# PATH INTEGRAL SOLUTION FOR A PARTICLE WITH POSITION DEPENDENT MASS

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The problem of the particle with variable mass is considered by the approach of path integral. The Green's function related to this problem is reduced to that of a particle with a constant mass. As examples, simple cases are considered.

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## 1. Introduction

We know that a simple description of the motion of a particle interacting with an external environment consists of replacing the mass by a so-called effective mass, this effective mass is in general variable and dependent on the position.

From the classical point of view it is known that the Lagrangian or the Hamiltonian which can be constructed and associated to the equation of motion relating the particles with variable masses, is not unique. Among various Lagrangians or Hamiltonians, there exists a form where the kinetic energy has the standard form in $\frac{1}{2}m(x)\dot{x}^2$ or in $p^2/(2m(x))$ (mass being variable).

From the quantum point of view and at the level of the Hamiltonian, the replacement of the classical variables $x$ and $p$ by operators $\hat{x}$ and $\hat{p}$ poses the problem of the order. Thus, the Hamiltonian operator and the Schrödinger equation are not unique and it is not possible, in spite of the limit $\hbar \to 0$ that all the Hamiltonian operators give the classical Hamiltonian, to remove this ambiguity of the order in $\hat{H}$, except some physical conditions such as, for example, the hermiticity condition.

The problem of variable mass can be formulated by the path integral approach where the associated propagator takes the standard form of $\sum\limits_{\text{path}} \exp(\frac{i}{\hbar}S(\text{path}))$ ($S$ being the action). For this purpose it is necessary to start with a Hamiltonian operator where the order is fixed, and then to use the usual process: application of the Trotter formula, elimination of operators $\hat{x}$ and $\hat{p}$. The order problem in $\hat{H}$ is then transposed in the path integral on the various ways of carrying out the discretization (post-point or mid-point, ...). For example, if $\hat{H}$ is chosen [1] following the Weyl order, it will appear in the path integral formulation the Lagrangian of classical mechanics where the mid-point plays a central role. In addition, in some cases of potentials it is necessary to introduce regulating functions or to change the parameterization of paths by using a new time "$s$" instead of the usual time $t$ in order to obtain a regular expression. This procedure completed with a transformation on the coordinates enabled to solve practically all the problems of standard quantum mechanics [2].

In this paper we propose, using the path integral approach, to consider the problem of position dependent mass which is not sufficiently studied, despite the large literature which has been devoted to it [3–9].

Also, our aim in this paper is to show, by using the path integral formalism, how to transform the problem of a particle having a variable mass into a problem of particle with a constant mass and to establish the effective potential $V_{\text{eff}}$ in $\hbar^2$ which was induced.

For this purpose, we adapt the procedure of Duru–Kleinert related to particles having a constant mass in order to determine the corrections induced by the regularization and the transformation on the coordinate.

Finally, let us note that the problem of variable mass has been also studied by the so-called supersymmetric approach and that connection with the mass constant problem was shown by using the transformations on the coordinates and on the wave functions [10].

## 2. Green's function

First, let us consider the Green's function operator $G$, solution of the formal equation

$$(E - \hat{H})\,\hat{G} = i\hbar I\,, \tag{1}$$

where

$$\hat{H} = \hat{T} + \hat{V}\,, \tag{2}$$

is the Hamiltonian operator with the kinetic term

$$\begin{aligned} T &= \frac{1}{4}\left[m^{\alpha}(x)\,\hat{p}m^{\beta}(x)\,\hat{p}m^{\gamma}(x) + m^{\gamma}(x)\,\hat{p}m^{\beta}(x)\,\hat{p}m^{\alpha}(x)\right], & (3)\\ \alpha+\beta+\gamma &= -1\,, & (4) \end{aligned}$$

and $V$ is the potential term, $\alpha$, $\beta$ and $\gamma$ are parameters.

In order to make the kinetic term constant, we choose two arbitrary regulating functions $f_{\mathrm{l}}(x)$, $f_{\mathrm{r}}(x)$ whose product is $f(x)$ and we introduce them as follows

$$f_{\mathrm{l}}(E-\hat{H})f_{\mathrm{r}}f_{\mathrm{r}}^{-1}G = i\hbar f_{\mathrm{l}}\,, \tag{5}$$

then, we obtain for $G$, an equivalent expression

$$G = f_{\mathrm{r}}\frac{i\hbar}{f_{\mathrm{l}}(E-\hat{H})f_{\mathrm{r}}}f_{\mathrm{l}}\,. \tag{6}$$

In the configurations space, the Green's function becomes

$$\begin{aligned} G\left(x_b,x_a;E\right) &= \langle x_b|\, f_{\mathrm{r}}\frac{i\hbar}{f_{\mathrm{l}}(E-\hat{H})f_{\mathrm{r}}}f_{\mathrm{l}}\,|x_a\rangle \\ &= f_{\mathrm{r}}(x_b)f_{\mathrm{l}}(x_a)\langle x_b|\,\frac{i\hbar}{f_{\mathrm{l}}(E-\hat{H})f_{\mathrm{r}}}\,|x_a\rangle \\ &= f_{\mathrm{r}}(x_b)f_{\mathrm{l}}(x_a)\int_0^\infty dS\,\langle x_b|\,\exp\left(-\frac{i}{\hbar}f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)S\right)|x_a\rangle\,, \end{aligned} \tag{7}$$

where in the last line, the exponential form is introduced in order to pass to the path integral formulation.

Let us subdivide the time interval $S$ into $N$ intervals having a length each one equal to $\sigma = S/N$,

$$\begin{aligned} G\left(x_b,x_a;E\right) &= f_{\mathrm{r}}(x_b)f_{\mathrm{l}}(x_a)\int_0^\infty dS\lim_{N\to\infty}\langle x_b|\exp\left(-\frac{i}{\hbar}f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)\sigma\right) \\ &\times\exp\left(-\frac{i}{\hbar}f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)\sigma\right)\ldots\exp\left(-\frac{i}{\hbar}f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)\sigma\right)|x_a\rangle\,, \end{aligned} \tag{8}$$

and with the completeness relation

$$\int dx_n\,|x_n\rangle\langle x_n| = 1\,, \tag{9}$$

$G$ becomes

$$\begin{aligned} G\left(x_b,x_a;E\right) &= f_{\mathrm{r}}(x_b)f_{\mathrm{l}}(x_a)\int_0^\infty dS\lim_{N\to\infty}\prod_{n=1}^{N}\int dx_n \\ &\times\prod_{n=1}^{N+1}\langle x_n|\,\exp\left(-\frac{i}{\hbar}f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)\sigma\right)|x_{n-1}\rangle\,. \end{aligned} \tag{10}$$

First, let us consider the term $[f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)]$, by using the commutation relations since the mass is not constant

$$[\,m^{\alpha}\,(x)\,,\hat{p}\,] = i\hbar\,\alpha m'\,(x)\,m^{\alpha-1}\,(x)\;, \tag{11}$$

the kinetic term is arranged by moving $\hat{p}$

$$\begin{aligned} &f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x) \\ &= f_{\mathrm{l}}(x)\Big[\tfrac{1}{4}\left[m^{\alpha}\hat{p}m^{\beta}\hat{p}m^{\gamma}+m^{\gamma}\hat{p}m^{\beta}\hat{p}m^{\alpha}\right]+V(x)-E\Big]f_{\mathrm{r}}(x)\,. \end{aligned} \tag{12}$$

The term $\langle x_n|\exp(-\frac{i}{\hbar}f_{\mathrm{l}}(x)(\hat{H}-E)f_{\mathrm{r}}(x)\sigma)|x_{n-1}\rangle$ is then calculated and in the exponent, it appears the following expression

$$\begin{aligned} \to f_{\mathrm{l}}(x_n)\Big\{&\tfrac{1}{4}\Big[m^{\alpha}(x_n)\hat{p}^2m^{\beta}\,(x_{n-1})m^{\gamma}(x_{n-1})+m^{\gamma}(x_n)\hat{p}^2m^{\beta}(x_{n-1})m^{\alpha}(x_{n-1}) \\ &+i\hbar\beta\Big\{m^{\alpha}(x_n)\hat{p}m'(x_{n-1})m^{\beta-1}(x_{n-1})m^{\gamma}(x_{n-1}) \\ &+\;m^{\gamma}(x_n)\hat{p}m'(x_{n-1})m^{\beta-1}(x_{n-1})m^{\alpha}(x_{n-1})\Big\}\Big] \\ &+V(x_{n-1})-E\Big\}f_{\mathrm{r}}(x_{n-1})\,, \end{aligned} \tag{13}$$

in the 2nd step, and in order to eliminate the operators, we introduce the completeness relation

$$\int dp_n\,|p_n\rangle\langle p_n| = 1\,, \tag{14}$$

with the scalar product

$$\langle\,p_n\,|x_n\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,e^{-\frac{i}{\hbar}p_nx_n} \tag{15}$$

and after having performed the integrations on the canonical variables $p_n$, the Green's function in the general case of the problem with position dependent mass becomes

$$\begin{aligned} G(x_b,x_a;E) &= f_{\mathrm{r}}(x_b)f_{\mathrm{l}}(x_a)\int_0^{\infty}dS\;\lim_{N\to\infty}\int\prod_{n=1}^{N}dx_n \\ &\times\prod_{n=1}^{N+1}\frac{1}{\sqrt{2i\pi\hbar\sigma f_{\mathrm{l}}(x_n)f_{\mathrm{r}}(x_{n-1})\left(m_n^{\alpha}m_{n-1}^{\beta}m_{n-1}^{\gamma}+m_n^{\gamma}m_{n-1}^{\beta}m_{n-1}^{\alpha}\right)\Big/2}} \\ &\times e^{\frac{i}{\hbar}\sum_{n=1}^{N+1}\left[\frac{\left\{\Delta x_n-\frac{i\hbar\beta\sigma f_{\mathrm{l}}(x_n)f_{\mathrm{r}}(x_{n-1})}{2}\frac{\left(m_n^{\alpha}m'_{n-1}m_{n-1}^{\beta-1}m_{n-1}^{\gamma}+m_n^{\gamma}m'_{n-1}m_{n-1}^{\beta-1}m_{n-1}^{\alpha}\right)}{2}\right\}^2}{\frac{(m_n^{\alpha}m_{n-1}^{\beta}m_{n-1}^{\gamma}+m_n^{\gamma}m_{n-1}^{\beta}m_{n-1}^{\alpha})}{2}2\sigma f_{\mathrm{l}}(x_n)f_{\mathrm{r}}(x_{n-1})}\right]} \\ &\times e^{\left[\frac{i}{\hbar}\sum_{n=1}^{N+1}\sigma f_{\mathrm{l}}(x_n)f_{\mathrm{r}}(x_{n-1})\{E-V(x_{n-1})\}\right]}\,. \end{aligned} \tag{16}$$

Let us simplify the study of the variable mass problem by considering the simple case of the following Hamiltonian

$$\hat{H} = \frac{1}{2}\,\hat{p}\frac{1}{m(x)}\,\hat{p} + V(x)\,, \tag{17}$$

*i.e.* we fix the parameters as follows

$$\alpha = \gamma = 0\,, \qquad \beta = -1\,, \tag{18}$$

since the regulating functions are arbitrary, we choose

$$f_{\mathrm{r}}(x) = f_{\mathrm{l}}(x) = f^{1/2}(x)\,. \tag{19}$$

Thus the kinetic term can be rearranged

$$\begin{aligned}\sqrt{f(x)}\,\hat{p}\frac{1}{m(x)}\,\hat{p}\sqrt{f(x)} &= \sqrt{f(x)}\,\hat{p}\frac{1}{\sqrt{m(x)}}\frac{1}{\sqrt{m(x)}}\,\hat{p}\sqrt{f(x)} \\ &= \sqrt{\frac{f(x)}{m(x)}}\,\hat{p}^2\sqrt{\frac{f(x)}{m(x)}} + \frac{\hbar^2}{2}f(x) \\ &\quad \times \left(\frac{3}{2}\frac{m'^2(x)}{m^3(x)} - \frac{m''(x)}{m^2(x)}\right)\,,\end{aligned} \tag{20}$$

and following this rearrangement, it appears a potential in $\hbar^2$ and $G$ takes the following form

$$\begin{aligned}G\left(x_b, x_a; E\right) &= \sqrt{f(x_b)f(x_a)}\lim_{N\to\infty}\int_0^\infty dS\int\prod_{n=1}^{N}dx_n \\ &\quad \times \exp\frac{i}{\hbar}\sum_{n=1}^{N+1}\sqrt{f(x_n)}(E - V(x_{n-1}))\sqrt{f(x_{n-1})}\;\sigma \\ &\quad \times \prod_{n=1}^{N+1}\langle x_n|\,\exp\frac{i}{\hbar}\frac{\sigma}{2}\left(\sqrt{\frac{f(x_n)}{m(x_n)}}\hat{p}^2\sqrt{\frac{f(x_{n-1})}{m(x_{n-1})}}\right) \\ &\quad \times \exp-\frac{i}{\hbar}\frac{\hbar^2}{4}\sigma f(x_n)\left(\frac{m''(x_n)}{m^2(x_n)} - \frac{3}{2}\frac{m'^2(x_n)}{m^3(x_n)}\right)|x_{n-1}\rangle,\end{aligned} \tag{21}$$

then after elimination of the operators

$$
\begin{aligned}
G(x_b, x_a; E) \;=\; & \sqrt{f(x_b) f(x_a)} \lim_{N\to\infty} \int_0^{\infty} dS \left[ \prod_{n=1}^{N} \int dx_n \right] \\
& \times \exp\left( \frac{i}{\hbar} \sum_{n=1}^{N+1} \sqrt{f(x_n)} (E - V(x_{n-1})) \sqrt{f(x_{n-1})}\, \sigma \right) \\
& \times \prod_{n=1}^{N+1} \int \frac{dp_n}{\sqrt{2\pi\hbar}} \exp\left( \frac{i}{\hbar} \left(p_n \left(x_n - x_{n-1}\right)\right) \right) \\
& \times \exp\left( \frac{i}{\hbar} \frac{\sigma}{2} \left[ \left( \sqrt{\frac{f(x_n)}{m(x_n)}} \sqrt{\frac{f(x_{n-1})}{m(x_{n-1})}} p_n^2 \right) \right] \right) \\
& \times \exp\left( \frac{i}{\hbar} \left[ -\, \sigma \, \frac{\hbar^2}{4} \, f(x_n) \left( \frac{m''(x_n)}{m^2(x_n)} - \frac{3}{2} \frac{m'^2(x_n)}{m^3(x_n)} \right) \right] \right), \qquad (22)
\end{aligned}
$$

and by using the integrals

$$
\int dp \exp(-ap^2 + bp) = \sqrt{\frac{\pi}{a}} \exp\left( -\frac{b^2}{4a} \right) , \qquad (23)
$$

in order to eliminate the variables $p_n$, we obtain finally

$$
\begin{aligned}
& G\left(x_b, x_a; E\right) = \sqrt{f(x_b) f(x_a)} \lim_{N\to\infty} \int_0^{\infty} dS \left[ \prod_{n=1}^{N} \int dx_n \right] \\
& \times \prod_{n=1}^{N+1} \frac{1}{\sqrt{2i\pi\hbar\sigma \sqrt{f(x_n) f(x_{n-1}) / m(x_n) m(x_{n-1})}}} \\
& \times \exp \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[ \frac{\Delta x^2}{2\sigma} \sqrt{\frac{m(x_n) m(x_{n-1})}{f(x_n) fr(x_{n-1})}} \right] \\
& \times \exp \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[ \sigma f(x_n) \left\{ (E - V(x_n)) - \frac{\hbar^2}{4} \left( \frac{m''(x_n)}{m^2(x_n)} - \frac{3}{2} \frac{m'^2(x_n)}{m^3(x_n)} \right) \right\} \right] . \qquad (24)
\end{aligned}
$$

Let us make the change on the regulating function $f \longrightarrow g$

$$
g(x) = \frac{f(x)}{m(x)} . \qquad (25)
$$

With this change $G$ becomes

$$\begin{aligned} G(x_b, x_a; E) &= \left[m(x_b)\, g^{1/2}(x_b) m(x_a) g^{1/2}(x_a)\right]^{\frac{1}{2}} \\ &\times \lim_{N\to\infty} \int_0^\infty ds \int \prod_{n=1}^{N} dx_n \prod_{n=1}^{N} \frac{1}{\sqrt{g(x_n)}} \prod_{n=1}^{N+1} \left[\frac{g(x_{n-1})}{g(x_n)}\right]^{-\frac{1}{2}} \\ &\times \exp \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[\frac{\Delta x^2}{2\sigma\sqrt{g(x_n)g(x_{n-1})}} - \sigma W_1(x_n)\right], \end{aligned} \tag{26}$$

where

$$W_1(x) = g(x)\left[m(x)(V(x) - E) + \frac{\hbar^2}{4}\left(\frac{m''(x)}{m(x)} - \frac{3}{2}\frac{m'^2(x)}{m^2(x)}\right)\right]. \tag{27}$$

At this level, we notice that the kinetic term still has an inconvenient form containing a space dependent mass. This space dependence can be removed by a coordinate transformation

$$x = F(y)\,. \tag{28}$$

Obviously, this transformation generates three corrections:

— the first related to the measure,

— the second, to the action

— and the third correction related to the factor in front of the Green's function.

The postpoint expansion of $\Delta x_n$ reads at each $n$

$$\Delta x = F(y_n) - F(y_{n-1}) = \frac{\partial F}{\partial y}\Delta y - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}\Delta y^2 + \frac{1}{6}\frac{\partial^3 F}{\partial y^3}\Delta y^3 + \ldots\,. \tag{29}$$

The choice of $F$ is arbitrary, we impose the following condition

$$\left(\frac{\partial F}{\partial y}\right)^2 = g\,, \tag{30}$$

thereafter, the mass being in the kinetic term is constant ($= 1$).

First, let us develop the exponential with the kinetic term. We have

$$\exp\left(\frac{i}{\hbar}\frac{\Delta x^2}{2\sigma\sqrt{g(x_n)g(x_{n-1})}}\right) = \exp\left(\frac{i}{\hbar}\frac{\Delta y^2}{2\sigma}\right)\left[1 + C_{\text{act}}\right], \tag{31}$$

where

$$C_{\rm act} = \frac{i}{\hbar}\frac{\Delta y^2}{2\sigma}\left\{-\frac{1}{4}\left(\frac{\partial^2 F/\partial y^2}{\partial F/\partial y}\right)^2 + \frac{1}{6}\frac{\partial^3 F/\partial y^3}{\partial F/\partial y}\Delta y^2 + \ \ldots\right\}$$

is the first correction.

The measure induce also a correction

$$\prod_{n=1}^{N}\int dx_n = \prod_{n=2}^{N+1}\int d\left(\Delta x_n\right) = \prod_{n=2}^{N+1}\int Jd\left(\Delta y_n\right)\,,$$

where $J$ is the Jacobian of the transformation

$$J = \frac{\partial\Delta x}{\partial\Delta y} = \frac{\partial F}{\partial y}\left(1 + C_{\rm meas}\right)$$

and

$$C_{\rm meas} = -\frac{\partial^2 F/\partial y^2}{\partial F/\partial y}\Delta y + \frac{1}{2}\frac{\partial^3 F/\partial y}{\partial F/\partial y}\Delta y^2 + \ \ldots$$

is the 2nd correction.

Also, the prefactor in the Green's function contribute by a correction $C_f$ which is obtained in the development of

$$\left(\frac{g(x_{n-1})}{g(x_n)}\right)^{-\frac{1}{2}} = 1 + C_f\,,$$

where

$$C_f = \frac{\partial^2 F/\partial y^2}{\partial F/\partial y}\Delta y + \left[\left(\frac{\partial^2 F/\partial y^2}{\partial F/\partial y}\right)^2 - \frac{1}{2}\frac{\partial^3 F/\partial y}{\partial F/\partial y}\right]\Delta y^2 + \ldots$$

is the 3rd correction.

By combining this three corrections we obtain the total correction $C_{\rm T}$ defined by

$$1 + C_{\rm T} = \left(1 + C_{\rm meas}\right)\left(1 + C_f\right)\left(1 + C_{\rm act}\right)\,. \tag{32}$$

The corrections terms are evaluated perturbatively using the expectation values

$$\left\langle(\Delta y)^{2n}\right\rangle = \left(i\frac{\hbar\sigma}{m}\right)^n (2n-1)\,, \tag{33}$$

and $C_{\rm T}$ is replaced by the following effective potential

$$V_{\rm eff} = -\hbar^2\left[\frac{1}{4}\frac{\partial^3 F/\partial y^3}{\partial F/\partial y} - \frac{3}{8}\left(\frac{\partial^2 F/\partial y^2}{\partial F/\partial y}\right)^2\right]\,. \tag{34}$$

The Green's function relating to the nonrelativistic problem with position dependent mass is finally the following

$$G\left(x_b, x_a; E\right) = \left(m_b F_b' m_a F_a'\right)^{1/2} \int_0^\infty dS \left( \int Dy(s) e^{\frac{i}{\hbar}\int_0^S ds\left(\frac{\dot{y}^2}{2} - W_2\right)} \right), \quad (35)$$

where

$$\begin{aligned} W_2 \; = \; & \left(\frac{\partial F}{\partial y}\right)^2 \left[ m(x)(V(x)-E) + \frac{\hbar^2}{4}\left(\frac{m''(x)}{m(x)} - \frac{3}{2}\frac{m'^2(x)}{m^2(x)}\right)\right] \\ & + \frac{\hbar^2}{4}\left(\frac{\partial^3 F/\partial y^3}{\partial F/\partial y} - \frac{3}{2}\left(\frac{\partial^2 F/\partial y^2}{\partial F/\partial y}\right)^2\right). \end{aligned} \quad (36)$$

In order to illustrate our calculations, let us make two applications.

### *2.1. Applications*

Let us consider the cases treated in [12]

**1st case:** $m(x) = cx^2$ and $V(x) = A/(cx^4) + B(cx^2)$.

With this choice, the Green's function relating to the problem with position dependent mass can be reduced to that of a particle of mass $= 1$ and subjected to the action of the combination of a harmonic force and a centrifugal barrier.

In this case, it is sufficient to choose an identical transformation

$$x = y = F(y)$$

the Green's function has the following expression

$$\begin{aligned} G\left(x_b, x_a; E\right) &= \left(c x_b x_a\right) \left[ \int_0^\infty dS \left( \int Dx(s)\; e^{\frac{i}{\hbar}\int_0^S \left(\frac{\dot{x}^2}{2} - cEx^2 + \frac{A+g}{x^2} + B\right) ds} \right) \right] \\ &= \left(c x_b x_a\right)\left(x_b x_a\right)^{\frac{1}{2}} \left(\frac{\omega}{i\hbar}\right) \\ &\times \int_0^\infty dS \left[ \frac{e^{\left(\frac{i}{\hbar}\frac{B}{\omega}\omega S\right)}}{\sin(\omega S)} \exp\left(\frac{i\omega}{2\hbar}\left(x_b^2 + x_a^2\right)\cot(\omega S)\right) I_\nu\left(\frac{\omega x_b x_a}{i\hbar \sin(\omega S)}\right)\right], \end{aligned} \quad (37)$$

where $I_\nu$ is the Bessel function with $\nu = \left[-2A/\hbar^2 - 7/4\right]^{\frac{1}{2}}$.

In order to extract the energy spectrum and the corresponding wave functions, let us separate the variables $x_b, x_a$ and $S$ with the help of the Hill Hardy formula [13] by putting

$$X = \frac{\omega}{\hbar} x_a^2 , \qquad Y = \frac{\omega}{\hbar} x_b^2 , \qquad Z = e^{-2i\omega S} . \quad (38)$$

Then

$$G\left(x_b, x_a; E\right) = c\left(x_b x_a\right)^{\frac{3}{2}} \sum_{n=0}^{\infty}$$
$$\times \exp\left(-\frac{\omega}{2\hbar}x_a^2\right)\left[\frac{2\omega}{\hbar}\frac{n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2}\left[\frac{\omega}{\hbar}x_a^2\right]^{\upsilon/2} L_n^{\upsilon}\left(\frac{\omega}{\hbar}x_a^2\right)$$
$$\times \exp\left(-\frac{\omega}{2\hbar}x_b^2\right)\left[\frac{2\omega}{\hbar}\frac{n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2}\left[\frac{\omega}{\hbar}x_b^2\right]^{\upsilon/2} L_n^{\upsilon}\left(\frac{\omega}{\hbar}x_b^2\right)$$
$$\times \int_0^{\infty} \exp -i\omega S\left(1+2n+\upsilon-\frac{\beta}{\hbar\omega}\right)dS\,, \tag{39}$$

and which is reduced as

$$G\left(x_b, x_a; E\right) = \sum_{n=0}^{\infty}\left[\frac{2\omega}{\hbar}\frac{c\ x_a^3\ n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2}\left[\frac{\omega}{\hbar}x_a^2\right]^{\upsilon/2}$$
$$\times\ \exp\left(-\frac{\omega}{2\hbar}x_a^2\right) L_n^{\upsilon}\left(\frac{\omega}{\hbar}x_a^2\right)\left[\frac{2\omega}{\hbar}\frac{c\, x_b^3\, n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2}\left[\frac{\omega}{\hbar}x_b^2\right]^{\upsilon/2}$$
$$\times\ \exp\left(-\frac{\omega}{2\hbar}x_b^2\right) L_n^{\upsilon}\left(\frac{\omega}{\hbar}x_b^2\right)\int_0^{\infty} \exp -i\omega S\left(1+2n+\upsilon-\frac{\beta}{\hbar\omega}\right)dS\,, \tag{40}$$

with $cE = -\frac{1}{2}\omega^2$.

Let us integrate on $S$

$$\int_0^{\infty} \exp -i\omega S\left(1+2n+\upsilon-\frac{\beta}{\hbar\omega}\right)dS = \frac{1}{i\omega\left(1+2n+\upsilon-\frac{\beta}{\hbar\omega}\right)}\,, \tag{41}$$

from the poles, we obtain the energy spectrum

$$E_n = -\frac{\beta^2}{2c(2n+\upsilon+1)^2\hbar^2}\,, \qquad \text{with} \qquad n = 0, 1, 2\ \ldots\,, \tag{42}$$

and using the standard form of the Green's function

$$G\left(x_b, x_a; E\right) = i\hbar\sum_{n=0}^{\infty}\frac{\Psi_n^*(x_b)\ \Psi_n(x_a)}{E-E_n}\,, \tag{43}$$

we can extract from residues, the corresponding wave functions:

$$\begin{aligned}\Psi_n(x) &= \left[\frac{2\omega\beta}{\hbar^3}\frac{x^3}{(2n+\upsilon+1)^2}\frac{n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2}\left[\frac{\omega}{\hbar}x^2\right]^{\upsilon/2} \\ &\quad\times\exp\left(-\frac{\omega}{2\hbar}x^2\right)L_n^{\upsilon}\left(\frac{\omega}{\hbar}x^2\right)\end{aligned}$$

suitably normalized.

**2nd case:** $m = m_0\exp(cx)$ and $V = V_0\exp(cx)$.

With the transformation $x \to y$ defined by

$$x = F(y) = \ln y^{2/c}\,, \tag{44}$$

the Green's function becomes

$$\begin{aligned}G\left(x_b,x_a;E\right) &= \frac{2}{c}\frac{m_0\exp(c/2\ (x_b+x_a))}{(y_b y_a)^{1/2}}\int_0^\infty dS \\ &\times\left[\int Dy(s)\exp\frac{i}{\hbar}\int_0^S\left(\frac{\dot{y}^2}{2}+\frac{4E_0m_0}{c^2}-\frac{4V_0m_0}{c^2}y^2+\left(\frac{\hbar^2}{2}-\frac{\hbar^2}{8}\right)\frac{1}{y^2}\right)ds\right],\end{aligned} \tag{45}$$

which has the same form as the Green's function relating to a particle of mass $= 1$ subjected to the action of a harmonic force and a centrifugal barrier.

The Green's function being known

$$\begin{aligned}G\left(x_b,x_a;E\right) &= \frac{2}{c}\,\frac{m_0\exp(c/2\,(x_b+x_a))}{(y_b y_a)^{1/2}}\left(\frac{\omega}{i\hbar}\right)\left[y_b y_a\right]^{\frac{1}{2}} \\ &\quad\times\int_0^\infty dS\,\frac{\exp\left(i/\hbar\right)(4E_0m_0/c^2)S}{\sin(\omega s)} \\ &\quad\times\exp\frac{i\omega}{2\hbar}\left(y_b^2+y_a^2\right)\cot\left(\omega S\right)I_\upsilon\left[\frac{\omega y_b y_a}{i\hbar\sin(\omega s)}\right],\end{aligned} \tag{46}$$

with

$$\upsilon = \frac{i}{\sqrt{2}}\,. \tag{47}$$

By using the same formula of separation of variables [13] the Green's function is finally written:

$$
\begin{aligned}
G\left(x_b, x_a; E\right) = & \frac{2m_0 \exp\left[c/2\left(x_b + x_a\right)\right]}{c} \\
& \times \sum_{n=0}^{\infty} \exp\left(\frac{-\omega}{2\hbar} y_a^2\right) \left[\frac{2\omega}{\hbar} \frac{n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2} \left[\frac{\omega}{2\hbar} y_a^2\right]^{\upsilon/2} L_n^{\upsilon}\left(\frac{\omega}{\hbar} y_a^2\right) \\
& \times \exp\left(\frac{-\omega}{2\hbar} y_b^2\right) \left[\frac{2\omega}{\hbar} \frac{n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2} \left[\frac{\omega}{2\hbar} y_b^2\right]^{\upsilon/2} L_n^{\upsilon}\left(\frac{\omega}{\hbar} y_b^2\right) \\
& \times \int_0^{\infty} dS \exp\left[-i\omega\left(1 + 2n + \upsilon - \frac{4Em_0}{\hbar\omega c^2}\right) s\right] . \qquad (48)
\end{aligned}
$$

It is then easy to extract the energies spectrum,

$$
E_n = \sqrt{\frac{v_0}{2m_0}}\, \hbar c\, \left(2n + 1 + \upsilon\right) , \qquad n = 0, 1, 2 \ldots , \qquad (49)
$$

as well as the corresponding wave functions which are also normalized

$$
\Psi_n(x) = \left[\frac{2\omega}{\hbar} \frac{n!}{\Gamma\left(n+\upsilon+1\right)}\right]^{1/2} \left[\frac{\omega}{2\hbar}\, e^{cx}\right]^{\upsilon/2} \exp\left(\frac{cx_b}{2} - \frac{\omega}{2\hbar}\, e^{cx}\right) L_n^{\upsilon}\left(\frac{\omega}{\hbar}\, e^{cx}\right) , \qquad (50)
$$

with $\omega = (2\sqrt{2m_0 v_0})/c$.

## 3. Conclusion

In this paper we showed how to treat the problem of a particle having a position dependent mass (variable mass) by the use of Duru and Kleinert procedure related to particles having a constant mass and to determine the corrections induced by the combination of the path-dependent time reparametrization and a coordinate transformation. We have also shown how to transform a problem of position dependent mass into a problem of constant mass and how to obtain the relation which exists between the two Green's functions (variable mass and constant mass). For that, in order to regularize the kinetic energy we introduced the functions $\left(f_{\mathrm{r}} = f_{\mathrm{l}} = f^{1/2}\right)$ and we tacked account the terms in $(\Delta y)^2$ (order of $\sigma$) and after transformation we obtained for the classical trajectory in a time interval $\sigma$, an unique action (or Lagrangian).

Our Green's function obtained is thus completely symmetrical in respect to the initial and final points (this is not the case of propagator [1] for example).

Finally, for the general case of variable mass depend on the position and of time the study is in progress and the results can be found elsewhere.

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