COMPACT SHOCK IMPULSES IN MODELS WITH V-SHAPED POTENTIALS

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A new class of solutions in the signum-Klein–Gordon model is presented. Our solutions merge properties of shock waves and compactons that appear in scalar field models with V-shaped potentials.

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1. Introduction

Scalar field-theoretic models play an important role in contemporary theoretical physics. They have a wide range of applications from condensed matter physics [1] to cosmology [2]. An interesting new class of scalar field models has been proposed in [3]. Such models are called models with V-shaped potentials due to their common property of a non-vanishing first derivative of the potential at the minimum. This feature introduces a qualitatively new behaviour of fields close to the vacuum. It turns out that for models with V-shaped potentials the field approaches its vacuum value in a polynomial (quadratic) way. As a consequence, topological defects (*e.g.* kinks) have compact support (so-called *compactons*), see [4]. A recently found non-topological object which has the form of an oscillon is also compact, [5]. The other group of compact solutions which seem to be very interesting from a physical point of view are Q-balls, see [6, 7]. The possibility of applying Q-balls to boson stars and black holes looks especially attractive [8]. Also models with standard (smooth) field potential can support compact objects if they have non-standard kinetic (*e.g.* quartic) terms [9–13]. These models, known also as K-field models, are studied in the context of the global expansion of the universe (K-essence). In fact, historically, the first non-topological compact solution was found by Roseanau and Hyman in a modified KdV system [14].

The simplest V-shaped scalar field model is the signum-Gordon model [15,16]. It has been shown that, apart from self-similar solutions, it supports also so-called shock waves, *i.e.* a class of solutions with a field discontinuity that propagates with velocity $v = 1$. The violation of the scaling symmetry of the signum-Gordon model has been studied for both self-similar solutions [17] and shock waves [18]. In the following paper we concentrate on the case of shock waves. The genesis of this paper comes from the observation that a specific class of solutions in [18] has not been considered. The new solutions are quite interesting because they merge properties of shock waves (discontinuity of the field at one end) and compactons (quadratic approach to a vacuum value at the other end). Such a solution is compact in an arbitrary stage of evolution so we call it *a shock impulse* instead of *a shock wave*.

Our paper is organized as follows. In Section 2 we briefly recall the signum-Klein–Gordon model. Section 3 is devoted to the presentation of a new class of solutions. In the last section we summarize our paper.

2. The signum-Klein–Gordon model

The Lagrangian of a (1+1) dimensional signum-Klein–Gordon model reads

$$L = \tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{1}{2}(\partial_x\phi)^2 - V(\phi)\,, \tag{1}$$

where $\phi(x,t)$ is a scalar field with an interaction given by the potential

$$V(\phi) = |\phi| - \tfrac{1}{2}\lambda\phi^2\,. \tag{2}$$

In [18] we have proposed a mechanical system which in the limit of an infinite number of degrees of freedom is described by the Lagrangian (1). The quadratic interaction term has been chosen for simplicity. It is one of the simplest terms that can be used to violate an exact scaling symmetry. For details see [17] and [18]. The Euler–Lagrange equation reads

$$\phi_{tt} - \phi_{xx} + \mathrm{sign}\phi - \lambda\phi = 0\,, \tag{3}$$

where we assume (for physical reasons) that $\mathrm{sign}(0) = 0$. It is clear that $\phi = 0$ is a solution of (3). This solution is necessary in the construction of compactons. The Ansatz $\phi(x,t) = \Theta(\pm z)W(z)$ gives discontinuous solutions, where "+" refers to solutions outside the light cone, "−" refers to solutions inside the light cone and $z = (x^2 - t^2)/4$. It has been already discussed in [18] that the velocity $v = 1$ is distinguished by the model and it is the unique admissible velocity for the propagation of field discontinuities.

3. Compact shock impulses

We are interested in the case $\lambda \equiv \rho^2 > 0$. For solutions outside the light cone our Ansatz takes the form

$$\phi(x,t) = \Theta(z)W(z), \qquad \text{where} \qquad z = \tfrac{1}{4}(x^2 - t^2)\,. \tag{4}$$

In the new variable y, related to z by the formula $z = \frac{1}{4}y^2$, the equation (3) takes the form

$$g'' + \frac{1}{y}g' + \rho^2 g = \operatorname{sign}(g)\,, \tag{5}$$

where $g(y) \equiv W(z(y))$. It could be helpful to consider equation (5) as an equation of motion of a fictious particle in the potential $U(g) = \frac{1}{2}\rho^2 g^2 - |g|$. As we mentioned already in [18], equation (5) has partial solutions

$$g_k(y) = (-1)^k \left(\frac{1}{\rho^2} - \mu_k J_0(\rho y) - \nu_k Y_0(\rho y)\right)\,, \tag{6}$$

where $g_k > 0$ for $k = 0, 2, 4, \ldots$ and $g_k < 0$ otherwise. J_0 and Y_0 are Bessel functions. The partial solution not considered in [18] is a constant zero solution. Because of $\operatorname{sign}(0) = 0$, $g(y) = 0$ is a solution of (5). Such a solution corresponds to the solution $\phi = 0$ in a physical system, so it is well motivated from a physical point of view.

3.1. Single-zero solution

The solution $g_0(y)$ can be parametrized by only one parameter because at $y = 0$ the Bessel function Y_0 has a singularity, so for physical reasons we set $\nu_0 = 0$. The second coefficient μ_0 can be expressed by the value of $g_0(0)$, which results in

$$g_0(y) = \frac{1}{\rho^2} - \left(\frac{1}{\rho^2} - g_0(0)\right) J_0(\rho y)\,. \tag{7}$$

As was explained in a previous paper, depending on the values $g_0(0)$, the solution $g_0(y)$ can be positive or can be matched with a negative partial solution $g_1(y)$. The intermediate case allows for matching the solution $g_0(y)$ with the trivial solution $g(y) = 0$. In this case $g_0(y)$ reaches its zero value quadratically. This happens for

$$g_0(0) = \frac{1}{\rho^2}\left(1 - \frac{1}{J_0(j_1^1)}\right)\,, \tag{8}$$

where j_1^1 is the first zero of $J_1(y)$. We name this value $g_0^{\rm crit}$. Approximately $\rho^2 g_0^{\rm crit} = 3.482872$. In this case the full compact impulse reads

$$g(y) = \begin{cases} \frac{1}{\rho^2}\left(1 - \frac{J_0(\rho y)}{J_0(j_1^1)}\right) & \text{if } 0 \le y \le j_1^1/\rho\,, \\ 0 & \text{if } y > j_1^1/\rho\,. \end{cases}$$

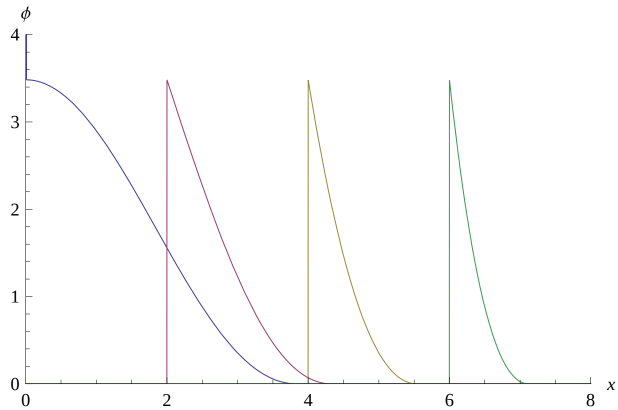

Fig. 1. Evolution of a compact impulse $\phi(x,t)$ at moments $t = 0, 2, 4, 6$ for $\rho = 1$.

The matching point x_0 moves with subluminal velocity $\dot{x}_0 = t/\sqrt{\bar{c}_0^2 + t^2}$, where $\bar{c}_0 = j_1^1/\rho$. For $t \to \infty$ it is reached by the discontinuity $x = t$. It comes from the fact that the length of the compact shock impulse $L_{\rm c} = \sqrt{\bar{c}_0^2 + t^2} - t$ for $t \to \infty$ has a leading term $L_{\rm c} = \frac{\bar{c}_0^2}{2}\frac{1}{t} + \mathcal{O}\left(\frac{1}{t^3}\right)$.

3.2. Multi-zero solutions

As was discussed in [18], for $g_0(0) > g_0^{\rm crit}$, negative partial solutions are present. If $g_0(0)$ is not too big a full compact impulse $g(y)$ is composed of two pieces $g_0(y)$ and $g_1(y)$. Again, for a certain value of $g_0(0)$ a partial solution $g_1(y)$ reaches its zero $\bar{c}_1$ quadratically, therefore it could be matched with a constant zero partial solution. At the point c_0 $g_1(y)$ is matched with $g_0(y)$. We have used a special notation $\bar{c}_k$ for those zeros for which the field reaches its zero value in a quadratic way, whereas all other zeros we label by c_k. A general higher-rank multi-zero solution consists of partial solutions $g_0, \ldots, g_k$ and $g = 0$ and has a matching points $c_0, \ldots, c_{k-1}$ and $\bar{c}_k$.

As an example let us study in details a solution for $k = 1$. There are three partial solutions $g_0(y)$, $g_1(y)$ and a constant solution $g(y) = 0$, and two matching points c_0 and $\bar{c}_1$. The partial solutions obey the following matching conditions

$$g_0(c_0) \; = \; 0 = g_1(c_0)\,, \qquad g_0'(c_0) = g_1'(c_0)\,, \tag{9}$$

$$g_1(\bar{c}_1) \;=\; 0\,, \qquad\qquad g_1'(\bar{c}_1) = 0\,. \tag{10}$$

The first condition, $g_0(c_0) = 0$, and conditions (10) give us

$$g_0(y) \;=\; \frac{1}{\rho^2}\left[1 - \frac{J_0(\rho y)}{J_0(\rho c_0)}\right]\,, \tag{11}$$

$$g_1(y) \;=\; -\frac{1}{\rho^2}\left[1 - \frac{Y_1(\rho\bar{c}_1)J_0(\rho y) - J_1(\rho\bar{c}_1)Y_0(\rho y)}{Y_1(\rho\bar{c}_1)J_0(\rho\bar{c}_1) - J_1(\rho\bar{c}_1)Y_0(\rho\bar{c}_1)}\right]\,, \tag{12}$$

where the parameter $g_0(0)$ is expressed in terms of c_0. The other conditions in (9) give algebraic equations for c_0 and $\bar{c}_1$. These equations can be rewritten in the following form

$$\frac{Y_1(\rho\bar{c}_1)J_0(\rho c_0) - J_1(\rho\bar{c}_1)Y_0(\rho c_0)}{Y_1(\rho\bar{c}_1)J_0(\rho\bar{c}_1) - J_1(\rho\bar{c}_1)Y_0(\rho\bar{c}_1)} \;=\; 1\,, \tag{13}$$

$$\frac{Y_0(\rho c_0)}{J_0(\rho c_0)} + \frac{Y_1(\rho c_0)}{J_1(\rho c_0)} - 2\frac{Y_1(\rho\bar{c}_1)}{J_1(\rho\bar{c}_1)} \;=\; 0\,. \tag{14}$$

Equations (13) and (14) can be solved numerically. We will not present formulas for bigger k because they are too complicated.

In Figs 2 and 3 we present two examples of multi-zero solutions $g(y)$ with three and four segments, respectively. For higher-range solutions, $k > 2$, instead of solutions we present only plots of numerical values of $g_0(0)$ and $\bar{c}_k$ as functions of k, see Figs 4 and 5.

The numerical results suggest, surprisingly, that, at least for not too large k, both the sequence of $g_0^2(0)(k)$ and the sequence of zeros $\bar{c}_k$ behave linearly (arithmetic sequences). Linear fits give us:

- $\rho = 0.8$

$$\begin{aligned} g_0^2(0)(k) &= 30.89 + 35.28k\,, \\ \bar{c}_k &= 4.57 + 5.61k\,, \end{aligned}$$

- $\rho = 1.0$

$$\begin{aligned} g_0^2(0)(k) &= 12.65 + 14.44k\,, \\ \bar{c}_k &= 3.67 + 4.49k\,, \end{aligned}$$

- $\rho = 1.2$

$$\begin{aligned} g_0^2(0)(k) &= 6.09 + 6.97k\,, \\ \bar{c}_k &= 3.06 + 3.74k\,. \end{aligned}$$

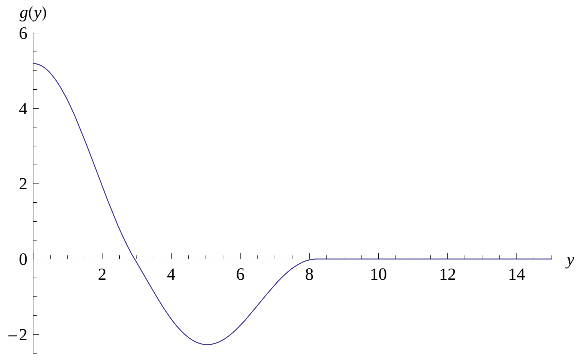

Fig. 2. Three segment ($k = 1$) multi-zero solution $g(y)$ for $\rho = 1$.

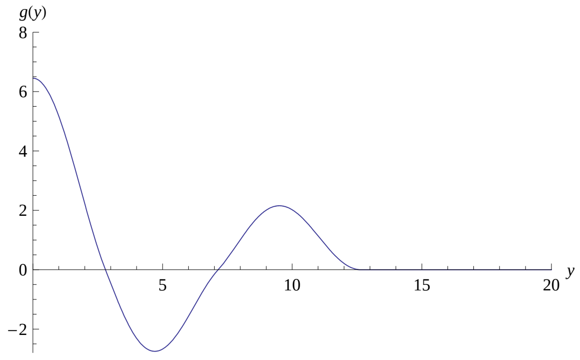

Fig. 3. Four segment ($k = 2$) multi-zero solution $g(y)$ for $\rho = 1$.

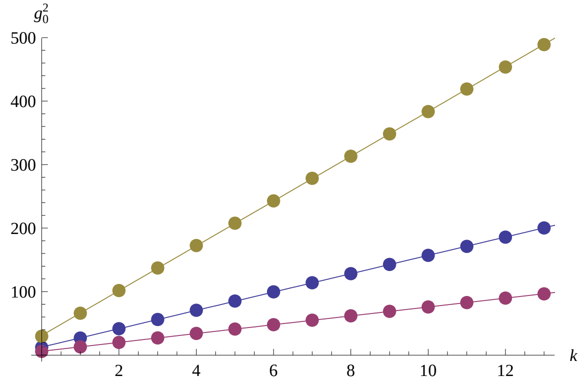

Fig. 4. Values of $g_0^2(0)$ for consecutive multi-zero compact solutions $g(y)$. Straight lines represent linear fits to numerical data. The upper line has been obtained for $\rho = 0.8$, the middle line for $\rho = 1.0$ and the bottom line corresponds to $\rho = 1.2$.

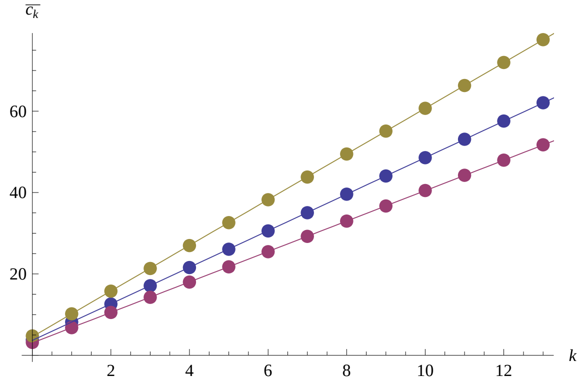

Fig. 5. The consecutive zeros $\bar{c}_k$ and their linear fits. The upper line has been obtained for $\rho = 0.8$, the middle line for $\rho = 1.0$ and the bottom line corresponds to $\rho = 1.2$.

This means that the size of a compact impulse (as a function of y) or the size of an initial compacton configuration (as a function of x) changes linearly with a number k. The size of the compacton for a fixed k shrinks when ρ increases.

We want to emphasize that linear dependence $\bar{c}_k$ and $g_0(0)$ is completely unexpected. The expressions for solutions for higher k and relations between coefficients are very complicated and one cannot expect such a simple relationship.

4. Summary

We have presented a new class of solutions in the (1+1) dimensional signum-Klein–Gordon model which was not been considered in [18]. The solutions which appear in models with V-shaped potentials are usually either shock or compacton type. Our solutions merge both of these properties. The compact shock impulses, presented above, have on one end a wave front where a scalar field is discontinuous and on the other end they approach their vacuum value quadratically.

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