

POTENTIAL AND LIMITATIONS
OF THE HBT METHOD* **K. ZALEWSKI[†]

The H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences
Radzikowskiego 152, 31-342 Kraków, Poland
zalewski@th.if.uj.edu.pl

(Received October 13, 2009)

The HBT method is used to get information about the sizes, shapes and sometimes also about the time evolution of the homogeneity regions in hadroproduction processes. Homogeneity region \mathbf{K} is the region, where the hadrons with momentum \mathbf{K} are produced. The shape and size of homogeneity region \mathbf{K} is described by the Wigner function $W(\mathbf{K}, \mathbf{X})$ evaluated in the interaction representation after all the hadrons had been produced. Additional information about the evolution in time is contained in the emission function $S(K, X)$. A theorem is presented and discussed which specifies which of the parameters characterizing the Wigner function can and which cannot be measured using the HBT method. In order to obtain the complete Wigner function additional assumptions are needed. For instance, it is enough to know the distribution of the centers of the homogeneity regions $\langle \mathbf{X} \rangle_{\mathbf{K}}$. In order to find the emission function further assumptions are required. No systematic analysis is available, but some instructive examples are discussed.

PACS numbers: 25.75.Gz, 13.65.+i

1. Introduction

The HBT method is a somewhat controversial [1] name for the use of Bose–Einstein correlations to get information about the interaction regions, defined below, in multiparticle production processes. For a recent review see [2]. In spite of its long history since the seminal paper of the Goldhabers, Lee and Pais [3], the method remains a field of active research. In the present paper we discuss what can and what cannot be measured using the HBT method. We also describe some open problems.

* Presented at the XXXIII International Conference of Theoretical Physics, “Matter to the Deepest”, Ustroń, Poland, September 11–16, 2009.

** This work has been partly supported by the Polish Ministry of Science and Higher Education grant N N202 125437(2009–2012).

[†] Corresponding address: Institute of Physics, Reymonta 4, 30-059 Kraków, Poland.

The interaction region is interpreted as follows. Consider a high energy collision of two heavy nuclei where many hadrons are produced. The trajectory of each hadron begins somewhere — in the point where the hadron got created. This is a classical picture. It can be improved by replacing the points of creation by small regions in space. All these points, or regions, averaged over many similar collisions, form the interaction region. Thus, the interaction region is, in general, not the volume where the fluid (strongly interacting quark–gluon plasma?) created in the collision evolves. For instance, in some models it is a two-dimensional shell.

If we were not constrained by the laws of Nature, the best description of the interaction region would be the time dependent probability distribution in phase space, *i.e.* the probability density for a particle of momentum \mathbf{p} to be created at space-time point X . Since, however, according to quantum mechanics it is not possible to measure simultaneously and precisely the position and the momentum of a particle, one must compromise. In ordinary quantum mechanics, where the number of particles is conserved, the Wigner function is considered to be the best replacement for the phase space probability distribution. In hadroproduction the situation is more complicated, because the number of hadrons increases from zero to some final number. The analogue of the Wigner function applicable in this case is known as the emission function. It is implicit already in the work of Shuryak [4], but has been first explicitly defined and used by Pratt [5]. For a fairly recent discussion see [2].

2. Emission function

The emission function can be defined¹ by the formula (*cf. e.g.* [2])

$$S(p, X) = \int d^4Y \sum_R T_R^* \left(X + \frac{Y}{2} \right) T_R \left(X - \frac{Y}{2} \right) e^{-ipY}, \quad (1)$$

where

$$X = \frac{1}{2}(x_1 + x_2), \quad Y = x_1 - x_2. \quad (2)$$

$T_R(Z)$ is the probability amplitude for producing a hadron at space-time point Z . This, of course, depends on the state of the surrounding, denoted here by R , of Z . By analogy with thermodynamics, the averaging over R is done on the product $T_R^* T_R$ and not on the single probability amplitudes T_R . The integrand depends on two moments of time: $X^0 \pm \frac{1}{2}Y^0$. Sometimes (see *e.g.* [5]) incoherence in time of the production process is assumed:

¹ In order to make the emission function closer to the Wigner function, we have changed the sign of the exponent with respect to the one given in [2].

$$\sum_R T_R^* \left(X + \frac{Y}{2} \right) T_R \left(X - \frac{Y}{2} \right) = \delta(Y^0) \bar{\Sigma}(X, \mathbf{Y}). \quad (3)$$

Then there is only one time, occurring on both sides of equation (1), and the emission function becomes closer to the Wigner function (see Section 4).

Note that in (1) there is neither integration nor differentiation with respect to the components of p . Thus, momentum appears only as a parameter. This is related to the fact, discussed in the following section, that it is not possible to measure the whole interaction region. The most one can hope for is to measure the *homogeneity regions*. Homogeneity region \mathbf{K} is the region where the hadrons with momentum \mathbf{K} got created. Thus, the problem is to find the profiles of the homogeneity regions²:

$$p_{\mathbf{K}}(\mathbf{x}) \sim \int_{-\infty}^{+\infty} dX^0 S(K, X). \quad (4)$$

This limitation of the HBT method was noticed by Bowler [6]. The name homogeneity region was introduced by Sinyukow [7].

The emission function is related to the single particle density matrix in the momentum representation by the formula

$$\rho(\mathbf{K}, \mathbf{q}) \sim \int d^4X S(K, X) e^{iqX}, \quad (5)$$

where

$$K = \frac{1}{2}(p_1 + p_2), \quad q = p_1 - p_2. \quad (6)$$

The density matrix does not depend on time, because it refers to a time when all the hadrons are already present and propagate freely (To some extent one can include final state interactions see *e.g.* [2]). It is written in the interaction picture.

Formula (5) rises two problems. How to measure $\rho(\mathbf{K}, \mathbf{q})$? The diagonal elements are given by the single particle momentum distribution, but the textbook advice to look for distributions of other measurable quantities in order to find the out of diagonal elements is inapplicable here, because momenta are all we know how to measure. A brilliant partial solution to this problem [3] is discussed in the following section. Since Wigner's function is the Fourier transform of the density matrix, knowing the density matrix

² Here and in the following \sim means: equal up to a known proportionality constant irrelevant for our discussion.

is enough to find the size and shape of a homogeneity region. The emission function, however, contains additional interesting information about the time evolution of the hadronization process.

The second problem is, how to solve for S equation (5) for a given density matrix? At first sight it might seem that it is enough to invert the Fourier transformation, but in order to do that one would have to know $\rho(\mathbf{K}, \mathbf{q})$ for all the four-vectors q and not only for all the three-vectors \mathbf{q} as is the case. Here not much is known. We discuss the problem in Section 4.

An important assumption concerning the emission function is the *smoothness assumption*, see e.g. [2]. According to this assumption, the dependence of $S(K, X)$ on K is so weak that we can replace³ K by p_1 or p_2 , or replace K_0 , by $\sqrt{m^2 + \mathbf{K}^2}$, without changing significantly the results. With this assumption many objections can be explained away. For instance, two apparently very different versions of the HBT method can be shown to be equivalent [2], or the question can be answered: why the arguments of the emission function, which are half sums just like the arguments of the Wigner function, can be interpreted as particle momentum and position? The smoothness assumption should hold for pairs of momenta p_1, p_2 important for the analysis. For high energy collisions of heavy ions the relevant momentum differences are small and the assumptions seems justified; for e^+e^- annihilations or pp scattering they are much bigger and the assumption is doubtful [8]. One of the outstanding open problems is, why the HBT method is applied with comparable success to heavy ion, e^+e^- and pp collisions?

3. Measuring the density matrix

Much information about the single particle density matrices can be obtained, under certain assumptions, by studying the distributions of momenta for sets of $n = 1, 2, \dots$ identical mesons, for instance π^- mesons [3]. The formulae used to relate the n -particle momentum distributions $P(\mathbf{p}_1, \dots, \mathbf{p}_n)$ to the single particle density matrix elements are:

$$\begin{aligned} P_1(\mathbf{p}_1) &= N_1 \rho(\mathbf{p}_1; \mathbf{p}_1), \\ P_2(\mathbf{p}_1, \mathbf{p}_2) &= P(\mathbf{p}_1)P(\mathbf{p}_2) + N_{12} |\rho(\mathbf{p}_1; \mathbf{p}_2)|^2, \\ P_3(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) &= P(\mathbf{p}_1)P(\mathbf{p}_2)P(\mathbf{p}_3) + P(\mathbf{p}_1, \mathbf{p}_2)P(\mathbf{p}_3) + P(\mathbf{p}_2, \mathbf{p}_3)P(\mathbf{p}_1) \\ &\quad + P(\mathbf{p}_3, \mathbf{p}_1)P(\mathbf{p}_2) + N_{123} \Re[\rho(\mathbf{p}_1; \mathbf{p}_2)\rho(\mathbf{p}_2; \mathbf{p}_3)\rho(\mathbf{p}_3; \mathbf{p}_1)] \end{aligned} \quad (7)$$

and so on, where \Re stand for *real part of* and N_α are normalization constants, irrelevant for our discussion. The first formula follows from the definition of the density matrix. The others are derived like the second one, which

³ Care is taken to minimize the error. E.g. the product $S(K, x_1)S(K, x_2)$ is replaced by $S(p_1, x_1)S(p_2, x_2)$, so that the errors of the two substitutions partly cancel, see [2].

was obtained, for a specific model and in a different notation, in [3]: For two totally uncorrelated particles, the probability distribution for their momenta would be proportional to $\rho(\mathbf{p}_1; \mathbf{p}_1)\rho(\mathbf{p}_2; \mathbf{p}_2)$; when the two particles are identical and have spin zero, symmetrization introduces the correction term $\rho(\mathbf{p}_1; \mathbf{p}_2)\rho(\mathbf{p}_2; \mathbf{p}_1) = |\rho(\mathbf{p}_1; \mathbf{p}_2)|^2$.

There is a problem with the consistency of equations (7). Suppose that exactly two particles are produced. Then $P_2(\mathbf{p}_1, \mathbf{p}_2)$ integrated over \mathbf{p}_2 should give $P_1(\mathbf{p}_1)$. Actually, the integral of the first term on the right-hand-side of the formula for P_2 yields P_1 and the second term supplies an unwanted correction. The dependence of this correction on \mathbf{p}_1 is usually different than that in $P_1(\mathbf{p}_1)$. Thus, it is not possible to compensate it by a normalizing factor. The only way out is to make this correction negligibly small. Since $\rho(\mathbf{p}_1; \mathbf{p}_1)$ is known and fixed, one must assume that $\rho(\mathbf{p}_1; \mathbf{p}_2)$ decreases rapidly with increasing $|\mathbf{q}|$.

The model described here has a number of other difficulties which, however, can be relieved by a suitable handling of the experimental data and/or of the comparison between theory and experiment. We will describe them very briefly. A much more detailed description with many references can be found in the review [2]. The model assumes no correlations except the Bose–Einstein correlations. The remedy is to construct a sample which contains all the correlations except the Bose–Einstein correlations and compare it with the full sample which contains all the correlations including the Bose–Einstein correlations. The relative momentum \mathbf{q} of two charged particles with similar momenta is strongly affected by Coulomb, and in some situations also by strong, interactions. This is the famous problem of final state interactions and is usually handled by introducing a suitable \mathbf{q} -dependent correction term. Many final hadrons are secondaries originating from decays of resonances. The short-lived resonances are no problem, but the long-lived ones mimic much bigger interaction regions. Actually, they produce in the ratio $\frac{P_2(\mathbf{p}_1, \mathbf{p}_2)}{P_1(\mathbf{p}_1)P_2(\mathbf{p}_2)}$ a narrow peak at small $|\mathbf{q}|$ which is below experimental resolution and is usually corrected for by changing the normalization factor N_{12} . Finally, there are purely experimental problems like correcting for momentum resolution, particle misidentification *etc.* We assume in the following that the consistency condition is satisfied and that the experimental data have been fully corrected, so that relations (7) can be applied.

The obvious question is, can one solve equations (7) for the single particle density matrix $\rho(\mathbf{p}_1; \mathbf{p}_2)$? The answer is negative. A simple calculation [9], [10] shows that the predicted momentum distributions (for $n = 1, 2, \dots$) do not change when $\rho(\mathbf{p}_1; \mathbf{p}_2)$ changes as follows:

$$\rho(\mathbf{p}_1; \mathbf{p}_2) \rightarrow \rho(\mathbf{p}_2; \mathbf{p}_1) \quad \text{and/or} \quad (8)$$

$$\rho(\mathbf{p}_1; \mathbf{p}_2) \rightarrow e^{if(\mathbf{p}_1)}\rho(\mathbf{p}_1; \mathbf{p}_2)e^{-if(\mathbf{p}_2)}, \quad (9)$$

where $f(\mathbf{p})$ is an arbitrary real function of a single momentum. This group of transformations includes all the modifications of $\rho(\mathbf{p}_1; \mathbf{p}_2)$ which affect none of the momentum distributions. The first ambiguity is not much of a problem. It corresponds to the space inversion of the homogeneity region. If the homogeneity region is symmetric this has no effect. If it is not symmetric, one usually can find a physical argument to choose one of the two possibilities. The second ambiguity, however, is much more serious. Let us take an example. Choosing

$$f(\mathbf{p}) = \frac{1}{2} \sum_{i=x,y,z} a_i p_i^2, \quad (10)$$

where a_i are arbitrary real numbers, one finds that this implies the modification

$$S(K, X) \rightarrow S(K, \mathbf{X} + \mathbf{a}(\mathbf{K}), X^0), \quad (11)$$

where the components of $\mathbf{a}(\mathbf{K})$ are $a_i K_i$. Thus, every homogeneity region, except the $\mathbf{K} = \mathbf{0}$ one, gets shifted. Since the possible shifts are a class of functions of \mathbf{K} , the relative positions of the homogeneity regions can be almost arbitrarily changed. This, incidentally, is a proof that at best one can hope to measure the sizes and shapes of the individual homogeneity regions. Putting all the homogeneity regions on top of each other, one could get a lower bound for the size of the overall interaction region, but there is no upper bound.

The previous example is a very special case of a general theorem stating what can and what cannot be measured using the HBT method [11]. The formulation uses cumulants, so let us quote their definition. For the profile of any homogeneity region $p_{\mathbf{K}}(\mathbf{x})$ we can define its characteristic function $\langle e^{i\mathbf{t}\cdot\mathbf{x}} \rangle$ which is, of course, a function of the vector \mathbf{t} . Consider now the power series expansion

$$\log \langle e^{i\mathbf{t}\cdot\mathbf{x}} \rangle = \sum_{n_x, n_y, n_z} K(n_x, n_y, n_z) \frac{(it_x)^{n_x}}{n_x!} \frac{(it_y)^{n_y}}{n_y!} \frac{(it_z)^{n_z}}{n_z!}, \quad (12)$$

where the summation is over three sets of all non-negative integers. The coefficients $K(n_x, n_y, n_z)$ are the cumulants of the probability distribution $p_{\mathbf{K}}(\mathbf{x})$. The number $n_x + n_y + n_z$ is the order of the cumulant. The cumulant is even (odd) when its order is even (odd).

The theorem consists of three points:

- Every even cumulant of $p_{\mathbf{K}}(\mathbf{x})$ can be measured.
- No odd cumulant of $p_{\mathbf{K}}(\mathbf{x})$ can be measured.

- When the first order cumulants, *i.e.* $\langle \mathbf{X} \rangle_{\mathbf{K}}$, are given, they fix the phase $f(\mathbf{p})$ up to an irrelevant constant and, for $\langle \mathbf{X} \rangle_{\mathbf{K}} \neq \mathbf{0}$, eliminate the freedom of inversion. Thus, the full $p_{\mathbf{K}}(\mathbf{x})$ become measurable.

Let us consider a few applications of this theorem. The second order cumulants are elements of the covariance matrix $\langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle_{\mathbf{K}}$ where $x_i = x, y, z$. Experimentalist routinely measure the HBT radii and use them to calculate the elements of the covariance matrices of the homogeneity regions. The theorem shows that this interpretation is not affected by the ambiguity (9). In the imaging method (see [12] and references given there) one determines from the measured momentum distributions the probability distributions $\bar{p}_{\mathbf{K}}(\mathbf{x}_1 - \mathbf{x}_2)$ for pairs of points within a homogeneity region. This is much less information than given by the profiles of the homogeneity regions. For instance, the distributions $\bar{p}_{\mathbf{K}}$ are not sensitive to the averages $\langle \mathbf{X} \rangle_{\mathbf{K}}$. However, one finds that the distribution $\bar{p}_{\mathbf{K}}$ depends only on the even cumulants of the distribution $p_{\mathbf{K}}$ and that it depends on all of them. Therefore, it is reliably measurable and it gives much more information than the measurements of the HBT radii.

On the other hand, the centers of the homogeneity regions $\langle \mathbf{X} \rangle_{\mathbf{K}}$ are first order cumulants and cannot be reliably measured. This is a much shorter derivation of the conclusion from example (10) that the relative positions of the homogeneity regions are not measurable.

Of special interest is the third point. The collisions of heavy ions are usually described using rather simple-minded tools, like Euler's hydrodynamics. It is much easier to believe that such analyses give reasonably the centers of the homogeneity regions than that they reproduce all the intricacies of their shapes. According to the third point, however, once the distribution of the centers is known, everything else can be unambiguously measured without further model assumptions.

4. Emission function and Wigner function

Much less work has been done on the ambiguities in the solution of equation (5) for $\rho(\mathbf{K}, \mathbf{q})$ known. Instead, one usually derives from a model, or guesses, an emission function and checks whether it yields the momentum distributions in agreement with experiment. Since a general analysis has not yet been performed, we present here only some partial results.

Let us assume that in some reference frame all the hadrons got created simultaneously at time $t = 0$. Then the emission function can be written in the form

$$S(K, X) = \delta(t) \bar{S}(K, X), \quad (13)$$

where the function $\bar{S}(K, X)$ should be found from equation (5). Substituting into (5), one finds, a unique solution

$$S(K, X) \sim \delta(t)W(\mathbf{K}, \mathbf{X}), \quad (14)$$

where W denotes the Wigner function. Thus, in this case the emission function is as good as the Wigner function. A solution of this type always exists, assuming that the hadronization process is over at $t = 0$, but usually it is not the realistic solution we are looking for.

Assuming (in some reference frame) no coherence in time, one can relate the emission function to the Wigner function by the formula [13]

$$S(K, X) \sim \frac{\overline{W}_{dt}(\mathbf{K}, X)}{dt}. \quad (15)$$

\overline{W}_{dt} is the Wigner function for the particles produced in the time interval dt around $t = X^0$. This Wigner function is normalized to the number of particles produced in the time interval dt and not to one as is usual for Wigner functions. To mark this difference we introduced the overline. With this normalization \overline{W}_{dt} tends to zero with $dt \rightarrow 0$ and it has to be divided by dt to give a finite result. Note that this solution for $S(K, X)$ does not depend on K^0 . The equation for $S(K, X)$ can be obtained by integrating both sides of (15) over time. The result is

$$W(\mathbf{K}, X) \sim \int dt S(K, X). \quad (16)$$

Solution (15) is one among the infinity of solutions of (16) and the HBT method gives no hint how to find it.

However, (16) can be used as a sum rule for $S(K, X)$. If assumptions are made about the time dependence of $S(K, X)$, it is possible to get one parameter of the assumed time distribution from experiment. An example is contained in [14]. This paper proposed a once very famous formula to measure both the size and the life-time of the interaction region by the HBT method. The formula was used for years by many experimental groups. The authors assumed that all the particle sources are produced simultaneously at $t = 0$, that there is no coherence in time and that each source decays exponentially with the same life time. Then, in agreement with our analysis, they were able to determine this life time of the sources.

Let us consider a generalization of the previous case. Suppose that $S(K, X)$ does not depend on \mathbf{q}^2 . This is more general than the previous assumption that it does not depend on K^0 . For instance, it is enough to assume that, in the spirit of the smoothness assumption, K^0 in $S(K, X)$ can be replaced by $\sqrt{m^2 + \mathbf{K}^2}$. Applying the Fourier transformation to both

sides of equation (5) we convert the density matrix into the Wigner function and using the identity

$$Kq = 0 = K^0 q^0 - \mathbf{K} \cdot \mathbf{q}, \quad (17)$$

to eliminate q^0 , we find after trivial integrations

$$W(\mathbf{K}, \mathbf{X}) \sim \int dt S \left(K, \mathbf{X} + \frac{\mathbf{K}}{K^0} t, t \right). \quad (18)$$

In the $\mathbf{K} = \mathbf{0}$ frame one recovers formula (16). Actually, the use of this frame has a number of advantages (see *e.g.* [2]) and should be strongly recommended.

5. Conclusions

It is not possible to measure the size and/or shape of the full interaction region using the HBT method. Therefore, efforts concentrate on measurements of the homogeneity regions. From the classical point of view, the best description of the size and shape would be a probability distribution for producing a particle with momentum \mathbf{K} at space-time point X . As known from quantum mechanics, however, this is not possible. The usual translation into quantum physics is to look for the Wigner function $W(\mathbf{K}, \mathbf{X})$. When written in the interaction picture and after all the hadrons have been produced, this Wigner function does not depend on time. Its interpretation as a probability distribution $p_{\mathbf{K}}(\mathbf{X}) = W(\mathbf{K}, \mathbf{X})$ is only an approximation, *e.g.* because Wigner functions can be negative in certain regions. It seems, however, that at least for high-energy heavy ion scattering this approximation is reasonable (for a recent discussion see [15]).

The theorem presented and discussed in Section 3 states that: The even cumulants of $p_{\mathbf{K}}(\mathbf{X})$ can be measured using the HBT method. This, in particular, justifies the measurements of the HBT radii and the imaging method. The odd cumulants cannot be measured, which explains, in particular, why one has to study the separate homogeneity regions instead of the whole interaction region. The theorem also shows that when the distribution $\langle \mathbf{x} \rangle_{\mathbf{K}}$ is known, it is possible to measure the full Wigner functions for all the homogeneity regions and consequently also the full interaction region.

In order to learn about the time evolution of the homogeneity regions one uses emission functions. Even when the Wigner function is known, the evaluation of the emission function requires additional assumptions. When these assumptions are too strong, one obtains no more than the input. This is illustrated by the example leading to formula (14). With a judicious choice of assumptions, however, one can obtain additional information. An example

is the model of Kopylov and Podgoretskii [14], where the life time of the hadron sources can be measured. Of course, such results are no more credible than the assumptions used to derive them.

Nowadays people usually propose full models of hadronization, which among other things predict also the results of the HBT measurements. The comparison with these experimental measurements is used to test the model and sometimes also to fix some of its parameters. The credibility of the model, and consequently of the description of the interaction region that it offers, is based on a comparison of its prediction with a variety of experimental data and not just on the study of the HBT results. The question how much one can learn about the interaction region in a (as nearly as possible) model-independent way remains, however, interesting.

REFERENCES

- [1] G.I. Kopylov, M.I. Podgoretsky, *Zhur. Exp. Theor. Phys.* **69**, 414 (1975).
- [2] M.A. Lisa, S. Pratt, R. Soltz, W. Wiedemann, *Ann. Rev. Nucl. Part. Sci.* **55**, 357 (2005).
- [3] G. Goldhaber, S. Goldhaber, W. Lee, A. Pais, *Phys. Rev.* **120**, 300 (1960).
- [4] E.V. Shuryak, *Phys. Lett.* **44B**, 387 (1973).
- [5] S. Pratt, *Phys. Rev. Lett.* **53**, 1219 (1984).
- [6] M.G. Bowler, *Z. Phys.* **C29**, 617 (1985).
- [7] Yu.M. Sinyukov, in *Hadronic Matter Theory and Experiment*, eds. J. Letessier, H.H. Gutbrod, J. Rafelski, Plenum Press, New York 1955, p. 309.
- [8] S. Pratt, *Phys. Rev.* **C56**, 1095 (1997).
- [9] A. Białas, K. Zalewski, *Phys. Rev.* **D72**, 036009 (2005).
- [10] K. Zalewski, *Phys. Rev.* **D74**, 114022 (2006).
- [11] K. Zalewski, *Phys. Rev.* **D77**, 074006 (2008).
- [12] P. Danielewicz, S. Pratt, *Phys. Rev.* **C75**, 034907 (2007).
- [13] K. Zalewski, *Acta Phys. Pol. B* **34**, 3379 (2003).
- [14] G.I. Kopylov, M.I. Podgoretsky, *Yad. Phys.* **18**, 656 (1973).
- [15] K. Zalewski, *Acta Phys. Pol. B* **39**, 181 (2008).