PHASE SPACES OF TWISTED LIE-ALGEBRAICALLY DEFORMED RELATIVISTIC AND NONRELATIVISTIC SYMMETRIES

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The twisted Lie-algebraically deformed relativistic and nonrelativistic phase spaces are constructed with the use of Heisenberg double procedure. The corresponding Heisenberg uncertainty principles are discussed as well.

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1. Introduction

Recently, there were found formal arguments, based mainly on Quantum Gravity [1, 2] and String Theory models [3, 4], indicating that space-time at Planck-length should be noncommutative, *i.e.* it should have a quantum nature. On the other side, the main reason for such considerations follows from many phenomenological suggestions, which state that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincaré invariance still remains valid at larger distances [5–8].

It is well known that a proper modification of the Poincaré Hopf algebra can be realized in the framework of Quantum Groups [9]. Hence, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [10, 11]), one can distinguish two kinds of quite interesting quantum spaces. First of them corresponds to the well-known canonical type of noncommutativity

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}\,, \tag{1}$$

with antisymmetric constant tensor $\theta^{\mu\nu}$. Its relativistic and nonrelativistic Hopf-algebraic realizations have been discovered with the use of twist procedure (see [9]) of classical Poincaré [12, 13] and Galilei [14, 15] Hopf structures, respectively.

The second class of mentioned deformations introduces the Lie-algebraic type of space-time noncommutativity

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho} x_\rho \,, \tag{2}$$

with particularly chosen coefficients $\theta_{\mu\nu}^{\rho}$ being constants. It is represented by two Lie-algebraically deformed Poincaré Hopf algebras. First of them, so-called κ-Poincaré algebra $\mathcal{U}_\kappa(\mathcal{P})$, has been proposed in [16, 17] as a result of contraction limit of q-deformed anti-De-Sitter Hopf structure. It leads to the κ-Minkowski space-time [18, 19]

$$[x_0, x_i] = \frac{i}{\kappa} x_i \,, \qquad [x_i, x_j] = 0 \,, \tag{3}$$

with mass-like deformation parameter κ. Besides, it also gives a formal framework for such theoretical constructions as Double Special Relativity (see *e.g.* [20–22]), which postulates two observer-independent scales, of velocity, describing the speed of light, and of mass, which can be identify with κ-parameter — the fundamental Planck mass.

The κ-deformed dual Poincaré quantum group $\mathcal{P}_\kappa$ has been provided in [23], while the κ-deformed Galilei Hopf algebra $\mathcal{U}_\kappa(\mathcal{G})$ and the corresponding dual quantum group $\mathcal{G}_\kappa$, have been discovered in [24] and [25] by nonrelativistic contraction (see [26–28]) of their relativistic counterparts.

The second type of deformation associated with noncommutativity (2) is generated (similar to the canonical deformation (1)) by twist procedure [9]. The corresponding Hopf algebras have been proposed at relativistic and nonrelativistic level in [29] (see also [30]) and [14], while their dual quantum groups — in [29] and [15], respectively.

The basic properties of the Lie-algebraically twisted symmetries have been investigated recently in a context of nonrelativistic particle subjected to the external constant force [31], and in the case of harmonic oscillator model [32]. In particular, there was demonstrated that such a kind of quantum space-time produces additional acceleration, as well as the velocity and position-dependent forces, acting additionally on a moving particle.

In this article we introduce the relativistic and nonrelativistic phase spaces corresponding to the twisted Lie-algebraically deformed Hopf algebras [14, 30] and [15]. In the case of relativistic symmetries, we use the so-called Heisenberg double procedure [9, 33], which assumes that the momentum and position sectors of considered phase spaces can be identified with the translation generators of $\mathcal{U}_\xi(\mathcal{P})$ and $\mathcal{P}_\xi$ Hopf algebras, respectively; the cross-relations, *i.e.* the commutation relations between momentum and position variables are given by dual parings of corresponding generators. In the case of nonrelativistic symmetries the corresponding phase spaces are obtained by proper nonrelativistic contractions of their relativistic counterparts.

It should be noted that the deformed phase space $\chi_\kappa(\mathcal{P})$ has been already constructed in the case of κ-Poincaré algebra [33], while its basic physically implications have been investigated in the series of papers [33–35]. Particularly, there was studied its role in a context of Quantum Gravity (see *e.g.* [35]), Doubly Special Relativity Theory [36, 37], Statistical Physics[1] [38–41] and Friedman–Robertson–Walker cosmological model [41, 42].

The main motivation for the present studies is twofold. First of all, such a construction completed our knowledge about the whole considered Hopf structure, *i.e.* about the Hopf algebra $\mathcal{U}(\mathcal{A})$, its dual quantum group $\mathcal{A}$, and the corresponding phase space $\chi(\mathcal{A})$. On the other side, the recovered phase spaces give a background for the studies on physical implications of the Lie-algebraically twisted Poincaré and Galilei Hopf algebras. Following the mentioned above κ-Poincaré program [33–42] one can applied the provided phase spaces to the (for example) Quantum Gravitational or Cosmological considerations, respectively.

The paper is organized as follows. In Section 2 we recall necessary facts concerning the twisted Lie-algebraically deformed Poincaré Hopf algebras and their dual quantum group [30]. Section 3 is devoted to the corresponding twisted phase spaces provided with the use of Heisenberg double procedure. The proper Heisenberg uncertainty principles are discussed in Section 4. The nonrelativistic phase spaces (and the corresponding Heisenberg uncertainty principles) are derived and discussed in Section 5. The results are summarized in the last section.

2. Twisted Lie-algebraically deformed Poincaré Hopf algebra and its dual quantum group

In this section we recall the results of paper [29] (see also [30]) concerning the Lie-algebraically twisted Poincaré Hopf algebra $\mathcal{U}_\xi(\mathcal{P})$ and its dual quantum group $\mathcal{P}_\xi$, with mass-like deformation parameter ξ. Both structures are described by the following Abelian r-matrix

$$\mathrm{r}_\xi = \frac{1}{2\xi}\zeta^\lambda\, P_\lambda \wedge M_{\alpha\beta}\,, \quad \alpha,\ \beta\ -\ \text{fixed}\,,\ \zeta^\lambda \text{ denotes dimensionless fourvector}\,, \tag{4}$$

satisfying the classical Yang–Baxter equation [9]. After twist procedure[2] the algebraic sector of $\mathcal{U}_\xi(\mathcal{P})$ algebra remains classical

$$[M_{\mu\nu}, M_{\rho\sigma}] \;=\; i\left(\eta_{\mu\sigma}\, M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma}\right)\,, \tag{5}$$

$$[M_{\mu\nu}, P_\rho] \;=\; i\left(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu\right)\,, \qquad [P_\mu, P_\nu] = 0\,, \tag{6}$$

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[1] See *e.g.* deformed black body radiation law [41].

[2] Due to the formula (4) the corresponding twist factor has the form $\mathcal{F}_\xi = \exp \frac{i}{2\xi}(\zeta^\lambda\, P_\lambda \wedge M_{\alpha\beta})$.

while the coproduct becomes deformed

$$\begin{aligned}
\Delta_\xi\left(P_\mu\right) &= \Delta_0(P_\mu) + (-i)^\gamma \sinh\left(\frac{i^\gamma}{2\xi}\zeta^\lambda P_\lambda\right) \wedge \left(\eta_{\alpha\mu}P_\beta - \eta_{\beta\mu}P_\alpha\right) \\
&\quad + \left(\cosh\left(\frac{i^\gamma}{2\xi}\zeta^\lambda P_\lambda\right) - 1\right) \perp \left(\eta_{\alpha\alpha}\eta_{\alpha\mu}P_\alpha + \eta_{\beta\beta}\eta_{\beta\mu}P_\beta\right) , \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\Delta_\xi(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + M_{\alpha\beta} \wedge \frac{1}{2\xi}\zeta^\lambda \left(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu\right) \\
&\quad + i\left[M_{\mu\nu}, M_{\alpha\beta}\right] \wedge (-i)^\gamma \sinh\left(\frac{i^\gamma}{2\xi}\zeta^\lambda P_\lambda\right) \\
&\quad + \left[\left[M_{\mu\nu}, M_{\alpha\beta}\right], M_{\alpha\beta}\right] \perp (-1)^{1+\gamma}\left(\cosh\left(\frac{i^\gamma}{2\xi}\zeta^\lambda P_\lambda\right) - 1\right) \\
&\quad + M_{\alpha\beta}(-i)^\gamma \sinh\left(\frac{i^\gamma}{2\xi}\zeta^\lambda P_\lambda\right) \perp \frac{1}{2\xi}\zeta^\lambda \left(\psi_\lambda P_\alpha - \chi_\lambda P_\beta\right) \\
&\quad + \frac{1}{2\xi}\zeta^\lambda \left(\psi_\lambda \eta_{\alpha\alpha}P_\beta + \chi_\lambda \eta_{\beta\beta}P_\alpha\right) \wedge M_{\alpha\beta}(-1)^{1+\gamma} \\
&\quad \times \left(\cosh\left(\frac{i^\gamma}{2\xi}\zeta^\lambda P_\lambda\right) - 1\right) , \qquad (8)
\end{aligned}$$

where $\Delta_0(a) = a \otimes 1 + 1 \otimes a$, $a \wedge b = a \otimes b - b \otimes a$, $a \perp b = a \otimes b + b \otimes a$, $\psi_\gamma = \eta_{j\gamma}\eta_{li} - \eta_{i\gamma}\eta_{lj}$, $\chi_\gamma = \eta_{j\gamma}\eta_{ki} - \eta_{i\gamma}\eta_{kj}$, $\eta_{\mu\nu} = (-,+,+,+)$, and $\gamma = 0$ when $M_{\alpha\beta}$ is a boost or $\gamma = 1$ for a space rotation.

The dual quantum group $\mathcal{P}_\xi$ has been discovered with the use of FRT procedure [43]. In terms of (dual) base $\{\Lambda^\beta{}_\alpha, a^\mu\}$ it is given by the following algebraic sector

$$[a^\mu, a^\nu] = \frac{i}{\xi}\zeta^\nu\Big(\delta^\mu{}_\alpha a_\beta - \delta^\mu{}_\beta a_\alpha\Big) + \frac{i}{\xi}\zeta^\mu\left(\delta^\nu{}_\beta a_\alpha - \delta^\nu{}_\alpha a_\beta\right) , \quad [\Lambda^\mu_\nu, \Lambda^\rho_\tau] = 0 , \qquad (9)$$

$$\left[a^\mu, \Lambda^\nu{}_\rho\right] = \frac{i}{\xi}\zeta^\lambda \Lambda^\mu{}_\lambda \left(\eta_{\beta\rho}\Lambda^\nu{}_\alpha - \eta_{\alpha\rho}\Lambda^\nu{}_\beta\right) + \frac{i}{\xi}\zeta^\mu \left(\delta^\nu{}_\beta \Lambda_{\alpha\rho} - \delta^\nu{}_\alpha \Lambda_{\beta\rho}\right) , \qquad (10)$$

and the primitive coproducts

$$\Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu , \qquad \Delta(a^\mu) = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes 1 . \qquad (11)$$

3. Relativistic phase spaces from Heisenberg double procedure

The knowledge of Hopf algebra $\mathcal{U}_\xi(\mathcal{P})$ and its dual quantum group $\mathcal{P}_\xi$ allows us to find the corresponding ξ-deformed phase space $\chi_\xi(\mathcal{P})$. As it was mentioned in Introduction, in accordance with Heisenberg double procedure [9, 34], the position sector of such a phase space can be identified with translations a^μ (see relations (9)), while the momentum part — with generators P_μ (see formula (6)). In order to find the so-called cross-relations, *i.e.* the commutation relations between positions and momenta one should use the formula (see *e.g.* [34])

$$[\mathcal{Q}, \mathcal{R}] = \mathcal{R}_{(1)} \left\langle \mathcal{Q}_{(1)}, \mathcal{R}_{(2)} \right\rangle \mathcal{Q}_{(2)} - \mathcal{R}\mathcal{Q} \,, \tag{12}$$

where $\langle .,. \rangle$ denotes paring between generators $\mathcal{R} \in \{M_{\mu\nu}, P_\rho\}$ and $\mathcal{Q} \in \{\Lambda^{\beta}{}_{\alpha}, a^\mu\}$

$$\langle \Lambda^{\mu}{}_{\nu}, 1 \rangle = \delta^{\mu}{}_{\nu} \,, \quad \left\langle \Lambda^{\mu}{}_{\nu}, M^{\alpha\beta} \right\rangle = i\left(\eta^{\alpha\mu}\delta^{\beta}{}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}{}_{\nu}\right) \,, \quad \langle a^\mu, P_\nu \rangle = i\delta^{\mu}{}_{\nu} \,, \tag{13}$$

and where we use Sweedler (shorthand) notation for coproduct $\Delta(\mathcal{R}) = \sum \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}$.

Let us start with "rotation-like"[3,4] twist carrier $\{M_{kl}, P_\gamma \,;\ \gamma \neq\ k,\, l,\, 0\}$ ($\lambda = \gamma$, $\alpha = k$, $\beta = l$ in the formula (4)). Then, in accordance with the above prescription, we get the corresponding phase space (see (7), (11) and (12))[5]

$$\begin{aligned}
(i) \quad [x_0, x_i] &= [\,x_k, x_l\,] = [p_\mu, p_\nu] = 0 \,; \qquad i = k, l, \gamma \,, \\
[x_k, x_\gamma] &= \frac{i}{\xi} x_l \,, \qquad [x_l, x_\gamma] = -\frac{i}{\xi} x_k \,, \\
[x_0, p_i] &= [x_i, p_0] = [x_k, p_\gamma] = [x_l, p_\gamma] = 0 \,, \\
[x_0, p_0] &= -i \,, \qquad [x_\gamma, p_\gamma] = i \,, \\
[x_\gamma, p_k] &= \frac{i}{2\xi} p_l \,, \qquad [x_\gamma, p_l] = -\frac{i}{2\xi} p_k \,, \\
[x_l, p_l] &= i \cos\left(\frac{p_\gamma}{2\xi}\right) = [x_k, p_k] \,, \\
[x_k, p_l] &= i \sin\left(\frac{p_\gamma}{2\xi}\right) = -[x_l, p_k] \,.
\end{aligned} \tag{14}$$

In the case of carrier $\{M_{kl}, P_0\}$ we obtain

$$\begin{aligned}
(ii) \quad [x_0, x_a] &= [x_k, x_l] = [p_\mu, p_\nu] = 0 \,; \qquad a \neq k, l, 0 \,, \\
[x_0, x_k] &= \frac{i}{\xi} x_l \,, \qquad [x_0, x_l] = -\frac{i}{\xi} x_k \,, \qquad [x_k, x_a] = [x_l, x_a] = 0 \,,
\end{aligned}$$

[3] Below, we consider three kinds of twist factor (4), providing the three different types of Lie-algebraic space-time noncommutativity (see [29, 30]).

[4] By "rotation-like" twist carrier we mean the carrier containing space rotation generator M_{kl}.

[5] We put the nonzero components of fourvector ζ equal one.

$$
\begin{aligned}
[x_0, p_a] &= [x_a, p_0] = [x_k, p_a] = [x_l, p_a] = 0\,, \\
[x_0, p_0] &= -i\,, \qquad [x_a, p_a] = i\,, \qquad [x_a, p_k] = [x_a, p_l] = 0\,, \\
[x_k, p_0] &= 0\,, \qquad [x_l, p_0] = 0\,, \\
[x_0, p_k] &= -\frac{i}{2\xi}p_l\,, \qquad [x_0, p_l] = \frac{i}{2\xi}p_k\,, \\
[x_l, p_l] &= i\cos\left(\frac{p_0}{2\xi}\right) = [x_k, p_k]\,, \\
[x_k, p_l] &= i\sin\left(\frac{p_0}{2\xi}\right) = -[x_l, p_k]\,,
\end{aligned}
\tag{15}
$$

while for "boost-like"[6] carrier $\{M_{k0}, P_l\ ;\ k \neq l\}$, we have

$$
\begin{aligned}
(iii) \quad [x_0, x_a] &= [x_0, x_k] = [p_\mu, p_\nu] = 0\,; \qquad a \neq k, l, 0\,, \\
[x_k, x_a] &= [x_l, x_a] = 0\,, \qquad [x_0, x_l] = \frac{i}{\xi}x_k\,, \qquad [x_l, x_k] = -\frac{i}{\xi}x_0\,, \\
[x_l, p_k] &= \frac{i}{2\xi}p_0\,, \qquad [x_0, p_0] = -i\cosh\left(\frac{p_l}{2\xi}\right)\,, \\
[x_a, p_a] &= i\,, \qquad [x_l, p_l] = i\,, \qquad [x_a, p_0] = [x_k, p_l] = [x_0, p_l] = 0\,, \\
[x_0, p_k] &= i\,\sinh\left(\frac{p_l}{2\xi}\right)\,, \qquad [x_k, p_k] = i\cosh\left(\frac{p_l}{2\xi}\right)\,, \\
[x_k, p_0] &= -i\,\sinh\left(\frac{p_l}{2\xi}\right)\,, \qquad [x_l, p_0] = \frac{i}{2\xi}p_k\,, \\
[x_k, p_a] &= [x_l, p_a] = [x_0, p_a] = [x_a, p_l] = [x_a, p_k] = 0\,.
\end{aligned}
\tag{16}
$$

The relations (14)–(16) describe three relativistic phase spaces $\chi_\xi(\mathcal{P})$ associated with the Lie-algebraically deformed Poincaré Hopf algebra $\mathcal{U}_\xi(\mathcal{P})$ and with its (dual) quantum group $\mathcal{P}_\xi$. Of course, for deformation parameter ξ running to infinity the above phase spaces become classical. It should be also noted that for very particular choice of twist factors (the choice of indices α, β, γ) one can recover the phase space proposed in [44].

4. Heisenberg uncertainty principle

Let us now turn to the Heisenberg uncertainty relations associated with the above phase spaces. If we introduce the dispersion of observable $\hat{a}$ in a quantum mechanical sense by (see *e.g.* [35])[7]

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[6] By "boost-like" twist carrier we mean the carrier containing boost generator M_{k0}.

[7] We put $\hbar = 1$.

$$\Delta(\hat{a}) = \sqrt{\langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2}\,, \qquad \Delta(\hat{a})\Delta(\hat{b}) \geq \tfrac{1}{2}|\langle \hat{c} \rangle|\,, \tag{17}$$

where $\hat{c} = [\hat{a}, \hat{b}]$, than, we get the following generalized (deformed) Heisenberg relations for all considered above carriers:

- for "rotation-like" carrier $\{\, M_{kl}, P_\gamma\,;\ \gamma \neq\ k,\, l,\, 0\}$

$$\begin{aligned}
(i)\quad \Delta(x_k)\Delta(x_\gamma) &\geq \frac{|\langle x_l \rangle|}{2\xi}\,, \qquad \Delta(x_l)\Delta(x_\gamma) \geq \frac{|\langle x_k \rangle|}{2\xi}\,,\\
\Delta(x_k)\Delta(p_k) &\geq \frac{\left|\left\langle \cos\left(\frac{p_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(x_l)\Delta(p_l) \geq \frac{\left|\left\langle \cos\left(\frac{p_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,,\\
\Delta(x_0)\Delta(p_0) &\geq \tfrac{1}{2}\,, \qquad \Delta(x_\gamma)\Delta(p_\gamma) \geq \tfrac{1}{2}\,,\\
\Delta(x_\gamma)\Delta(p_k) &\geq \frac{|\langle p_l \rangle|}{4\xi}\,, \qquad \Delta(x_\gamma)\Delta(p_l) \geq \frac{|\langle p_k \rangle|}{4\xi}\,,\\
\Delta(x_k)\Delta(p_l) &\geq \frac{\left|\left\langle \sin\left(\frac{p_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(x_l)\Delta(p_k) \geq \frac{\left|\left\langle \sin\left(\frac{p_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,,
\end{aligned} \tag{18}$$

- for twist carrier $\{M_{kl}, P_0\}$

$$\begin{aligned}
(ii)\quad \Delta(x_k)\Delta(x_0) &\geq \frac{|\langle x_l \rangle|}{2\xi}\,, \qquad \Delta(x_l)\Delta(x_0) \geq \frac{|\langle x_k \rangle|}{2\xi}\,,\\
\Delta(x_k)\Delta(p_k) &\geq \frac{\left|\left\langle \cos\left(\frac{p_0}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(x_l)\Delta(p_l) \geq \frac{\left|\left\langle \cos\left(\frac{p_0}{2\xi}\right)\right\rangle\right|}{2}\,,\\
\Delta(x_0)\Delta(p_0) &\geq \tfrac{1}{2}\,, \qquad \Delta(x_a)\Delta(p_a) \geq \tfrac{1}{2}\,,\\
\Delta(x_0)\Delta(p_k) &\geq \frac{|\langle p_l \rangle|}{4\xi}\,, \qquad \Delta(x_0)\Delta(p_l) \geq \frac{|\langle p_k \rangle|}{4\xi}\,,\\
\Delta(x_k)\Delta(p_l) &\geq \frac{\left|\left\langle \sin\left(\frac{p_0}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(x_l)\Delta(p_k) \geq \frac{\left|\left\langle \sin\left(\frac{p_0}{2\xi}\right)\right\rangle\right|}{2}\,,
\end{aligned} \tag{19}$$

- and for "boost-like" carrier $\{M_{k0}, P_l\,;\ k \neq\ l\}$

$$\begin{aligned}
(iii)\quad \Delta(x_k)\Delta(x_l) &\geq \frac{|\langle x_0 \rangle|}{2\xi}\,, \qquad \Delta(x_l)\Delta(x_0) \geq \frac{|\langle x_k \rangle|}{2\xi}\,,\\
\Delta(x_l)\Delta(p_k) &\geq \frac{|\langle p_0 \rangle|}{4\xi}\,, \qquad \Delta(x_0)\Delta(p_0) \geq \frac{\left|\left\langle \cosh\left(\frac{p_l}{2\xi}\right)\right\rangle\right|}{2}\,,\\
\Delta(x_l)\Delta(p_l) &\geq \tfrac{1}{2}\,, \qquad \Delta(x_a)\Delta(p_a) \geq \tfrac{1}{2}\,,
\end{aligned}$$

$$\Delta(x_0)\Delta(p_k) \ \geq \ \frac{\left|\left\langle \sinh\left(\frac{p_l}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(x_k)\Delta(p_k) \geq \frac{\left|\left\langle \cosh\left(\frac{p_l}{2\xi}\right)\right\rangle\right|}{2}\,,$$
$$\Delta(x_k)\Delta(p_0) \ \geq \ \frac{\left|\left\langle \sinh\left(\frac{p_l}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(x_l)\Delta(p_0) \geq \frac{|\langle p_k\rangle|}{4\xi}\,, \qquad (20)$$

respectively.

Obviously, for deformation parameter ξ approaching infinity the above relations become classical. It should be also noted that for momentum variables $p_\gamma = p_0 = p_k = 2\xi n\pi$ ($n = 0,\ \pm1,\ \pm2,\ \ldots$) all terms containing "sinus/cosinus" and "sinh / cosh" functions disappear, *i.e.* the deformation of Heisenberg uncertainty relations (18)–(20) becomes "minimal".

5. Nonrelativistic phase spaces and Heisenberg uncertainty principle

5.1. Nonrelativistic phase spaces

In this section we provide three nonrelativistic phase spaces (see (24)–(26)) with the use of contraction procedures of their relativistic counterparts (14)–(16). In a first step of our contraction scheme, we introduce the following redefinition of the relativistic phase space variables and the deformation parameter ξ, respectively

$$(i) \quad x_i = y_i\,, \qquad x_0 = ct\,, \qquad p_0 = \frac{\pi_0}{c}\,, \qquad p_i = \pi_i\,, \qquad \xi = \xi\,, \quad (21)$$

$$(ii) \quad x_i = y_i\,, \qquad x_0 = ct\,, \qquad p_0 = \frac{\pi_0}{c}\,, \qquad p_i = \pi_i\,, \qquad \xi = \frac{\hat{\xi}}{c}\,, \quad (22)$$

$$(iii) \quad x_i = y_i\,, \qquad x_0 = ct\,, \qquad p_0 = \frac{\pi_0}{c}\,, \qquad p_i = \pi_i\,, \qquad \xi = c\bar{\xi}\,. \quad (23)$$

Next, in a second step, we rewrite the phase spaces (14)–(16) in terms of t, y_i, π_0, π_i variables and deformation parameters ξ, $\hat{\xi}$, $\bar{\xi}$, and we take the (nonrelativistic) limit $c \to \infty$. In such a way we get the following Galilean phase spaces in the first case (see "rotation-like" carrier)

$$\begin{aligned}
(i) \quad [t, y_i] \ &= \ [y_k, y_l] = [\pi_\mu, \pi_\nu] = 0\,; \qquad i = k, l, \gamma\,,\\
[y_k, y_\gamma] \ &= \ \frac{i}{\xi} y_l\,, \qquad [y_l, y_\gamma] = -\frac{i}{\xi} y_k\,,\\
[t, \pi_i] \ &= \ [y_i, \pi_0] = [y_k, \pi_\gamma] = [y_l, \pi_\gamma] = 0\,,\\
[t, \pi_0] \ &= \ -i\,, \qquad [y_\gamma, \pi_\gamma] = i\,,\\
[y_\gamma, \pi_k] \ &= \ \frac{i}{2\xi}\pi_l\,, \qquad [y_\gamma, \pi_l] = -\frac{i}{2\xi}\pi_k\,,
\end{aligned}$$

$$\begin{aligned}
[y_l, \pi_l] &= i\cos\left(\frac{\pi_\gamma}{2\xi}\right) = [y_k, \pi_k] \ , \\
[y_k, \pi_l] &= -i\sin\left(\frac{\pi_\gamma}{2\xi}\right) = -\,[y_l, \pi_k] \ ,
\end{aligned} \tag{24}$$

in the second case (corresponding to the twist carrier $\{M_{kl}, P_0\}$)

$$\begin{aligned}
(ii) \quad [t, y_a] &= [y_k, y_l] = [\pi_\mu, \pi_\nu] = 0\,; \qquad a \neq k, l, 0\,, \\
[t, y_k] &= \frac{i}{\hat{\xi}} y_l\,, \qquad [t, y_l] = -\frac{i}{\hat{\xi}} y_k\,, \qquad [y_k, y_a] = [y_l, y_a] = 0\,, \\
[t, \pi_a] &= [y_a, \pi_0] = [y_k, \pi_a] = [y_l, \pi_a] = 0\,, \\
[t, \pi_0] &= -i\,, \qquad [y_a, \pi_a] = i\,, \qquad [y_a, \pi_k] = [y_a, \pi_l] = 0\,, \\
[y_k, \pi_0] &= [y_l, \pi_0] = 0\,, \\
[t, \pi_k] &= -\frac{i}{2\hat{\xi}}\pi_l\,, \qquad [t, \pi_l] = \frac{i}{2\hat{\xi}}\pi_k\,, \\
[y_l, \pi_l] &= i\cos\left(\frac{\pi_0}{2\hat{\xi}}\right) = [y_k, \pi_k] \ , \\
[y_k, \pi_l] &= i\sin\left(\frac{\pi_0}{2\hat{\xi}}\right) = -\,[y_l, \pi_k] \ ,
\end{aligned} \tag{25}$$

and in the last case (for "boost-like" carrier $\{M_{k0}, P_l\,;\ k \neq l\}$)

$$\begin{aligned}
(iii) \quad [t, y_a] &= [t, y_k] = [\pi_\mu, \pi_\nu] = 0\,; \qquad a \neq k, l, 0\,, \\
[y_k, y_a] &= [y_l, y_a] = 0\,, \qquad [t, y_l] = 0\,, \qquad [y_l, y_k] = -\frac{i}{\hat{\xi}} t\,, \\
[y_l, \pi_k] &= 0\,, \qquad [t, \pi_0] = -i\,, \qquad [t, \pi_k] = 0\,, \qquad [y_k, \pi_k] = i\,, \\
[y_a, \pi_a] &= i\,, \qquad [y_l, \pi_l] = i\,, \qquad [y_k, \pi_l] = [t, \pi_l] = 0\,, \\
[y_k, \pi_a] &= [y_l, \pi_a] = [t, \pi_a] = [y_a, \pi_l] = [y_a, \pi_k] = 0\,, \\
[y_a, \pi_0] &= [y_k, \pi_0] = [y_l, \pi_0] = 0\,,
\end{aligned} \tag{26}$$

respectively.

It should be noted that all above phase spaces can be get by direct application of Heisenberg double procedure as well. As it was mentioned in Introduction, the corresponding Hopf structures (*i.e.* the corresponding Galilei Hopf algebras $\mathcal{U}.(\mathcal{G})$ and their dual quantum groups $\mathcal{G}.$) have been obtained in [14, 15] with the use of contractions of their relativistic counterparts $\mathcal{U}_\xi(\mathcal{P})$ and $\mathcal{P}_\xi$. It should be mentioned, however, that such a treatment is more complicated technically than one used in presented subsection.

5.2. Heisenberg uncertainty principle

Let us now consider the Heisenberg uncertainty relations corresponding to the nonrelativistic phase spaces (24)–(26). Using (17), one can check that they take the form

$$
\begin{aligned}
(i)\quad \Delta(y_k)\Delta(y_\gamma) &\geq \frac{|\langle y_l\rangle|}{2\xi}\,, \qquad \Delta(y_l)\Delta(y_\gamma) \geq \frac{|\langle y_k\rangle|}{2\xi}\,,\\
\Delta(y_k)\Delta(\pi_k) &\geq \frac{\left|\left\langle \cos\left(\frac{\pi_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(y_l)\Delta(\pi_l) \geq \frac{\left|\left\langle \cos\left(\frac{\pi_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,,\\
\Delta(t)\Delta(\pi_0) &\geq \tfrac{1}{2}\,, \qquad \Delta(y_\gamma)\Delta(\pi_\gamma) \geq \tfrac{1}{2}\,,\\
\Delta(y_\gamma)\Delta(\pi_k) &\geq \frac{|\langle \pi_l\rangle|}{4\xi}\,, \qquad \Delta(y_\gamma)\Delta(\pi_l) \geq \frac{|\langle \pi_k\rangle|}{4\xi}\,,\\
\Delta(y_k)\Delta(\pi_l) &\geq \frac{\left|\left\langle \sin\left(\frac{\pi_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,, \qquad \Delta(y_l)\Delta(\pi_k) \geq \frac{\left|\left\langle \sin\left(\frac{\pi_\gamma}{2\xi}\right)\right\rangle\right|}{2}\,,
\end{aligned}
\tag{27}
$$

for the first deformation

$$
\begin{aligned}
(ii)\quad \Delta(y_k)\Delta(t) &\geq \frac{|\langle y_l\rangle|}{2\hat{\xi}}\,, \qquad \Delta(y_l)\Delta(t) \geq \frac{|\langle y_k\rangle|}{2\hat{\xi}}\,,\\
\Delta(y_k)\Delta(\pi_k) &\geq \frac{\left|\left\langle \cos\left(\frac{\pi_0}{2\hat{\xi}}\right)\right\rangle\right|}{2}\,, \qquad \Delta(y_l)\Delta(\pi_l) \geq \frac{\left|\left\langle \cos\left(\frac{\pi_0}{2\hat{\xi}}\right)\right\rangle\right|}{2}\,,\\
\Delta(t)\Delta(\pi_0) &\geq \tfrac{1}{2}\,, \qquad \Delta(y_a)\Delta(\pi_a) \geq \tfrac{1}{2}\,,\\
\Delta(t)\Delta(\pi_k) &\geq \frac{|\langle \pi_l\rangle|}{4\hat{\xi}}\,, \qquad \Delta(t)\Delta(\pi_l) \geq \frac{|\langle \pi_k\rangle|}{4\hat{\xi}}\,,\\
\Delta(y_k)\Delta(\pi_l) &\geq \frac{\left|\left\langle \sin\left(\frac{\pi_0}{2\hat{\xi}}\right)\right\rangle\right|}{2}\,, \qquad \Delta(y_l)\Delta(\pi_k) \geq \frac{\left|\left\langle \sin\left(\frac{\pi_0}{2\hat{\xi}}\right)\right\rangle\right|}{2}\,,
\end{aligned}
\tag{28}
$$

in the second case, and

$$
\begin{aligned}
(iii)\quad \Delta(y_k)\Delta(y_l) &\geq \frac{|\langle t\rangle|}{2\bar{\xi}}\,, \qquad \Delta(t)\Delta(\pi_0) \geq \tfrac{1}{2}\,, \qquad \Delta(y_l)\Delta(\pi_l) \geq \tfrac{1}{2}\,,\\
\Delta(y_a)\Delta(\pi_a) &\geq \tfrac{1}{2}\,, \qquad \Delta(y_k)\Delta(\pi_k) \geq \tfrac{1}{2}\,,
\end{aligned}
\tag{29}
$$

for the last twist factor.

Of course, for deformation parameters ξ, $\hat{\xi}$ and $\bar{\xi}$ approaching infinity the relations (24)–(26) as well as (27)–(29) become classical. Moreover, for momentum variables $\pi_\gamma = \pi_0 = \pi_k = 2\alpha n\pi$ ($n = 0,\ \pm1,\ \pm2,\ \ldots; \alpha = \xi, \hat{\xi}, \bar{\xi}$) the oscillating and expanding terms in the above relations disappear.

6. Final remarks

In this article we construct six relativistic and nonrelativistic phase spaces corresponding to the Lie-algebraically deformed Poincaré and Galilei Hopf algebras, respectively. The considered phase spaces are provided with the use of Heisenberg double procedure [9].

It should be noted that presented results compleat our studies on the Lie-algebraically twisted groups at both levels — at the level of relativistic and nonrelativistic symmetries as well. Moreover, the provided phase spaces constitute the background for future construction of basic dynamical models associated with twisted symmetries. As it was mentioned in Introduction, such investigations have been already performed in the case of nonrelativistic particle moving in a field of constant force [31], and in the case of harmonic oscillator model [32]. However, used in [31,32] phase spaces have been taken *ad hoc*, *i.e.* without any formal, quantum group-like construction, such as, for example, Heisenberg double procedure. The studies in this direction are in progress.

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