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SEMI-CLASSICAL UNIVERSE NEAR INITIAL SINGULARITY

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The properties of the quantum universe on extremely small spacetime scales are studied in the semi-classical approach to the well-defined quantum model. It is shown that near the initial cosmological singularity point quantum gravity effects $\sim \hbar$ exhibit themselves in the form of additional matter source with the negative pressure and the equation of state as for ultrastiff matter. The analytical solution of the equations of theory of gravity, in which matter is represented by the radiation and additional matter source of quantum nature, is found. It is shown that in the stage of the evolution of the universe, when quantum corrections $\sim \hbar$ dominate over the radiation, the geometry of the universe is described by the metric which is conformal to a metric of a unit four-sphere in a five-dimensional Euclidean flat space. In the radiation dominated era the metric is found to be conformal to a unit hyperboloid embedded in a five-dimensional Lorentz-signatured flat space. The origin of the universe can be interpreted as a quantum transition of the system from the region in a phase space with a trajectory in imaginary time into the region, where the equations of motion have the solution in real time. Near the boundary between two regions the universe undergoes almost an exponential expansion which passes smoothly into the expansion under the action of radiation dominating over matter. As a result of such a quantum transition the geometry of the universe changes. This agrees with the hypothesis about the possible change of geometry after the nucleation of expanding universe from 'nothing'.

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1. Introduction

It is accepted that the present-day Universe as a whole can be considered as a cosmological system described by the standard model based on general relativity [1, 2]. According to the Standard Big Bang Model [2, 3], the early Universe was very hot and dense. In order to describe that era one must treat the gravitational degrees of freedom and matter fields quantum mechanically [4, 5]. The fact that in the course of its evolution the Universe has passed through a stage with quantum degrees of freedom before turning into the cosmological system, whose properties are described well by general relativity, means that a consistent description of the Universe as a nonstationary cosmological system should be based on quantum general relativity in the form admitting the passage to general relativity in semi-classical limit $\hbar \to 0$ [4]. One of the possible versions of such a theory with a well-defined time variable was proposed in Refs [6, 7] in the case of homogeneous, isotropic and closed universe filled with primordial matter in the form of a uniform scalar field and relativistic matter associated with a reference frame. As calculations have demonstrated [6, 7], the equations of the quantum model may be reduced to the form in which the matter energy density in the universe has a component in the form of a condensate of massive quanta of a scalar field. Under the semi-classical description this component behaves as an antigravitating fluid. Such a property has a quantum nature and it is connected with the fact that the states with all possible masses of a condensate contribute to the total wave function of the quantum universe. If one discards the corresponding quantum corrections, the quantum fluid degenerates into a dust, *i.e.* matter component of the energy density commonly believed to make a dominant contribution to the mass-energy of ordinary matter in the present Universe in the standard cosmological model. Let us note that the presence of a condensate in the universe, as well as the availability of a dust representing an extreme state of a condensate, is not presupposed in the initial Lagrangian of the theory. An antigravitating condensate arises out of a transition from classical description of gravitational and matter fields to their quantum description achieved by canonical quantization. If one supposes that the properties of our Universe are described in an adequate manner by such a quantum theory, an antigravitating condensate being found out can be associated with dark energy [7]. Assuming that particles of a condensate can decay to baryons, leptons (or to their antiparticles) and particles of dark matter, one can describe the percentage of baryons, dark matter and dark energy observed in the present Universe [8].

In semi-classical limit the negative pressure fluid arises as a remnant of the early quantum era. This antigravitating component of the energy density does not vanish in the limit $\hbar \to 0$. In addition to this component, the stress-energy tensor contains the term vanishing after the transition to general relativity, *i.e.* to large spacetime scales. However, on small spacetime scales quantum corrections $\sim \hbar$ turn out to be significant. As it is shown in this paper, the effects stipulated by these corrections determine the equation of state of matter and geometry near the initial cosmological singularity point. They define a boundary condition that should be imposed on the wave function in the origin so that a nucleation of the universe from the initial cosmological singularity point becomes possible.

In this paper we use the modified Planck system of units. The $l_{\rm P} = \sqrt{2G\hbar/(3\pi c^3)}$ is taken as a unit of length, the $\rho_{\rm P} = 3c^4/(8\pi G l_{\rm P}^2)$ is a unit of energy density and so on. All relations (with the exception of Appendix A) are written for dimensionless values.

2. Equations of motion in quantum model

Let us consider the homogeneous, isotropic and closed universe which is described by the Robertson–Walker metric with the cosmic scale factor $a(\tau)$, where τ is the proper time. We assume that the universe is originally filled with the uniform scalar field ϕ and a perfect fluid which defines so called *material reference frame* [6, 9]. The perfect fluid is taken in the form of relativistic matter (radiation) with the energy density $\rho_\gamma = E/a^4$, where $E = \text{const}$. The scalar field oscillates with a small amplitude near the minimum of its potential energy density (potential) $V(\phi)$ at the point $\phi = \sigma$, $\{dV(\phi)/d\phi\}_\sigma = 0$, while in general case $V(\sigma) \neq 0$. Then the equations of the quantum model under consideration are reduced to the following spectral problem [6, 7]

$$\left(-\partial_a^2 + U_k(a) - E\right) f(a) = 0\,, \tag{1}$$

where

$$U_k(a) = a^2 - 2aM_k - a^4 V(\sigma) \tag{2}$$

is the effective potential, $M_k = m_\sigma(k+\frac{1}{2})$ with $m_\sigma^2 = \{d^2V(\phi)/d\phi^2\}_\sigma > 0$ and $k = 0, 1, 2, \ldots$ describes an amount of matter (mass) in the universe in the form of a condensate of quantized scalar field as an aggregate of massive excitation quanta of the spatially coherent oscillations of the scalar field about the equilibrium state σ. The solutions of Eq. (1) determine the states of the universe in an effective potential well (2) at given mass of a condensate M_k. The wave function $f(a)$ and the eigenvalue E depend on quantum number k, which is equal to the quantity of quanta of a condensate, and second quantum number $n = 0, 1, 2, \ldots$, marking the states (levels) in the potential well (2). If $V(\sigma) = 0$, the spectrum of these states is discrete [6]. If $V(\sigma) \neq 0$, the states in the well (2) will be, generally speaking, quasistationary, since there exists a nonzero probability of tunneling through the barrier into the region where its dynamics is determined by the condition $a^2V(\sigma) > 1 - 2M_k/a$. However, if $V(\sigma) \ll 1$, such states can be approximately considered as stationary within the lifetime[1] of a system inside the barrier. The wave function may be normalized to unity in the region limited by the barrier.

[1] It can reach values comparable with the age of our Universe [10].

The incorporation of the reference frame through the introduction of relativistic matter makes it possible to define a time variable and describe the evolution of quantum universe by the time-dependent Schrödinger type equation with a time-independent Hamiltonian. The wave function of stationary states is characterized by the parameter E which has a definite value [6, 7].

In the quantum model under consideration the following relation between the matrix elements is fulfilled [7]

$$\left\langle f \right| - \frac{i}{N}\frac{d}{d\eta}\,\partial_a \left| f \right\rangle = \langle f \left| \left(a - 2a^3 V(\sigma) - 4M_k \right) \right| f \rangle \qquad \text{for} \qquad k \gg 1\,, \quad (3)$$

where η is the time variable which is connected with the proper time τ by the differential equation $d\tau = aN d\eta$, $N = (g^{00})^{-1/2}$ is the lapse function that specifies the time reference scale[2].

Choosing the function $f(a)$ in the form

$$f(a) = A(a)\, e^{iS(a)}\,, \quad (4)$$

where A and S are real functions of a, and assuming that the matter-energy in semi-classical universe can be written in the form of perfect fluid source [2] we obtain from (1) and (3) the equations for A and S

$$\frac{1}{a^4}\,(\partial_a S)^2 - \rho + \frac{1}{a^2} - \frac{1}{a^4}\,\frac{\partial_a^2 A}{A} - \frac{i}{a^4}\,\frac{\partial_a\left(A^2 \partial_a S\right)}{A^2} = 0\,, \quad (5)$$

$$\frac{1}{a^2}\,\frac{d}{d\tau}\,(\partial_a S) + \frac{1}{2}\,(\rho - 3p) - \frac{1}{a^2} - \frac{i}{a^2}\,\frac{d}{d\tau}\left(\frac{\partial_a A}{A}\right) = 0\,, \quad (6)$$

where

$$\rho = \rho_k + \rho_\sigma + \rho_\gamma\,, \qquad p = p_k + p_\sigma + p_\gamma \quad (7)$$

are the energy density and the isotropic pressure as the sum of the components

$$\rho_k = \frac{2M_k}{a^3}\,, \qquad \rho_\sigma \equiv V(\sigma) \equiv \frac{\Lambda}{3}\,, \qquad \rho_\gamma = \frac{E}{a^4}\,, \quad (8)$$

Λ is the cosmological constant. The equations of state are

$$p_k = -\rho_k\,, \qquad p_\sigma = -\rho_\sigma\,, \qquad p_\gamma = \tfrac{1}{3}\,\rho_\gamma\,. \quad (9)$$

The equations of state for the vacuum component $\rho_\sigma =$ const. and relativistic matter ρ_γ are dictated by the formulation of the problem. The vacuum-type equation of state of a condensate with the density ρ_k, which does not remain

[2] Even for the Planck mass $M_k = m_{\mathrm{P}} = 1$ the number $k \sim 10^{18}$, if $m_\sigma \sim 10^{-18}$ (~ 10 GeV) [6].

constant throughout the evolution of the universe, but decreases according to a power law with the increase of a, follows from the condition of consistency of Eqs (5) and (6).

From Eqs (5)–(9) we can conclude that a condensate behaves as an anti-gravitating medium. Its anti-gravitating effect has a purely quantum nature. Its appearance is determined by the fact that the total wave function of the universe $\psi = \sum_k f_k u_k$ is a superposition of quantum states with all possible values of the quantum number k, u_k is the wave function, which describes the quantum states of a scalar field. In the quantum model with the potential $V(\phi)$, which has a harmonic oscillator form near the point $\phi = \sigma$, the contribution into the right-hand side of Eq. (3) is made by the states with k and $k \pm 2$ only. If one discards the contributions from the states with $k \pm 2$, a condensate turns into an aggregate of separate macroscopic bodies with zero pressure (dust) [7]. The existence of this limit argues in favour of reliability of this quantum model.

In the classical limit ($\hbar = 0$), when all terms with the amplitude A are discarded (see Appendix A), Eqs (5) and (6) reduce to the Einstein–Friedmann equations which predict an accelerating expansion of the universe in the era with $\rho_k > \frac{2}{3}\,\rho_\gamma$, even if $\Lambda = 0$. Since $\rho_\gamma \sim a^{-4}$ decreases with a more rapidly then $\rho_k \sim a^{-3}$ (or even $\sim a^{-2}$ [7]), the era of accelerating expansion should begin with increasing a, even if the state with $\rho_k < \frac{2}{3}\,\rho_\gamma$ and $\Lambda \sim 0$ existed in the past, when the expansion was decelerating.

3. Semi-classical universe

Separating in Eq. (5) the real and imaginary parts and setting each one of them equal to zero separately, from the equation for the imaginary part $\partial_a \left(A^2 \partial_a S\right) = 0$, we find the amplitude A,

$$A = \frac{\text{const.}}{\sqrt{\partial_a S}}\,. \tag{10}$$

Substituting this solution into the real part of Eq. (5) and into Eq. (6), we arrive at the equations for the phase S,

$$\frac{1}{a^4}\,(\partial_a S)^2 - \rho + \frac{1}{a^2} = \frac{1}{a^4}\left\{\frac{3}{4}\left(\frac{\partial_a^2 S}{\partial_a S}\right)^2 - \frac{1}{2}\,\frac{\partial_a^3 S}{\partial_a S}\right\}, \tag{11}$$

$$\frac{1}{a^2}\,\frac{d}{d\tau}\,(\partial_a S) + \frac{1}{2}\,(\rho - 3p) - \frac{1}{a^2} = -\,\frac{i}{2a^2}\,\frac{d}{d\tau}\left(\frac{\partial_a^2 S}{\partial_a S}\right). \tag{12}$$

These equations are exact. If the solution of these nonlinear equations is found, it would be possible, theoretically, to restore the wave function (4).

Rewriting Eqs (11) and (12) in ordinary units (see Appendix A), we find that the right-hand sides of Eqs (11) and (12) are proportional to $\hbar^2$ and $\hbar$, respectively. It means that the right-hand sides of these equations can be considered as small quantum corrections to the equations of general relativity.

Quantum corrections are essential in the region of extremely small values of the scale factor. Substituting the standard model solution of the Einstein–Friedmann equations (see Appendix B)

$$a(\tau) = \beta\, \tau^{\alpha}\,, \tag{13}$$

$\alpha > 0$ and $\beta > 0$ are constants, into the expression for the generalized momentum

$$\partial_a S = -a\, \frac{da}{d\tau} \equiv -a\dot{a}\,, \tag{14}$$

we find that this solution satisfies Eqs (11) and (12) in the region, where quantum corrections play a main role[3]. Using Eq. (13) in order to calculate the right-hand sides of Eqs (11) and (12) we obtain the following equations

$$\frac{1}{a^4}\,(\partial_a S)^2 = \rho + \frac{d_\alpha}{a^6} - \frac{1}{a^2}\,, \tag{15}$$

$$\frac{1}{a^2}\,\frac{d}{d\tau}\,(\partial_a S) + \frac{1}{2}\,(\rho - 3p) - \frac{1}{a^2} = -\,i\,\frac{b_\alpha}{2a^5}\,\partial_a S\,, \tag{16}$$

where we denote

$$d_\alpha \equiv \frac{(2\alpha-1)(4\alpha-1)}{4\alpha^2}\,, \qquad b_\alpha \equiv \frac{2\alpha-1}{\alpha}\,. \tag{17}$$

These equations take into account the presence of matter with the energy density ρ and describe the evolution of the universe in the semi-classical approximation.

Eq. (15) can be considered as the first of the Einstein–Friedmann equations for the matter which is characterized by the generalized energy density $\rho + d_\alpha a^{-6}$. It means that the quantum effects under consideration cause the appearance of an additional source of gravitational field which decreases with a as a^{-6}. If we regard Eq. (16) as the second of the Einstein–Friedmann equations with the generalized energy density and pressure, then comparing the quantum corrections with the energy density of the standard model (see Appendix B), we find that it should be set $\alpha = \frac{1}{3}$, and $d_{1/3} = -\frac{1}{4}$, $b_{1/3} = -1$.

Using the representation for the classical momentum (14), Eqs (15) and (16) can be reduced to the standard form

$$\left(\frac{\dot{a}}{a}\right)^2 = \rho + \rho_{\mathrm{u}} - \frac{1}{a^2}\,, \tag{18}$$

[3] See Eq. (30) below.

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left[\rho + \rho_{\rm u} + 3(p + p_{\rm u})\right], \tag{19}$$

where

$$\rho_{\rm u} = -\frac{1}{4a^6} \tag{20}$$

is the quantum correction ($\sim \hbar^2$) [4], which may be identified with the ultra-stiff matter with the equation of state[5]

$$p_{\rm u} = \rho_{\rm u}, \tag{21}$$

where $p_{\rm u}$ is the pressure. This 'matter' has quantum origin. It is formed due to nonzero derivatives of the amplitude A of the wave function (4) and is related nonlinearly with matter components contributing into the energy density ρ (7).

Let us note that when deriving Eq. (19) from Eq. (16) we have used Eq. (15), while in the right-hand side of Eq. (16) only the main term in the range of action of the quantum correction, $2Ea^2 < 1$, is taken into account. In the region $2Ea^2 > 1$ the density $\rho_{\rm u}$, and pressure $p_{\rm u}$ may be neglected. In this case equations correspond to the limit $\hbar \to 0$ [7]. They coincide formally with the equations of standard cosmology, but, unlike them, besides the familiar contributions into the energy density ρ (7) from the vacuum term ρ_σ and radiation ρ_γ, it contains a nontrivial contribution from antigravitating quantum fluid with the energy density ρ_k which does not vanish at $\hbar \to 0$.

Let us estimate the ratio of energy density ρ to $\rho_{\rm u}$. Passing to the ordinary units (see Appendix A), we have

$$\mathcal{R} \equiv \left[\frac{8\pi G}{3c^4}\,\rho\right] : \left[\left(\frac{2G\hbar}{3\pi c^3}\right)^2 \frac{1}{4a^6}\right]. \tag{22}$$

Substituting here the values of the fundamental constants we obtain

$$\mathcal{R} \sim 10^{81}\,\rho\, a^6, \tag{23}$$

where ρ is measured in GeV/cm^3 and a in cm. For our Universe today $\rho \sim 10^{-5}$ GeV/cm^3, $a \sim 10^{28}$ cm and

$$\mathcal{R}_{\rm today} \sim 10^{244}, \tag{24}$$

[4] Let us note that the presence of a minus sign in Eq. (20) is not extraordinary. So, according to quantum field theory, for instance, vacuum fluctuations make a negative contribution to the field energy per unit area (the Casimir effect).

[5] Specifying the form of the equation of state for the density $d_\alpha a^{-6}$ as $p_{\rm u} = w\rho_{\rm u}$, where w is constant which has to be found, and performing the corresponding calculations we obtain the equations from which it follows that $\alpha = \frac{1}{3}$ and $w = 1$.

i.e. the quantum correction may be neglected to an accuracy of $\sim O(10^{-244})$. In the Planck era $\rho \sim 10^{117}$ GeV/cm^3, $a \sim 10^{-33}$ cm and the relation

$$\mathcal{R}_{\text{Planck}} \sim 1 \tag{25}$$

shows that the densities ρ and ρ_{u} are of the same order of magnitude.

In Fig. 1 the energy density $\rho + \rho_{\text{u}}$ as a function of a for typical values of the parameters of the early universe is shown. It is evident that the range of action of the quantum correction is limited by the condition $a \lesssim 0.5$. A small variation of the parameters does not affect the result.

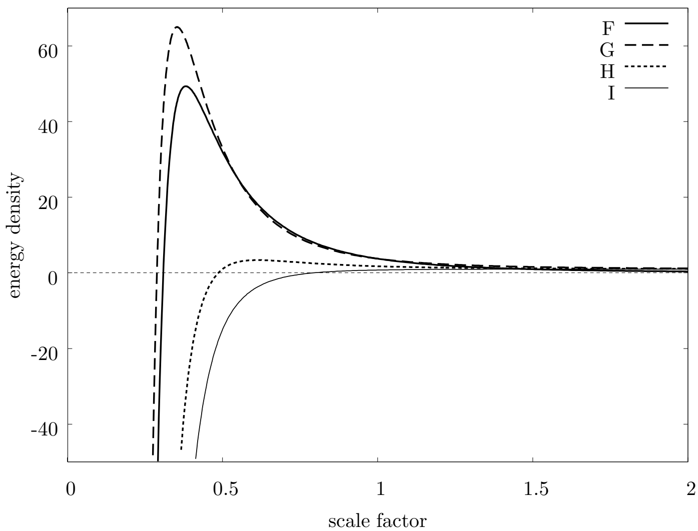

Fig. 1. The energy density $\rho{+}\rho_{\text{u}}$ *versus* the scale factor a. The following parameters are used: $F = \{M_k = 1,\, \rho_\sigma = 0,\, E = 2\}$, $G = \{M_k = 0,\, \rho_\sigma = 1,\, E = 1\}$, $H = \{M_k = 0,\, \rho_\sigma = 1,\, E = 3\}$, $I = \{M_k = 0,\, \rho_\sigma = 1,\, E = 0\}$. The curves G, H, and I correspond to the potential (2) with $M_k = 0$. The I describes the standard model with nonzero cosmological constant in the absence of matter.

4. Quantum effects on sub-Planck scales

Let us consider the solutions of Eqs (18) and (19) in the region $2Ea^2 < 1$. Here the contributions from the condensate, cosmological constant and curvature may be neglected. As a result the equations of the model take the form

$$\tfrac{1}{2}\,\dot{a}^2 + U(a) = 0\,, \qquad \ddot{a} = -\frac{dU}{da}\,, \tag{26}$$

where

$$U(a) \equiv \frac{1}{2}\left[\frac{1}{4a^4} - \frac{E}{a^2}\right]. \tag{27}$$

These equations are similar to ones of Newtonian mechanics. Using this analogy they can be considered as equations which describe the motion of a 'particle' with a unit mass and zero total energy under the action of the force $-\frac{dU}{da}$, $U(a)$ is the potential energy, and $a(\tau)$ is a generalized variable. A point $a_c = \frac{1}{2\sqrt{E}}$, where $U(a_c) = 0$, divides the region of motion of a 'particle' into the subregion $4Ea^2 < 1$, where the classical motion of a 'particle' is forbidden, and the subregion $4Ea^2 > 1$, where the classical trajectory of a 'particle' moving in real time τ exists. The quantum corrections are essential in the region $2Ea^2 < 1$. The boundary point $a_b = \frac{1}{\sqrt{2E}}$ corresponds to the minimum of the potential energy (27). The solution can be extended to the full range of values of the variable $a > a_b$.

In the subregion $4Ea^2 < 1$ there exists the classical trajectory of a 'particle' moving in imaginary time $t = -i\tau + \text{const.}$ in the potential $-U(a)$. Denoting the corresponding solution as $\tilde{a}$ we find

$$\tilde{a} = \frac{1}{2\sqrt{E}} \sin z\,, \tag{28}$$

$$t = \frac{1}{16E^{3/2}}\left[2z - \sin 2z\right]. \tag{29}$$

At small z, *i.e.* in the region $\tilde{a} \sim 0$, we have

$$\tilde{a} = \left(\frac{3}{2}\,t\right)^{1/3}. \tag{30}$$

According to the standard model solution (see Appendix B) the 'matter' near the point $\tilde{a} = 0$ is described by the equation of state of the ultrastiff matter (21).

In the subregion $4Ea^2 > 1$ the solution of the equations (26) can be written as

$$a = \frac{1}{2\sqrt{E}} \cosh\zeta\,, \tag{31}$$

$$\tau = \frac{1}{16E^{3/2}}\left[2\zeta + \sinh 2\zeta\right]. \tag{32}$$

At $\zeta \ll 1$ it follows from here that $\zeta \approx 4E^{3/2}\tau$, while the scale factor at $\tau \ll (4E^{3/2})^{-1}$ increases almost exponentially

$$a = \frac{1}{2\sqrt{E}}\left[1 + (2E)^3\tau^2 + \ldots\right] \approx \frac{1}{2\sqrt{E}} \exp\left\{(2E)^3\tau^2\right\}. \tag{33}$$

The almost exponential expansion of the early universe in that era is stipulated by the action of quantum effects which, according to Eqs (20) and (21), cause the negative pressure, $p_{\mathrm{u}} < 0$, *i.e.* produce an anti-gravitating effect on the cosmological system under consideration.

At $\zeta \gg 1$ the solution (31), (32) takes the form

$$a = \left(2\sqrt{E}\,\tau\right)^{1/2}. \tag{34}$$

It describes the radiation dominated era and corresponds to the time τ which satisfies the condition $\frac{1}{2}\ln(32E^{3/2}\tau) \gg 1$.

The solutions (28), (29) and (31), (32) are related between themselves through an analytic continuation into the region of complex values of the time variable,

$$t = -i\tau + \frac{\pi}{16E^{3/2}}, \qquad z = \frac{\pi}{2} - i\zeta. \tag{35}$$

The scale factors (28) and (31) are connected through the condition

$$a(\tau) = \tilde{a}\left(\frac{\pi}{16E^{3/2}} - i\tau\right), \tag{36}$$

which describes an analytic continuation of the time variable τ into the region of complex values of Euclidean time t.

The model determined by the equations (26) allows us to describe the origin (nucleation) of the universe as the transition from the state in the subregion $4Ea^2 < 1$ to the state in the subregion $4Ea^2 > 1$. The corresponding transition amplitude can be written as follows [12]

$$T \sim e^{-S_t}, \tag{37}$$

where S_t is the action on a trajectory in imaginary time t,

$$S_t = 2\int_{-\infty}^{\infty} dt U(\tilde{a}). \tag{38}$$

Let us proceed to the integration with respect to the time variable z. According to (28) the scale factor $\tilde{a}$ is a periodical function of z. Therefore, we shall restrict ourselves to the interval of integration $z = [-\frac{\pi}{2}, \frac{\pi}{2}]$. On the boundaries of this interval the $|\tilde{a}|$ takes the maximum possible value $a_c = \frac{1}{2\sqrt{E}}$, while the interval itself contains the point $\tilde{a} = 0$. Then

$$S_t = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \frac{dt}{dz} U(\tilde{a}(z)). \tag{39}$$

Using the explicit form of the solution (28), (29), we find

$$S_t = -\sqrt{E}\pi\,, \tag{40}$$

and the amplitude (37) becomes

$$T \sim e^{\sqrt{E}\pi}\,, \tag{41}$$

i.e. a 'particle' which is the equivalent of the universe leaves the subregion forbidden for classical motion with an exponential probability density. It is pushed out of forbidden subregion into the subregion of very small values of a in real time τ by the anti-gravitating forces stipulated by the negative pressure which cause quantum processes at $a \sim 0$ (see Eqs (20) and (21)). This phenomenon can be interpreted as the origin of the universe from the region $a \sim 0$. It is possible only if the probability density that the universe is in the state with $a \sim 0$ is nonzero. It means that the wave function of the universe at the point $a = 0$ must be nonvanishing.

5. Wave function near initial singularity

Let us determine the behaviour of the wave function $f(a)$ in the region $a \sim 0$ and calculate the nucleation rate of the universe from the point $a = 0$. The result will be compared with the transition amplitude (41).

From Eqs (4) and (10) it follows that

$$f(a) = \frac{f_0}{\sqrt{\partial_a S_{\mathrm{E}}}}\, e^{-S_{\mathrm{E}}}\,, \tag{42}$$

where $f_0 = \text{const.}$, $S_{\mathrm{E}} = -iS$ is the Euclidean action. Using Eqs (14) and (30) we find

$$\partial_a S_{\mathrm{E}} = \frac{1}{2a}\,. \tag{43}$$

The integration of this equation gives

$$S_{\mathrm{E}} = \frac{1}{2}\,\ln\left(\frac{a}{a_0}\right)\,, \tag{44}$$

where the integration constant is taken in the form $\ln a_0^{-1/2}$, for convenience. Then the wave function

$$f(a) = \sqrt{2\,a_0}\,f_0 \tag{45}$$

does not depend on a in the region $a \sim 0$, and the probability density that the universe may be found at the point $a = 0$ is nonzero,

$$|f(0)|^2 = 2\,a_0\,|f_0|^2\,. \tag{46}$$

The nucleation rate of the universe from the initial cosmological singularity point $a = 0$ can be written as follows [6]

$$\Gamma = \overline{v\,\sigma_{\rm r}}\,|f(0)|^2\,, \tag{47}$$

where the multiplier $\overline{v\,\sigma_{\rm r}}$ has to be calculated now with respect to imaginary time t,

$$\overline{v\,\sigma_{\rm r}} = \lim_{a\to 0}\left(\frac{da}{dt}\,\pi a^2\right) = \frac{\pi}{2}\,. \tag{48}$$

Taking into account Eq. (46) we obtain

$$\Gamma = \pi\,a_0\,|f_0|^2\,. \tag{49}$$

The constant $a_0\,|f_0|^2$ can be found as a result of exact integration of Eq. (1) with the boundary condition $f(0) = \text{const.}$, in accordance with Eq. (45). For the case $\rho_\sigma = 0$ we obtain that a rate of nucleation of the universe in the n-state follows the law [6]

$$\Gamma \simeq \frac{\sqrt{\pi}}{2}\,2^n\,\mathcal{P}(n)\,, \tag{50}$$

where $\mathcal{P}(n) = (\langle n\rangle^n/n!)\,\exp(-\langle n\rangle^n)$ is the Poisson distribution with the mean value $\langle n\rangle = M_k^2$. Summing over all values of the quantum number n the total nucleation rate appears to be exponentially high

$$\Gamma_{\rm tot} \simeq \frac{\sqrt{\pi}}{2}\,e^{M_k^2}\,. \tag{51}$$

Comparing the nucleation rate (51) with the transition amplitude (41) we see that both these quantities predict an exponential origin (nucleation) of the universe from the region $a \sim 0$ forbidden for classical motion.

6. Geometry

Let us consider how the geometry of the universe changes as a result of its transition from the region $2Ea^2 < \frac{1}{2}$ into $\frac{1}{2} < 2Ea^2 < 1$. In the model under consideration the metric has the form

$$ds^2 = d\tau^2 - a^2 d\Omega_3^2\,, \tag{52}$$

where $d\Omega_3^2$ is a line element on a unit three-sphere. According to the solutions (28), (29) and (31), (32) the metric (52) takes the form

$$ds_{\rm E}^2 = -\frac{1}{4E}\,\sin^2 z\left\{\frac{\sin^2 z}{4E^2}\,dz^2 + d\Omega_3^2\right\} \qquad \text{at} \qquad 2Ea^2 < \frac{1}{2} \tag{53}$$

and

$$ds_{\rm L}^2 = \frac{1}{4E}\cosh^2\zeta\left\{\frac{\cosh^2\zeta}{4E^2}\,d\zeta^2 - d\Omega_3^2\right\} \qquad \text{at} \qquad \frac{1}{2} < 2Ea^2 < 1\,, \quad (54)$$

where the interval with the Euclidean signature is denoted by the index E, and the one with the Lorentzian signature is marked by L. Introducing the new time variables ξ and ς according to

$$d\xi = \frac{1}{2E}\sin z dz\,, \qquad d\varsigma = \frac{1}{2E}\cosh\zeta d\zeta, \quad (55)$$

the metrics (53) and (54) can be reduced to the conformally flat form

$$ds_{\rm E}^2 = -\frac{1}{4E}\left[1-(2E\xi)^2\right]\left\{d\xi^2 + d\Omega_3^2\right\}\,, \quad (56)$$

$$ds_{\rm L}^2 = \frac{1}{4E}\left[1+(2E\varsigma)^2\right]\left\{d\varsigma^2 - d\Omega_3^2\right\}\,. \quad (57)$$

Both metrics are related between themselves through the analytic continuation into the region of complex values of the time variable $\varsigma = i\xi$.

The metric (56) is conformal to a metric of a unit four-sphere in a five-dimensional Euclidean flat space. With increasing a, the universe transits from the region $4Ea^2 < 1$ into the region $4Ea^2 > 1$, where the geometry is conformal to a unit hyperboloid embedded in a five-dimensional Lorentz-signatured flat space. Such a picture of change in spacetime geometry during the transition of the universe from the region near initial singularity into the region of real physical scales agrees with the hypothesis [13, 14], widely discussed in the literature for the de Sitter space, about possible change in four-space geometry after the spontaneous nucleation of the expanding universe from the initial singularity point. In this paper this phenomenon is demonstrated in the case of the early universe filled with the radiation and ultrastiff matter which effectively takes into account quantum effects on extremely small spacetime scales.

7. Conclusions

In this paper we study the properties of the quantum universe on extremely small spacetime scales in the semi-classical approach to the well-defined quantum model. We show that near the initial cosmological singularity point quantum gravity effects $\sim \hbar$ exhibit themselves in the form of additional matter source with the negative pressure and the equation of state as for ultrastiff matter. The analytical solution of the equations of theory of gravity, in which matter is represented by the radiation and additional matter source of quantum nature, is found. It is shown that in the stage of the evolution of the universe, when quantum corrections $\sim \hbar$ dominate over the radiation, the geometry of the universe is described by the metric which is conformal to a metric of a unit four-sphere in a five-dimensional Euclidean flat space. In the radiation dominated era the metric is found to be conformal to a unit hyperboloid embedded in a five-dimensional Lorentz-signatured flat space. One solution can be continued analytically into another.

The wave function of the universe in the initial cosmological singularity point is nonzero and the nucleation of the universe from this point becomes possible. The origin of the universe can be interpreted as a quantum transition of the system from the region in a phase space forbidden for classical motion, but where a trajectory in imaginary time exists, into the region, where the equations of motion have the solution which describes the evolution of the universe in real time. Near the boundary between two regions, from the side of real time, the universe undergoes almost an exponential expansion which passes smoothly into the expansion under the action of radiation dominating over matter which is described by the standard cosmological model. As a result of such a quantum transition the geometry of the universe changes. This agrees with the hypothesis about the possible change of geometry after the nucleation of expanding universe from 'nothing'.

Appendix A

In the ordinary units the wave function (4) has the form

$$f(a) = A(a)\, \exp\left\{\frac{i}{\hbar}\, \frac{3\pi c^3}{2G}\, S(a)\right\},$$

where S is measured in cm^2. Eqs (11) and (12) can be written as follows

$$\frac{1}{a^4}\,(\partial_a S)^2 - \frac{8\pi G}{3c^4}\,\rho + \frac{1}{a^2} = \left(\frac{2G\hbar}{3\pi c^3}\right)^2 \frac{1}{a^4}\left\{\frac{3}{4}\left(\frac{\partial_a^2 S}{\partial_a S}\right)^2 - \frac{1}{2}\,\frac{\partial_a^3 S}{\partial_a S}\right\},$$

$$\frac{1}{a^2}\,\frac{d}{d\tau}\,(\partial_a S) + \frac{4\pi G}{3c^4}\,(\rho - 3p) - \frac{1}{a^2} = -\frac{2G\hbar}{3\pi c^3}\,\frac{i}{2a^2}\,\frac{d}{d\tau}\left(\frac{\partial_a^2 S}{\partial_a S}\right).$$

Here a is measured in cm, ρ and p in GeV/cm^3, and c^4/G in GeV/cm. In the approximation $\hbar = 0$ these equations strictly pass into the equations of general relativity for generalized momentum $\partial_a S = -a\, da/d\tau$.

Appendix B

As it is well known [2, 3] the standard model solution of the Einstein–Friedmann equations for a single component domination in the energy density ρ have the form

$$\rho \sim a^{-3(1+w)}$$

for the equation of state $w = p/\rho = \text{const.}$ If $w \neq -1$ and the curvature term can be neglected the scale factor a is

$$a \sim \tau^{2/[3(1+w)]}\,.$$

According to accepted notations [3, 11] the value $w = -\frac{2}{3}$ describes the domain walls ($\rho \sim a^{-1}$, $a \sim \tau^2$); the value $w = -\frac{1}{3}$ corresponds to the strings ($\rho \sim a^{-2}$, $a \sim \tau$); $w = 0$ is a dust ($\rho \sim a^{-3}$, $a \sim \tau^{2/3}$), $w = \frac{1}{3}$ is a radiation ($\rho \sim a^{-4}$, $a \sim \tau^{1/2}$); $w = \frac{2}{3}$ is a perfect gas ($\rho \sim a^{-5}$, $a \sim \tau^{2/5}$), and $w = 1$ is an ultrastiff matter ($\rho \sim a^{-6}$, $a \sim \tau^{1/3}$). The special case with $w = -1$ and $\rho = \text{const.}$ describes the de Sitter vacuum.

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