_2\dot{\alpha}^4\ddot{\alpha}^2 + C_3\dot{\alpha}^2\ddot{\alpha}^3 + C_4\ddot{\alpha}^4\Big) \,, \end{aligned} \tag{46}$$

in which the index n of the coefficients Z_n counts the power of $\ddot{\alpha}$, while A_n, B_n or C_n denotes that the term originates from R, $\mathcal{R}^2$ or $\mathcal{R}^4$, respectively.

The exact expressions for the numerical coefficients Z_n are given by Eqs (III24)–(III40). We have

$$\begin{aligned}
\Lambda &= -3ka^{-2} - 24k^2\kappa^2 Ba^{-4} - \lambda_0' k^4\kappa^6 A_{\rm r}^3 a^{-8} + \Lambda_0\,, && (47)\\
A_0 &= -3\kappa^{-2} + 48kBa^{-2} + a_0' k^3\kappa^4 A_{\rm r}^3 a^{-6}\,, && (48)\\
B_0 &= b_0 k^2\kappa^4 A_{\rm r}^3 a^{-4}\,, \qquad B_2 = 24B + b_2' k^2\kappa^4 A_{\rm r}^3 a^{-4}\,, && (49)\\
C_n &= c_n\kappa^4 A_{\rm r}^3\,, \qquad K_n = k_n k\kappa^4 A_{\rm r}^3 a^{-2}\,, && (50)
\end{aligned}$$

where Λ_0 is the bare cosmological constant,

$$\lambda_0' = -156\zeta(3) + 15/16\,, \qquad (51)$$

$$a_0' = -504\zeta(3) + 15/4\,, \qquad (52)$$

$$b_0 = -1,426\zeta(3) + 25/4\,, \qquad b_2' = -366\zeta(3) + 3/8\,, \qquad (53)$$

$$\left.\begin{aligned}
c_0 &= -\tfrac{1}{2}\left[225\zeta(3) - 9/14\right], \qquad & c_2 &= -1,212\zeta(3) + 6\,,\\
c_3 &= -255\zeta(3) + 15/4\,, \qquad & c_4 &= \tfrac{1}{2}\left[27\zeta(3) + 15/8\right]
\end{aligned}\right\} \qquad (54)$$

and

$$k_0 = -\tfrac{1}{5}\left[5,574\zeta(3) - 21\right], \quad k_2 = -1,659\zeta(3) + 3/4\,, \quad k_3 = -309\zeta(3)\,. \qquad (55)$$

The Wheeler–DeWitt equation for the theory (46) has been derived in the form of a modified Schrödinger equation in the case $k = 0$, Eq. (II44), which, including also the field ϕ and the cosmological constant Λ, setting $k \neq 0$, and taking into account terms up to order $\mathcal{R}^4$, reads

$$\begin{aligned}
i\xi\frac{\partial\Psi}{\partial\alpha} \approx \Big[&-\frac{1}{4B_2}\left(1 - \lambda_2\xi^2 - \lambda_2'\xi^4\right) a_0^{-3} e^{-3\alpha}\frac{\partial^2}{\partial\xi^2} - i\left(\lambda_3 + \lambda_3'\xi^2\right) a_0^{-6} e^{-6\alpha}\frac{\partial^3}{\partial\xi^3}\\
&- \lambda_4 a_0^{-9}\, e^{-9\alpha}\frac{\partial^4}{\partial\xi^4} - \frac{1}{2} a_0^{-3}\, e^{-3\alpha}\frac{\partial^2}{\partial\phi^2} + \mathcal{V}(\alpha,\xi)\Big]\Psi\,, \qquad (56)
\end{aligned}$$

where the potential is given by Eq. (II45),

$$\mathcal{V}(\alpha,\xi) = \left[\kappa^{-2}\Lambda - \left(A_0\xi^2 + B_0\xi^4 + K_0\xi^6 + C_0\xi^8\right)\right] a_0^3\, e^{3\alpha}\,, \qquad (57)$$

and the constants λ_2,λ_2', λ_3, λ_3' and λ_4 are defined by

$$\begin{aligned}
\lambda_2 &= \frac{K_2}{B_2}\,,\\
\lambda_2' &= \frac{C_2}{B_2} - \left(\frac{K_2}{B_2}\right)^2 = -\left[\frac{101\zeta(3)}{2} - \frac{1}{4}\right]\frac{\kappa^4 A_{\rm r}^3}{B + b_2' k^2\kappa^4 A_{\rm r}^3 a^{-4}/24}\\
&\qquad - \left(\frac{\left[1.659\zeta(3) - 3/4\right] k\kappa^4 A_{\rm r}^3 a^{-2}}{24B + b_2' k^2\kappa^4 A_{\rm r}^3 a^{-4}}\right)^2, \qquad (58)
\end{aligned}$$

$$
\begin{aligned}
\lambda_3 = \frac{K_3}{8B_2^3}\,, \quad \lambda_3' &= \frac{C_3}{8B_2^3} - \frac{3K_2K_3}{8B_2^4} \\
&= -\left[\frac{255\zeta(3)-15/4}{110.592}\right]\frac{\kappa^4 A_{\rm r}^3}{(B+b_2'k^2\kappa^4A_{\rm r}^3a^{-4}/24)^3} \\
&\quad - \frac{927\,\zeta(3)[1.659\,\zeta(3)-3/4]k^2\kappa^8A_r^6a^{-4}}{8(24B+b_2'k^2\kappa^4A_r^3a^{-4})^4}
\end{aligned} \tag{59}
$$

and

$$
\begin{aligned}
\lambda_4 &= \frac{C_4}{16B_2^4} - \left(\frac{3K_3}{8B_2}\right)^2\frac{1}{B_2^3} = \left[\frac{27\zeta(3)+15/8}{10,616,832}\right]\frac{\kappa^4A_{\rm r}^3}{(B+b_2'k^2\kappa^4A_{\rm r}^3a^{-4}/24)^4} \\
&\quad - \left(\frac{927\zeta(3)k\kappa^4A_{\rm r}^3a^{-2}}{192B+8b_2'k^2\kappa^4A_{\rm r}^3a^{-4}}\right)^2\frac{1}{(24B+b_2'k^2\kappa^4A_r^3a^{-4})^3}\,.
\end{aligned} \tag{60}
$$

The "coordinate" in Eqs (56) and (57) is $\xi \equiv \dot{\alpha}$, and Eq. (56) is obtained by making the operator replacements

$$
\pi_\alpha \to -i\frac{\partial}{\partial\alpha}\,, \qquad \pi_\xi \to -i\frac{\partial}{\partial\xi}\,, \qquad \pi_\phi \to -i\frac{\partial}{\partial\phi}\,, \tag{61}
$$

where π_α, π_ξ and π_ϕ are the canonical momenta. The probabilistic interpretation of Ψ is rendered problematic by the presence of the higher-order operators $\partial^3/\partial\xi^3$ and $\partial^4/\partial\xi^4$, but it is still meaningful to study the potential (57). Further details are contained in paper II, where it was remarked that expression (57) becomes equal to the Lagrangian density in a maximally symmetric space-time, for which $\dot{\xi}=0$, when $\dot{\phi}=0$.

From Eqs (48), (49), (50) and (54) we see that the Z_0 are all negative semi-definite when $k \geq 0$ [1], as a consequence of which we have the important result that the potential (57) is positive semi-definite, being bounded from below with a minimum $\mathcal{V}(\alpha,0)=0$ at $\xi=0$, when $\Lambda \geq 0$ and $k \geq 0$.

If we assert that the vacuum wave function Ψ_0 is independent of both coordinates (α,ξ), implying a global vacuum state, then Ψ_0 has to satisfy the equation

$$
\mathcal{V}(\alpha,\xi)\Psi_0 = 0\,, \tag{62}
$$

from which it follows that the potential has to vanish,

$$
\mathcal{V}(\alpha,\xi) = 0\,, \tag{63}
$$

if the wave function is to be finite. We have used this argument previously [23] to prove that $\Lambda = 0$ for the gravitational theory including quadratic higher-derivative terms $\mathcal{R}^2$, obtained from Eqs (56) and (57) in the approximation of setting the $C_n = 0$, whereupon Eq. (56) reduces to the usual Schrödinger equation containing only the second derivative $\partial^2/\partial\xi^2$.

[1] Assuming that $3\kappa^{-2} - a_0'k^3\kappa^4A_{\rm r}^3a_0^{-6} \geq 48kBa^{-2}$.

5. Euclideanization

It is instructive to Euclideanize the Lagrangian density (46), by applying the Wick rotation (6), with the result that

$$\begin{aligned} \dot{\phi} &\to \mp i\phi' , \\ \xi &\to \mp i\tilde{\xi} , \\ \xi^2 &\to -\tilde{\xi}^2 \\ \text{and} \quad \dot{\xi} &\to -\tilde{\xi}' , \end{aligned} \tag{64}$$

where $\xi \equiv \dot{\alpha}$ and $\tilde{\xi} \equiv \alpha'$. Thus, adjusting the signs so that the coefficient of the Euclidean kinetic-energy density of the scalar field ϕ is positive, by choosing the sense of rotation (7), we find that

$$\begin{aligned} \mathcal{L}_{\mathrm{E}} = -\mathcal{L} = a_0^3\, e^{3\alpha} \Big(& \tfrac{1}{2}\phi'^{\,2} + \kappa^{-2}\Lambda + A_0\tilde{\xi}^2 - B_0\tilde{\xi}^4 - B_2\tilde{\xi}'^{\,2} + K_0\tilde{\xi}^6 + K_2\tilde{\xi}^2\tilde{\xi}'^{\,2} \\ & + K_3\tilde{\xi}'^3 - C_0\tilde{\xi}^8 - C_2\tilde{\xi}^4\tilde{\xi}'^{\,2} - C_3\tilde{\xi}^2\tilde{\xi}'^3 - C_4\tilde{\xi}'^4 \Big) . \end{aligned} \tag{65}$$

In the low-energy limit $\tilde{\xi}^2 \ll 1$, $\tilde{\xi}'^{\,2} \ll 1$, it appears that $\mathcal{L}_{\mathrm{E}}$ is bounded neither from above nor below, for we then have

$$\mathcal{L}_{\mathrm{E}} \approx a_0^3\, e^{3\alpha} \left(\tfrac{1}{2}\phi'^{\,2} + \kappa^{-2}\Lambda + A_0\tilde{\xi}^2 \right) , \tag{66}$$

where, from Eq. (48), $A_0 < 0$ for $k \leq 0$ or $\kappa^{-2} > 16Ba^{-2} + a_0'\kappa^4 A_{\mathrm{r}}^3 a^{-6}$ if $k = 1$, implying that $A_0\tilde{\xi}^2 < 0$, which would make the Wick rotation (7) ineffective.

When the higher-order corrections contained in expression (65) are taken into account, however, the situation becomes more complicated. Whilst the coefficient A_0 of the quadratic term $\tilde{\xi}^2$ is generally negative, the coefficients $-B_0$, K_0 and $-C_0$ of the terms $\tilde{\xi}^4$, $\tilde{\xi}^6$ and $\tilde{\xi}^8$, respectively, are all positive semi-definite, at least for $k \leq 0$, the dominant term $-C_0\tilde{\xi}^8$ at large $\tilde{\xi}^2$ being positive for all k. For $k \leq 0$ and $\tilde{\xi}' > 0$, the terms $K_2\tilde{\xi}^2\tilde{\xi}'^{\,2}$, $K_3\tilde{\xi}'^3$, $-C_2\tilde{\xi}^4\tilde{\xi}'^{\,2}$ and $-C_3\tilde{\xi}^2\tilde{\xi}'^3$ are positive semi-definite. The only negative contribution is $-C_4\tilde{\xi}'^4$, due to the fact that C_4, defined in Eqs (54), is positive, but it can be ignored because $C_4 \ll |C_n|$, $n = 0, 2, 3$ — specifically, we have

$$\begin{aligned} &\frac{|C_0|}{C_4} = \frac{225\zeta(3) - 9/14}{27\zeta(3) + 15/8} \approx 7.86 , \quad \frac{|C_2|}{C_4} = \frac{2[1,212\zeta(3) - 6]}{27\zeta(3) + 15/8} \approx 84.5 , \\ &\frac{|C_3|}{C_4} = \frac{2[255\zeta(3) - 15/4]}{27\zeta(3) + 15/8} \approx 17.6 \end{aligned} \tag{67}$$

and also

$$\frac{C_2}{C_3} = \frac{1,212\zeta(3) - 6}{255\zeta(3) - 15/4} \approx 4.79\,. \tag{68}$$

As a result, the negative divergence of $\mathcal{L}_{\rm E}$ is effectively prevented, for $\mathcal{L}_{\rm E}$ remains positive semi-definite as $\tilde{\xi} \to \infty$, when $k \leq 0$, provided that $\tilde{\xi}'/\tilde{\xi}^2 \leq \sqrt{|C_2|/C_4} \approx 9.19$ if $\tilde{\xi}' \geq 0$, or $|\tilde{\xi}'|/\tilde{\xi}^2 \leq C_2/C_3 \approx 4.79$ if $\tilde{\xi}' \leq 0$. The terms involving K_n are all subdominant to those involving C_n at large $\tilde{\xi}$, since they can be grouped as $(K_0 - C_0\tilde{\xi}^2)\tilde{\xi}^6$, $(K_2 - C_2\tilde{\xi}^4)\tilde{\xi}'^2$ and $(K_3 - C_3\tilde{\xi}^2)\tilde{\xi}'^3$, and therefore the same reasoning applies to the case $k = 1$.

The idea of using theories with higher-derivative terms $\mathcal{R}^2$ to improve the divergence behaviour of the Euclidean action was first suggested by Horowitz [24], and was further studied in Ref. [25].

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