SOLUTION OF DIRAC EQUATIONS WITH FIVE-PARAMETER EXPONENT-TYPE POTENTIAL

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Solving the Dirac equation with equal five-parameter exponent-type scalar and vector potentials in terms of the supersymmetric quantum mechanics method and shape invariance approach, we obtain the exact energy equation for the s-wave bound states. This work is performed under the conditions of the spin symmetry and pseudospin symmetry.

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1. Introduction

Pseudospin symmetry has been mainly observed at about thirty years ago, in the spherical atomic nuclei [1,2]. This symmetry was used for explaining quasi-degeneracy in heavy nuclei, between single nucleon doublets with quantum numbers $(n_r, l, j = l+\frac{1}{2})$ and $(n_r-1, l+2, j' = j+1 = l+\frac{3}{2})$, where n_r, l and j are the single-nucleon radials, orbital and total angular momentum quantum numbers, respectively. "Pseudo"-orbital angular momentum was defined as $\tilde{l} = l+1$. Afterwards, Ginocchio exhibited the "pseudo-orbital angular momentum" is nothing but the "orbital angular momentum" of the lower component of the Dirac spinor [3]. For example $(n_r s_{1/2}, (n_r-1)d_{3/2})$ we have $\tilde{l} = 1$, and $(n_r p_{3/2}, (n_r-1)f_{3/2})$ we have $\tilde{l} = 2$, *etc.* These doublets are almost degenerate in proportion to pseudospin $\tilde{s} = 1/2$, since $j = \tilde{l} + \tilde{s}$, for the two states in the doublet. In the work of [4] Ginocchio has shown that quasi-degenerate pseudospin doublets in nuclei arise from the near equality in the magnitudes of an attractive scalar potential, $S(r)$, and a repulsive vector potential, $V(r)$, *i.e.*, $S(r) \approx V(r)$. In this work he has shown that

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there are no bound Dirac valence state for realistic mean fields in the exact limit of pseudospin symmetry, $S(r) = V(r)$, and therefore nuclei would not exist. So a small breaking of pseudospin symmetry occurs in the nuclei [5]. This case is true for a spherical Coulomb potential and a spherical potential [6]. After the work of Ginocchio, some authors [7] showed that the pseudospin symmetry is exactly under a more general condition, $\frac{d\Sigma(r)}{dr} = 0$. In another term, with this condition, the Dirac Hamiltonian commutes with pseudospin generators. When $\Delta(r) = V(r) - S(r) = \text{constant}$, we have spin symmetry [8]. The Dirac Hamiltonian with spherically symmetric scalar and vector potentials is invariant under a SU(2) algebra for two limits: $S(r) = V(r) + C_{\rm S}$ and $S(r) = -V(r) + C_{\rm PS}$ where $C_{\rm S}$, $C_{\rm PS}$ are constants[1] [9]. In the recent years, the Dirac equation with vector and scalar potential has been solved with many relativistic typical potential under the condition of spin symmetry and pseudospin symmetry. Cheng *et al.* [10], using a Dirac Hamiltonian with scalar S and vector V potentials quadratic in space coordinates, have found a harmonic-oscillator-like second order equation which can be solved analytically for $\Delta = V - S - 0$ and also $\Sigma = S + V = 0$. Lisboa *et al.* [11] have studied the relativistic harmonic oscillator for spin-$\frac{1}{2}$ particles, and by setting either $\Sigma(r) = 0$ or $\Delta(r) = 0$ have obtained the bound state solutions for the Dirac equation. The authors of [12] have solved the Dirac equation with relativistic Woods–Saxon vector and scalar potentials. They have considered the cases of Pb^{208}, Ca^{48}, and Ca^{40} and drew the systematics for the variation of pseudospin splittings with the Woods–Saxon potential parameters and have shown that this symmetry is a good approximation in these cases. Some researchers [13] have obtained bound state solutions of the Pöschl–Teller with spin symmetry and pseudospin symmetry by solving the Dirac equation with attractive scalar and repulsive vector potentials. Recently, some authors in Ref. [14] have investigated the energy equations for the s-wave bound states of the five-parameter exponent-type potential in the Klein–Gordon theory with equally mixed potentials. In these works, they only considered the case of the spin symmetry limitation, *i.e.*, set the difference between the vector and scalar potentials to zero, $\Delta(r) = 0$. Therefore, following Ref. [14] it is interesting to study the pseudospin symmetry in the relativistic five-parameter exponent-type potential. In this work by the method of the super symmetric quantum mechanics and shape invariance we investigate the analytic solutions of Dirac equation for the five-parameter exponent-type vector and scalar potential. The five-parameter exponent-type potential is general solvable exponential-type potentials [15]. By choosing accurate parameters in the

[1] If one intends to get a bound state and the potentials S and V are asymptotically exponentially decreasing, they must go to zero as $r \to \infty$. This imply that, $C_{\rm S} = 0$ and $C_{\rm PS} = 0$.

five-parameter exponential-type potential, we can obtain many typical potentials, such as the Hulthén well, Eckart, Scarf II, generalized Pöschl–Teller and Pöschl–Teller II potentials, *etc.*, [15, 16]. The Hulthén potential [17] is a short-range potential in physics. This potential has been widely used in several branches of physics such as nuclear and particle physics [18], atomic physics [19], molecular physics [20], and chemical physics [21]. and its bound states and scattering properties have been investigated by a variety of techniques [22]. The well-known Eckart potential, introduced by Eckart [23] in 1930, is widely used in physics [24] and chemical physics [25]. The energy equation for the Eckart potential has been investigated in the Klein–Gordon and Dirac theory with equally mixed potentials by using the function analysis method [26, 27]. In Ref. [28], it has been shown that the Klein, Gordon and Dirac equations have the same spectra for the same potentials in spin or pseudospin conditions. The bound state energy spectra and the corresponding wavefunctions for the Scarf-type potential have been investigated by a variety of methods [29]. The Pöschl–Teller potential can be used as the electron–nucleus potential [30]. This potential can be described in the Heisenberg isotropic model with two interacting spins [31].

2. Bound state solutions

The Hamiltonian of a Dirac particle of mass M in an external scalar, $S(r)$, and vector, $V(r)$, potentials is given by:

$$H = \alpha \cdot P + \beta(M + S(r)) + V(r)\,, \tag{1}$$

where α and β are the Dirac matrices. The Dirac equation can be written as ($\hbar = c = 1$):

$$\{\alpha \cdot P + \beta[M + S(r)] + V(r)\}\,\Psi(r) = E\,\Psi(r)\,. \tag{2}$$

For spherical nuclei, the nucleon angular momentum J and spin matrix operator $\hat{K} = -\hat{\beta}(\hat{\sigma}\cdot\hat{L}+1)$ commute with the Dirac Hamiltonian. The eigenvalues of $\hat{K}$ are: $k = \pm(j+1)$ with $-$ for aligned spin ($s_{1/2}, p_{3/2}$, *etc.*) and $+$ for unaligned spin ($p_{1/2}, d_{3/2}$, *etc.*). Hence, we use the quantum number k since it is sufficient to label the orbitals. For a given $k = \pm1, \pm2, \dots\,,\ j = |k+\frac{1}{2}| - \frac{1}{2},\ \tilde{l} = |k-\frac{1}{2}| - \frac{1}{2}$. The wave functions can be classified according to their angular momentum j, k, and the radial quantum number n, and can be written in the following form:

$$\Psi_{nk} = \frac{1}{r}\left[F_{nk}(r)Y^{l}_{jm}(\theta,\phi), iG_{nk}(r)Y^{\tilde{l}}_{jm}(\theta,\phi)\right]\,, \tag{3}$$

where $F_{nk}(r)$ and $G_{nk}(r)$ are the upper and lower radial functions, $Y_{jm^l}(\theta,\phi)$ and $Y_{jm^{\tilde{l}}}(\theta,\phi)$ are the spherical harmonic functions, and m is the projection of angular momentum on the third axis. The orbital angular-momentum quantum numbers l and $\tilde{l}$ are the labels of upper and lower components, respectively. We replace Eq. (3) into Eq. (2) and obtain two coupled differential equations for the upper and lower radial wave functions, $F_{nk}(\mathrm{r})$ and $G_{nk}(r)$,

$$\left(\frac{d}{dr}+\frac{k}{r}\right)F_{nk}(r)=[M+E_{nk}+S(r)-V(r)]G_{nk}(r)\,, \tag{4}$$

$$\left(\frac{d}{dr}-\frac{k}{r}\right)G_{nk}(r)=[M-E_{nk}+S(r)+V(r)]F_{nk}(r)\,, \tag{5}$$

by substituting $G_{nk}(r)$ from Eq. (4) into Eq. (5) and $F_{nk}(r)$ from Eq. (5) into Eq. (4), we obtain the following two second-order differential equations for the upper and lower components,

$$\begin{aligned}&\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-(M+E_{nk}-\varDelta)(M-E_{nk}+\varSigma)\right.\\&\qquad\left.+\frac{\frac{d\varDelta}{dr}\left(\frac{d}{dr}+\frac{k}{r}\right)}{M+E_{nk}-\varDelta}\right)F_{nk}(r)=0\,,\end{aligned} \tag{6}$$

$$\begin{aligned}&\left(\frac{d^2}{dr^2}-\frac{\tilde{l}(\tilde{l}+1)}{r^2}-(M+E_{nk}-\varDelta)(M-E_{nk}+\varSigma)\right.\\&\qquad\left.-\frac{\frac{d\varSigma}{dr}\left(\frac{d}{dr}-\frac{k}{r}\right)}{M-E_{nk}+\varSigma}\right)G_{nk}(r)=0\,,\end{aligned} \tag{7}$$

where $\varDelta(r)=V(r)-S(r)$ and $\varSigma(r)=V(r)+S(r)$. In these equations, we have used the relations $k(k+1)=l(l+1)$ and $k(k-1)=\tilde{l}(\tilde{l}+1)$. We now, impose the condition of spin symmetry, $\varDelta(r)=C_{\mathrm{S}}=$ constant and pseudospin symmetry, $\varSigma(r)=C_{\mathrm{PS}}=$ constant, on this equation and solve them by using the shape invariant approach.

2.1. Spin symmetry

The condition of pseudospin symmetry is $\frac{d\varDelta}{dr}=0$ or $\varDelta=C_{\mathrm{S}}$, here we consider $C_{\mathrm{S}}=0$, substituting $\varDelta=0$ and $\varSigma=2V$ into equation (6), we have,

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-(M+E_{nk})(M-E_{nk}+2V)\right)F_{nk}(r)=0 \tag{8}$$

the eigen energies, E_{nk}, depend only on n and l, *i.e.*, $E_{nk}=E(n,l(l+1))$. For $l\neq 0$, the states with $j=l\pm\frac{1}{2}$ are degenerate. This is a SU(2) spin symmetry. We consider the five-parameter exponential-type potential model, which is written in the form of [15]:

$$
\begin{aligned}
V(r) \;=\; & \frac{Q_3^2 + g - \frac{Q_2^2}{q} + 2\alpha Q_2}{e^{2\alpha r} + q} + \frac{Q_2^2 - qQ_3^2 - 2\alpha q Q_2}{(e^{2\alpha r} + q)^2} \\
& + \frac{g\frac{Q_3}{Q_2} - \frac{Q_2 Q_3}{q} + \alpha Q_3}{e^{2\alpha r} + q} e^{\alpha r} + \frac{2Q_2 Q_3 - 2\alpha q Q_3}{(e^{2\alpha r} + q)^2} e^{\alpha r}\,,
\end{aligned} \tag{9}
$$

where the range of parameter q is $-1 \leq q < 0$ or $q > 0$. Substituting Eq. (9) into Eq. (8), we obtain a Schrödinger-like equation for the s-wave ($l = 0$, *i.e.*, $k = -1$),

$$
\begin{aligned}
&\Bigg[-\frac{d^2}{dr^2} + (M + E_{n,-1})\left(2\left(\frac{Q_3^2 + g - \frac{Q_2^2}{q} + 2\alpha Q_2}{e^{2\alpha r} + q} + \frac{Q_2^2 - qQ_3^2 - 2\alpha q Q_2}{(e^{2\alpha r} + q)^2}\right.\right. \\
&\left.\left. + \frac{g\frac{Q_3}{Q_2} - \frac{Q_2 Q_3}{q} + \alpha Q_3}{e^{2\alpha r} + q} e^{\alpha r} + \frac{2Q_2 Q_3 - 2\alpha q Q_3}{(e^{2\alpha r} + q)^2} e^{\alpha r}\right)\right)\Bigg] F_{n,-1}(r) = \tilde{E} F_{n,-1}(r)\,,
\end{aligned} \tag{10}
$$

where $\tilde{E} = E_{n,-1}^2 - M^2$. We solve Eq. (10) by using the basic concepts of the supersymmetric quantum mechanics formalism [32] and shape invariance approach [33]. The ground-state function for the upper component $F_{nk}(r)$ can be written in the form of

$$
F_{0,-1} = \exp(-\int W(r) dr)\,, \tag{11}
$$

where $W(r)$ is called a superpotential in supersymmetric quantum mechanics. Substituting Eq. (11) into Eq. (10), we obtain the following equation for $W(r)$

$$
\begin{aligned}
W(r)^2 - \frac{dW(r)}{dr} \;=\; & (M + E_{n,-1})\left(2\left(\frac{Q_3^2 + g - \frac{Q_2^2}{q} + 2\alpha Q_2}{e^{2\alpha r} + q}\right.\right. \\
& + \frac{Q_2^2 - qQ_3^2 - 2\alpha q Q_2}{(e^{2\alpha r} + q)^2} + \frac{g\frac{Q_3}{Q_2} - \frac{Q_2 Q_3}{q} + \alpha Q_3}{e^{2\alpha r} + q} e^{\alpha r} \\
&\left.\left. + \frac{2Q_2 Q_3 - 2\alpha q Q_3}{(e^{2\alpha r} + q)^2} e^{\alpha r}\right)\right) - \tilde{E}_0\,,
\end{aligned} \tag{12}
$$

where $\tilde{E}_0$ is the ground-state energy. Equation (12) is a non-linear Riccati equation. We consider the superpotential $W(r)$ as [15]

$$W(r) = \frac{g}{2Q_2} - \frac{Q_2}{2q} + \frac{Q_2}{e^{2\alpha r}+q} + \frac{Q_3}{e^{2\alpha r}+q} e^{\alpha r} . \tag{13}$$

By substituting this relation into Eq. (12) we obtain the following relation

$$\frac{g^2}{4Q_2^2} + \frac{Q_2^2}{4q^2} - \frac{g}{2q} = -\tilde{E}_0 . \tag{14}$$

By substituting Eqs (13) into (11), we obtain the ground-state function for $q > 0$ and $-1 \leq q < 0$, respectively

$$F_{0,-1}(r) = e^{-\left(\frac{g}{2Q_2}-\frac{Q_2}{2q}\right)r}\left(\frac{e^{2\alpha r}}{e^{2\alpha r}+q}\right)^{\frac{-Q_2}{2\alpha q}} \exp\left[-\frac{Q_3}{\alpha\sqrt{q}} \arg\tan\left(\frac{e^{\alpha r}}{\sqrt{q}}\right)\right], \tag{15}$$

$$F_{0,-1}(r) = e^{-\left(\frac{g}{2Q_2}-\frac{Q_2}{2q}\right)r}\left(\frac{e^{2\alpha r}}{e^{2\alpha r}+q}\right)^{-\frac{Q_2}{2\alpha q}} \exp\left[\frac{Q_3}{\alpha\sqrt{-q}} \arg\coth\left(\frac{e^{\alpha r}}{\sqrt{-q}}\right)\right]. \tag{16}$$

In this work, we will deal with bound-state solutions, *i.e.*, the radial part of the wave function Ψ_{nk} must satisfy the boundary conditions that $\frac{F_{nk}(r)}{r}$ becomes zero when $r \to \infty$, and $\frac{F_{nk}(r)}{r}$ is finite at $r = 0$. It is clear that, when $r \to \infty, F_{0,-1}(r)$ is finite, and $F_{0,-1}(r) = 0$ at the origin point $r = 0$, the radial wave function $\frac{F_{0,-1}(r)}{r}$ can satisfy the boundary conditions. In terms of the superpotential $W(r)$, the supersymmetric partner potentials are given by

$$\begin{aligned} V_2(r,a_2) &= W^2(r) - \frac{dW(r)}{dr} \\ &= \left(\frac{g}{2Q_2} - \frac{Q_2}{2q}\right)^2 + \frac{Q_3^2 + g - \frac{Q_2^2}{q} + 2\alpha Q_2}{e^{2\alpha r}+q} + \frac{Q_2^2 - qQ_3^2 - 2\alpha q Q_2}{(e^{2\alpha r}+q)^2} \\ &\quad + \frac{\left(g\frac{Q_3}{Q_2} - \frac{Q_2Q_3}{q} + \alpha Q_3\right)}{e^{2\alpha r}+q} e^{\alpha r} + \frac{2Q_2Q_3 - 2\alpha Q_3}{(e^{2\alpha r}+q)^2} e^{\alpha r} , \end{aligned} \tag{17}$$

$$\begin{aligned} V_1(r,a_1) &= W^2(r) + \frac{dW(r)}{dr} \\ &= \left(\frac{g}{2Q_2} - \frac{Q_2}{2q}\right)^2 + \frac{Q_3^2 + g - \frac{Q_2^2}{q} - 2\alpha Q_2}{e^{2\alpha r}+q} + \frac{Q_2^2 - qQ_3^2 + 2\alpha q Q_2}{(e^{2\alpha r}+q)^2} \\ &\quad + \frac{\left(g\frac{Q_3}{Q_2} - \frac{Q_2Q_3}{q} - \alpha Q_3\right)}{e^{2\alpha r}+q} e^{\alpha r} + \frac{2Q_2Q_3 + 2\alpha q Q_3}{(e^{2\alpha r}+q)^2} e^{\alpha r} . \end{aligned} \tag{18}$$

Considering $a_1 = Q_2$ and $a_2 = Q_2 + 2\alpha q$, we can easily show that the two partner potentials $V_1(r, a_1)$ and $V_2(r, a_2)$ satisfy the following relationship:

$$V_1(r, a_1) = V_2(r, a_2) + R(a_2)\,. \tag{19}$$

The remainder $R(a_2)$ is independent of r,

$$R(a_2) = \left(\frac{g}{2Q_2} - \frac{Q_2}{2q}\right)^2 - \left(\frac{g}{2(Q_2 + 2\alpha q)} - \frac{Q_2 + 2\alpha q}{2q}\right)^2 . \tag{20}$$

Equation (19) shows that the two partner potentials $V_1(r, a_1)$ and $V_2(r, a_2)$ are the shape invariant potentials. The energy spectra of the potential $V_2(r, a_2)$ can be determined by using the shape invariance approach [32]. The energy spectra of the potential $V_2(r, a_2)$ are given by

$$\tilde{E_0}^{(2)} = 0\,, \tag{21}$$

$$\begin{aligned} \tilde{E_n}^{(2)} &= \sum_{k=2} R(a_k) = R(a_2) + R(a_3) + \cdots + R(a_n) \\ &= \left(\frac{g}{2Q_2} - \frac{Q_2^2}{2q}\right)^2 - \left(\frac{g}{2(Q_2 + 2n\alpha q)} - \frac{Q_2 + 2n\alpha q}{2q}\right)^2 , \end{aligned} \tag{22}$$

where the quantum number $n = 0, 1, 2, \ldots$. From Eqs (12) and (17), we obtain the following relation:

$$\begin{aligned} V_{\text{eff}} = (M + E_{n,-1}) &\left(2\left(\frac{Q_3^2 + g - \frac{Q_2^2}{q} + 2\alpha Q_2}{e^{2\alpha r} + q} + \frac{Q_2^2 - qQ_3^2 - 2\alpha q Q_2}{(e^{2\alpha r} + q)^2}\right.\right. \\ &\left.\left. + \frac{g\frac{Q_3}{Q_2} - \frac{Q_2 Q_3}{q} + \alpha Q_3}{e^{2\alpha r} + q} e^{\alpha r} + \frac{2Q_2Q_3 - 2\alpha q Q_3}{(e^{2\alpha r} + q)^2} e^{\alpha r}\right)\right) = V_2(r, a_2) + \tilde{E}_0\,. \end{aligned} \tag{23}$$

From Eqs (14), (21), (22), and (23), we find the solution for Eq. (10),

$$\begin{aligned} \tilde{E_n} &= \tilde{E_n}^{(2)} + \tilde{E_0} = \sum_{k=2} R(a_k) - \left(\frac{g}{2Q_2} - \frac{Q_2}{2q}\right)^2 \\ &= R(a_2) + R(a_3) + \cdots + R(a_n) - \left(\frac{g}{2Q_2} - \frac{Q_2}{2q}\right)^2 \\ &= -\left(\frac{g}{2(Q_2 + 2n\alpha q)} - \frac{Q_2 + 2n\alpha q}{2q}\right)^2 . \end{aligned} \tag{24}$$

By using $\tilde{E} = E_{n,-1}^2 - M^2$, we obtain the energy equation for the five-parameter exponent-type potential potential with spin symmetry in the Dirac theory,

$$E_{n,-1}^2 - M^2 = -\left(\frac{g}{2(Q_2 + 2n\alpha q)} - \frac{Q_2 + 2n\alpha q}{2q}\right)^2 . \tag{25}$$

In order to generalize the results related to this potential to other potential, we use these definitions:

$$\begin{aligned}
P_3^2 + \tilde{g} - \frac{P_2^2}{q} + 2\alpha P_2 &= 2(M + E_{n,-1})\left(Q_3^2 + g - \frac{Q_2^2}{q} + 2\alpha Q_2\right) , && (26)\\
P_2^2 - qP_3^2 - 2\alpha q P_2 &= 2(M + E_{n,-1})\left(Q_2^2 - qQ_3^2 - 2\alpha q Q_2\right) , && (27)\\
\tilde{g}\frac{P_3}{P_2} - \frac{P_2P_3}{q} + \alpha P_3 &= 2(M + E_{n,-1})\left(g\frac{Q_3}{Q_2} - \frac{Q_2Q_3}{q} + \alpha Q_3\right) , && (28)\\
2P_2P_3 - 2\alpha q P_3 &= 2(M + E_{n,-1})(2Q_2Q_3 - 2\alpha q Q_3) && (29)
\end{aligned}$$

with this definition the (9) potential changes to new (30) potential

$$\begin{aligned}
V(r) \;=\; & \frac{P_3^2 + \tilde{g} - \frac{P_2^2}{q} + 2\alpha P_2}{e^{2\alpha r} + q} + \frac{P_2^2 - qP_3^2 - 2\alpha q P_2}{(e^{2\alpha r} + q)^2} \\
& + \frac{\tilde{g}\frac{P_3}{P_2} - \frac{P_2P_3}{q} + \alpha P_3}{e^{2\alpha r} + q} e^{\alpha r} + \frac{2P_2P_3 - 2\alpha q P_3}{(e^{2\alpha r} + q)^2} e^{\alpha r} .
\end{aligned} \tag{30}$$

The energy equation becomes as following :

$$E_{n,-1}^2 - M^2 = -\left(\frac{\tilde{g}}{2(P_2 + 2n\alpha q)} - \frac{P_2 + 2n\alpha q}{2q}\right)^2 . \tag{31}$$

The eigen function will be also obtained from the below relations for the cases of $q > 0$ and $-1 \le q < 0$, respectively,

$$\begin{aligned}
F_{0,-1} &= e^{-\left(\frac{g}{2P_2} - \frac{P_2}{2q}\right) r} \left(\frac{e^{2\alpha r}}{e^{2\alpha r} + q}\right)^{\frac{-P_2}{2\alpha q}} \exp\left[-\frac{P_3}{\alpha\sqrt{q}} \arg\tan\left(\frac{e^{\alpha r}}{\sqrt{q}}\right)\right], && (32)\\
F_{0,-1} &= e^{-(\frac{g}{2P_2} - \frac{P_2}{2q}) r} (\frac{e^{2\alpha}}{e^{2\alpha r} + q})^{-\frac{P_2}{2\alpha q}} \exp\left[\frac{P_3}{\alpha\sqrt{-q}} \arg\coth\left(\frac{e^{\alpha r}}{\sqrt{-q}}\right)\right]. && (33)
\end{aligned}$$

2.2. Pseudospin symmetry

The condition of pseudospin symmetry is $\frac{d\Sigma}{dr} = 0$ or $\Sigma = C_{\mathrm{PS}}$, here we consider $C_{\mathrm{PS}} = 0$, substituting $\Sigma = 0$ and $\Delta = 2V$ into Eq. (7) we have:

$$\left(\frac{d^2}{dr^2} - \frac{\tilde{l}(\tilde{l} + 1)}{r^2} - (M + E_{nk} - 2V)(M - E_{nk})\right) G_{nk}(r) = 0 . \tag{34}$$

Equation (34) shows that the energy eigenvalues, E_{nk}, depend only on n and $\tilde{l}$, *i.e.*, $E_{nk} = E(n, \tilde{l}(\tilde{l}+1))$. For $\tilde{l} \neq 0$, the states with $j = \tilde{l} \pm \frac{1}{2}$ are degenerate. This is a SU(2) pseudospin symmetry. By substituting Eq. (30) into Eq. (34) we can obtain a Schrödinger-like equation for the lower spinor component $G_{nk}(r)$ in the case of the s-wave ($\tilde{l} = 0$, *i.e.*, $k = 1$),

$$\left[-\frac{d^2}{dr^2} - (M - E_{n,-1})\left(2\left(\frac{P_3^2 + \tilde{g} - \frac{P_2^2}{q} + 2\alpha P_2}{e^{2\alpha r} + q} + \frac{P_2^2 - qP_3^2 - 2\alpha q P_2}{(e^{2\alpha r} + q)^2}\right.\right.\right.$$
$$\left.\left.\left. + \frac{\tilde{g}\frac{P_3}{P_2} - \frac{P_2 P_3}{q} + \alpha P_3}{e^{2\alpha r} + q} e^{\alpha r} + \frac{2P_2P_3 - 2\alpha q P_3}{(e^{2\alpha r} + q)^2} e^{\alpha r}\right)\right)\right] G_{n,-1}(r) = \tilde{E} G_{n,-1}(r)\,, \tag{35}$$

where $\tilde{E} = E_{n,-1}^2 - M^2$. Note that g and P_2 are different for spin and pseudospin cases as we can see in several examples in next section. Applying the same scheme used for solving Eq. (10), we can obtain the energy equation for the five-parameter exponent-type potential with pseudospin symmetry in the Dirac theory,

$$E_{n,-1}^2 - M^2 = -\left(\frac{\tilde{g}}{2(P_2 + 2n\alpha q)} - \frac{P_2 + 2n\alpha q}{2q}\right)^2, \tag{36}$$

where the quantum number $n = 0, 1, 2, 3, \ldots$.

Although the energy spectra of (31) and (36), related to spin and pseudospin symmetry conditions, respectively are formally the same, $\tilde{g}$ and P_2 parameters are deferent for these symmetries. This point is clearly identified in the following cases.

3. Discussion

By choosing appropriate parameters in the five parameter exponential-type potential model, we obtain four typical potential models and then argued their energy equations in the Dirac theory with spin symmetry and pseudospin symmetry conditions.

3.1. Hulthén potential

In Eq. (9), if we replace, $\alpha = \frac{1}{2a}, q = -1, Q_3 = 0$, and put:

$$g + Q_2^2 + \frac{Q_2}{a} = -V_0\,, \tag{37}$$

$$Q_2^2 + \frac{Q_2}{a} = 0 \tag{38}$$

the five-parameter exponential type given in Eq. (9) turns into the Hulthén potential [34].

$$V(r) = -\frac{V_0}{e^{\frac{r}{a}} - 1}\,. \tag{39}$$

Substituting Eqs (37) and (38) into Eqs (26) and (27), we obtain following parameters for spin symmetry condition,

$$P_2 = -\frac{1}{a}\,, \qquad \tilde{g} = -2V_0(E_{n,-1} + M) \tag{40}$$

and corresponding parameters for pseudospin symmetry condition are given by:

$$P_2 = -\frac{1}{a}\,, \qquad \tilde{g} = -2V_0(E_{n,-1} - M)\,. \tag{41}$$

Substituting Eqs (40), (41) into Eq. (31) or (36), we obtain the exact energy equation of the Hulthén potential in the Dirac theory with spin and pseudospin symmetry condition, respectively

$$\left(M^2 - E_{n,-1}^2\right)^{1/2} = \frac{aV_0(E_{n,-1} + M)}{n+1} - \frac{n+1}{2a}\,, \tag{42}$$

$$\left(M^2 - E_{n,-1}^2\right)^{1/2} = \frac{aV_0(E_{n,-1} - M)}{n+1} - \frac{n+1}{2a}\,, \tag{43}$$

where $n = 0, 1, 2, \ldots$. As one can see the only difference between Eqs (42), (43) is in the sign of mass term in the right hand sides of these equations. In fact as has been shown in [35] one can get from spin to pseudospin symmetry by a chiral transformation performed by γ^5. This amounts basically to change the mass sign in the formulas for the energy spectra.

3.2. Scarf II potential

If we replace $q = 1$, $g = 0$, and set:

$$\begin{aligned} Q_2^2 - Q_3^2 - 2\alpha Q_2 &= -4\left[B^2 - A(A+\alpha)\right]\,, && (44) \\ Q_2 Q_3 - \alpha Q_3 &= -2B(2A+\alpha) && (45) \end{aligned}$$

the potential (9) turns into the Scarf II potential [36]

$$V(r) = \left[B^2 - A(A+\alpha)\right]\sec^2\alpha r + B(2A+\alpha)\sec\alpha r\tanh\alpha r\,, \tag{46}$$

where $A > 0$. Substituting Eqs (44) and (45) into Eqs (26)–(29), and solving them, we obtain the $\tilde{g}$ and P_2 parameters for spin symmetry condition,

$$\tilde{g} \;=\; 0\,, \tag{47}$$

$$\begin{aligned} P_2 \;=\; \alpha \Bigg(1 - \sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(-B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2i(E_{n,-1}+M)(2AB+\alpha B)}{\alpha^2}} \\ - \sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(-B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2i(E_{n,-1}+M)(2AB+\alpha B)}{\alpha^2}} \Bigg)\,. \end{aligned} \tag{48}$$

and for pseudospin symmetry condition:

$$\tilde{g} \;=\; 0\,, \tag{49}$$

$$\begin{aligned} P_2 \;=\; \alpha \Bigg(1 - \sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(-B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2i(E_{n,-1}-M)(2AB+\alpha B)}{\alpha^2}} \\ - \sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(-B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2i(E_{n,-1}-M)(2AB+\alpha B)}{\alpha^2}} \Bigg)\,. \end{aligned} \tag{50}$$

Substituting Eqs (47), (48) and (49), (50) into Eq. (31) or (36), we get the exact energy equation for the Scarf II potential in the Dirac theory with spin and pseudospin symmetry condition, respectively,

$$\begin{aligned} M^2 - E_{n,-1}^2 = \alpha^2 \Bigg[n + \tfrac{1}{2} \\ -\tfrac{1}{2}\Bigg(\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(-B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2(E_{n,-1}+M)(2AB+\alpha B)}{i\alpha^2}} \\ + \sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(-B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2(E_{n,-1}+M)(2AB+\alpha B)}{i\alpha^2}} \Bigg) \Bigg]^2\,, \end{aligned} \tag{51}$$

$$\begin{aligned} M^2 - E_{n,-1}^2 = \alpha^2 \Bigg[n + \tfrac{1}{2} \\ -\tfrac{1}{2}\Bigg(\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(-B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2(E_{n,-1}-M)(2AB+\alpha B)}{i\alpha^2}} \\ + \sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(-B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2(E_{n,-1}-M)(2AB+\alpha B)}{i\alpha^2}} \Bigg) \Bigg]^2\,, \end{aligned} \tag{52}$$

3.3. Generalized Pöschl–Tellerr potential

By choosing $q = -1$, $g = 0$, and putting:

$$Q_2^2 + Q_3^2 + 2\alpha Q_2 \;=\; 4\left[B^2 + A(A+\alpha)\right]\,, \tag{53}$$

$$Q_2 Q_3 + \alpha Q_3 \;=\; -2B(2A+\alpha) \tag{54}$$

the potential (9) turns into the generalized Pöschl–Teller potential [36].

$$V(r) = (B^2 + A^2 + \alpha A)\cosh^2 \alpha r - B(2A + \alpha)\cosh \alpha r \coth \alpha r\,, \tag{55}$$

where $B > A$. With replacing Eqs (53) and (54) into Eqs (26)–(29), we can obtain for spin symmetry,

$$\tilde{g} = 0\,, \tag{56}$$

$$\begin{aligned} P_2 = \alpha \Bigg[-1 + \Bigg(\sigma & \\ \times\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2i(E_{n,-1}+M)(2AB+\alpha B)}{\alpha^2}} & \\ +\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2(E_{n,-1}+M)(2AB+\alpha B)}{\alpha^2}}\Bigg)\Bigg] & \end{aligned} \tag{57}$$

and for pseudospin symmetry:

$$\tilde{g} = 0\,, \tag{58}$$

$$\begin{aligned} P_2 = \alpha \Bigg[-1 + \Bigg(\sigma & \\ \times\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2i(E_{n,-1}-M)(2AB+\alpha B)}{\alpha^2}} & \\ +\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2(E_{n,-1}-M)(2AB+\alpha B)}{\alpha^2}}\Bigg)\Bigg]\,, & \end{aligned} \tag{59}$$

where $\sigma = \pm 1$. Substituting the Eqs (56), (57) and (58), (59) into Eq. (31) or (36), we get the exact energy equation for the generalized Pöschl–Teller potential in the Dirac theory with spin and pseudospin symmetry condition.

$$\begin{aligned} M^2 - E_{n,-1}^2 = \alpha^2 \Bigg[-n - \tfrac{1}{2} + \tfrac{1}{2}\Bigg(\sigma & \\ \times\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2(E_{n,-1}+M)(2AB+\alpha B)}{\alpha^2}} & \\ +\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}+M)(B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2(E_{n,-1}+M)(2AB+\alpha B)}{\alpha^2}}\Bigg)\Bigg]^2\,, & \end{aligned} \tag{60}$$

$$\begin{aligned} M^2 - E_{n,-1}^2 = \alpha^2 \Bigg[-n - \tfrac{1}{2} + \tfrac{1}{2}\Bigg(\sigma & \\ \times\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(B^2+A^2+\alpha A)}{\alpha^2} - \tfrac{2(E_{n,-1}-M)(2AB+\alpha B)}{\alpha^2}} & \\ +\sqrt{\tfrac{1}{4} + \tfrac{2(E_{n,-1}-M)(B^2+A^2+\alpha A)}{\alpha^2} + \tfrac{2(E_{n,-1}-M)(2AB+\alpha B)}{\alpha^2}}\Bigg)\Bigg]^2\,. & \end{aligned} \tag{61}$$

3.4. Pöschl–Teller II potential

If we choose $q = 1, \alpha \to 2\alpha, g = 0$, and put

$$Q_2^2 + Q_3^2 + 4\alpha Q_2 = 8[A(A+\alpha) + B(B-\alpha)]\,, \tag{62}$$
$$Q_2 Q_3 + 2\alpha Q_3 = -4[A(A+\alpha) - B(B-\alpha)] \tag{63}$$

the potential (9) turns into

$$V(r) = -A(A+\alpha)\text{sech}^2\alpha r + B(B-\alpha)\text{cosech}^2\alpha r\,, \tag{64}$$

where $B < A$. The potential given in Eq. (64) is the Pöschl–Teller II potential [36]. Applying Eqs (62) and (63), we can obtain the following expressions for $\tilde{g}$ and P_2 from Eqs (26)—(29), related to spin symmetry condition,

$$\tilde{g} = 0\,, \tag{65}$$

$$P_2 = 2\alpha\left[-1\left(\sigma + \sqrt{\frac{1}{4} + \frac{(E_{n,-1}+M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} - \frac{(E_{n,-1}+M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}} + \sqrt{\frac{1}{4} + \frac{(E_{n,-1}+M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} + \frac{(E_{n,-1}+M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}}\right)\right], \tag{66}$$

and to pseudospin symmetry condition:

$$\tilde{g} = 0\,, \tag{67}$$

$$P_2 = 2\alpha\left[-1\left(\sigma + \sqrt{\frac{1}{4} + \frac{(E_{n,-1}-M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} - \frac{(E_{n,-1}-M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}} + \sqrt{\frac{1}{4} + \frac{(E_{n,-1}-M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} + \frac{(E_{n,-1}-M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}}\right)\right], \tag{68}$$

where $\sigma = \pm 1$. Substituting the above expressions (65), (66) and (67), (68) into Eq. (31) or (36), we get the exact energy equation for the Pöschl–Teller II potential in the Dirac theory with spin and pseudospin symmetry condition, respectively,

$$M^2 - E_{n,-1}^2 = 4\alpha^2 \left[-n - \tfrac{1}{2} + \tfrac{1}{2} \left(\sigma \right.\right.$$
$$\times \sqrt{\tfrac{1}{4} + \tfrac{(E_{n,-1}+M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} - \tfrac{(E_{n,-1}+M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}}$$
$$\left.\left. + \sqrt{\tfrac{1}{4} + \tfrac{(E_{n,-1}+M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} + \tfrac{(E_{n,-1}+M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}} \right) \right]^2 , \quad (69)$$

$$M^2 - E_{n,-1}^2 = 4\alpha^2 \left[-n - \tfrac{1}{2} + \tfrac{1}{2} \left(\sigma \right.\right.$$
$$\times \sqrt{\tfrac{1}{4} + \tfrac{(E_{n,-1}-M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} - \tfrac{(E_{n,-1}-M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}}$$
$$\left.\left. + \sqrt{\tfrac{1}{4} + \tfrac{(E_{n,-1}-M)(A^2+\alpha A+B^2-\alpha B)}{\alpha^2} + \tfrac{(E_{n,-1}-M)(A^2+\alpha A-B^2+\alpha B)}{\alpha^2}} \right) \right]^2 . \quad (70)$$

4. Conclusions

In the present paper, we solved the Dirac equation for the five-parameter exponential-type potentials with spin symmetry and pseudospin symmetry for s-wave bound states by using the supersymmetric quantum mechanics and shape invariance method and obtained the energy equations for the s-wave bound states. Under the condition of spin symmetry and pseudospin symmetry, *i.e.*, $\Delta(r) = 0, \Sigma(r) = 0$, we reached the energy equation of the five-parameter exponential-type potentials potential in the Dirac theory. We could obtain some known exponential-type potentials, such as the Hulthén potential, Scarf II, generalized Pöschl–Teller and Pöschl–Teller II potentials, *etc.*, if we choose the accurate parameters in the five-parameter exponential-type potentials. Their energy equations in the Dirac theory with spin symmetry and pseudospin symmetry condition are the special cases of the energy equation for the five-parameter exponential-type potential.

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