RECURSIVE CALCULATION OF MULTI-GLUON ONE-LOOP AMPLITUDES* **

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An efficient numerical algorithm to evaluate one-loop amplitudes is presented. It expresses the amplitude in terms of universal tensor integrals and their process-dependent coefficients, both of which are calculated with recursive methods. It is shown by explicit calculations that for ordered QCD amplitudes with a number of external legs up to 10, its performance is competitive with other methods.

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1. Introduction

Data from the experiments at LHC for the study of elementary particles demand signals and potential backgrounds for new physics to be under control at sufficient accuracy [1]. Hard processes with high multiplicities involving many particles or partons cannot be neglected, and even have to be dealt with at the Next-to-Leading Order (NLO) level to, for example, reduce the scale dependence of observables and to have a better description of the shape of their distributions.

The evaluation of the one-loop amplitude is an essential part of an NLO calculation. Impressive results have been published in the past year for one-loop QCD amplitudes with very high numbers of partons [2–4]. They were obtained with the so-called unitarity-approach. Originally restricted to analytical calculations [5], the potential of this method in a numerical approach became, after the crucial input from [6], clear with the work of [7] and [9]. It is considered an alternative to the "traditional" approach involving

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tensor integrals. Both approaches expand the one-loop amplitude in terms of a basis set of one-loop functions. In the unitarity-approach, this set consists of scalar integrals up to 4-point or 5-point functions, and it aims at determining the coefficients directly. In the "tensor-approach", the basis set is larger and consists of tensor integrals or their coefficients functions when expanded in terms of Lorentz-covariant objects [10, 11]. Also these basis functions are eventually calculated by expressing them in terms of a smaller set of scalar integrals, but this happens in a, for the particular method, universal way, independent of the amplitude.

The success by [2–4] in dealing with very high multiplicities is explained by the fact that, for the considered ordered amplitudes, the asymptotic computational complexity coming with the unitarity-approach is polynomial. In the tensor-approach, the asymptotic computational complexity is at least exponential, because the number of tensor integrals grows exponentially with the multiplicity. However, most methods to evaluate tensor integrals are recursive, and the calculation of an amplitude may be organized such that each of them is evaluated only once. In such a scenario, the asymptotic computational complexity of evaluating all of them does not exceed their number, and merely adds a constant factor. Of course, in an expansion of a one-loop amplitude in terms of tensor integrals also the coefficients to be contracted with them have to be calculated. In the following, we will see how also these can be evaluated recursively, limiting the computational cost such that it allows for the calculation of ordered multi-gluon one-loop amplitudes up to 10 external legs.

2. Recursive evaluation of tensor integrals

We consider tensor integrals defined in D dimensions, but only with 4-dimensional components of the integration momentum in the numerator

$$\mathcal{T}^{\nu_1\nu_2\cdots\nu_r}_{n,r} = \int \frac{d^Dq}{\mathrm{i}\pi^{D/2}} \frac{q_4^{\nu_1} q_4^{\nu_2}\cdots q_4^{\nu_r}}{\prod_{j=1}^n \Big[(q+p_j)^2 - m_j^2\Big]}\,. \tag{1}$$

This will lead to a calculation of the one-loop amplitude within the scheme of [13], which asks for a finite counter term in order to arrive at gauge-invariant results and to cast the result into other schemes like 't Hooft–Veltman or FDH. This finite counter term is exactly given by the so-called $\mathcal{R}_2$-term, showing up explicitly in the OPP unitarity-approach as part of the rational terms [8], and which is shown how to be determined in [14].

The most straightforward way to calculate such tensor integrals for high n recursively is as follows. Using the fact that we can write

$$2(p_j-p_n)\,q = \Big[(q+p_j)^2 - m_j^2\Big] - \Big[(q+p_n)^2 - m_n^2\Big] + m_j^2 - p_j^2 - m_n^2 + p_n^2\,, \tag{2}$$

we have

$$2(p_j - p_n)_{\nu_r} \mathcal{T}_{n,r}^{\nu_1\nu_2\cdots\nu_r} = \mathcal{T}_{n-1,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}(j) - \mathcal{T}_{n-1,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}(n) + \left(m_j^2 - p_j^2 - m_n^2 + p_n^2\right) \mathcal{T}_{n,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}, \quad (3)$$

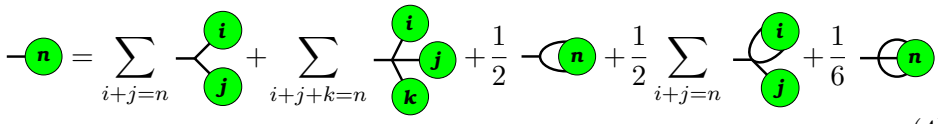
where $\mathcal{T}_{n-1,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}(j)$ is obtained from $\mathcal{T}_{n,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}$ by removing the j-th denominator. Choosing 4 different vectors p_j appearing in the denominators, we get 4 relations, enough to determine the 4 integrals $\mathcal{T}_{n,r}^{\nu_1\nu_2\cdots\nu_r}$ with the first $r-1$ Lorentz indices fixed. So given the numbers $\mathcal{T}_{n-1,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}(j)$ for 5 different values of j and the number $\mathcal{T}_{n,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}}$, the four numbers $\mathcal{T}_{n,r}^{\nu_1\nu_2\cdots\nu_{r-1}\nu_r}$ for $\nu_r = 0, 1, 2, 3$ can be determined.

In order for this to be possible, n has to be at least 5. The natural way to continue for $n \leq 4$ is to express the tensor integrals in terms of Passarino–Veltman functions, which can be determined recursively by inverting 3×3 and 2×2 systems for $n = 4$ and $n = 3$, respectively. Alternatively, the recursive method of [11], which asks for one denominator momentum less, can be applied for $n = 4$. In fact, this method is also preferred for the higher-n integrals, because it involves the Gram determinant of only 3 vectors, instead of 4, and is numerically more stable.

To obtain the results presented in this write-up, the high-n scalar integrals $\mathcal{T}_{n,r=0}$ were calculated with the OPP method [7]. Although not recursive, the method is very efficient, in particular if applied to scalar functions because they can be completely expressed in terms of scalar 4-point functions. The scalar 1-point, 2-point, 3-point and 4-point functions, finally, were evaluated with `OneLOop` [12]. Also the tensor 2-point functions were evaluated with this program.

3. Recursive relations for one-loop amplitudes

Amplitudes can be calculated recursively using the Dyson–Schwinger equation. Being an equation for the generating function of the Green's functions of the field theory under consideration, it generates recursive relations between the Green's functions themselves. In the application of amplitude calculation, it is best formulated in terms of *off-shell currents*, *i.e.* connected Green's functions with all legs referring to external particles put on-shell. For a bosonic field theory with 3-point and 4-point vertices, like for example purely gluonic QCD, the relations can be depicted graphically as follows:

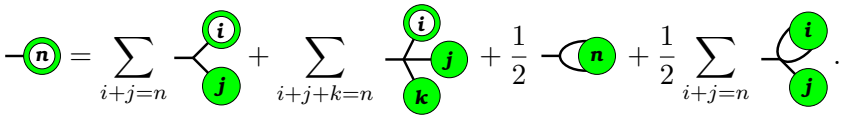


(4)

Here, blobs refer to off-shell currents, the symbol inside each blob refers to the number of on-shell legs implicitly attached, and the explicit legs sticking out are off-shell. The summations are over all possible ways to distribute the on-shell legs from the graph on the l.h.s. over the different blobs attached to a vertex. Off-shell legs from the same blob are considered equivalent, whereas on-shell legs, typically referring to external particles with different momenta, are considered inequivalent. The vertices are those from the usual Feynman rules. In the perturbative solution to the relations, each off-shell current can be represented as a sum of Feynman graphs [15].

As given above, the relations lead to off-shell currents including all loop orders. The relations are restricted to tree-level by removing the last three terms, *i.e.* the terms with the explicit loops, and relations of this type have been applied successfully to calculate tree-level amplitudes with many external legs [16]. Starting from the polarization vectors of the external particles as the one-particle currents, the relations dictate how to simply connect currents to vertices to obtain more-particle currents, and to finally end up with the complete amplitude.

Restricted to one-loop off-shell currents, the relations look as follows:



(5)

Now the solid blobs represent tree-level off-shell currents, and the blobs with a hole represent one-loop off-shell currents. The calculation of the one-loop currents following these relations is less straightforward than in the tree-level case, because one has to deal with the explicit loop integration appearing in the last two graphs. In these graphs, two off-shell legs of tree-level currents are connected to the vertex, which now implies a loop integration. To deal with this situation, the loop integrated off-shell currents can be expressed in terms of tensor integrals

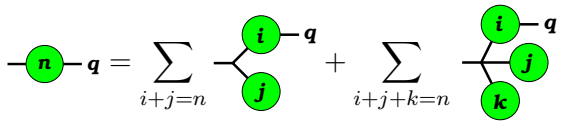
$$= \sum_{|\mathcal{D}|\leq n+1} \sum_{r=0}^{|\mathcal{D}|-1} \mathcal{G}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})\, \mathcal{T}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})\,, \tag{6}$$

$$= \sum_{|\mathcal{D}|\leq n+1} \sum_{r=0}^{|\mathcal{D}|} \mathcal{H}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})\, \mathcal{T}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})\,. \tag{7}$$

Here $\mathcal{T}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$ represents a tensor integral with a set $\mathcal{D}$ of denominator factors, and $\mathcal{G}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$ and $\mathcal{H}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$ its accompanying coefficients. Loop currents attached to the 4-point vertex (6) and the 3-point vertex (7) need to be distinguished in our example, because the latter vertex depends

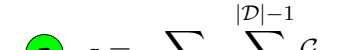
on the integration momentum. The number of elements $|\mathcal{D}|$ in the set $\mathcal{D}$ is at most $n+1$, and, in the Feynman gauge, the rank of the tensor integral does not exceed $|\mathcal{D}|$. The first sum in both equations is over all possible denominator structures occurring in the loop current.

Before loop integration, the l.h.s. of Eq. (6) satisfies the tree-level recursive relations. Cutting the loop which sticks out of the blob, we get



(8)

where the extra off-shell leg is indicated by the integration momentum q. On the other hand, un-integrating Eq. (6) we also have



$$= \sum_{|\mathcal{D}|\leq n+1} \sum_{r=0}^{|\mathcal{D}|-1} \mathcal{G}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) \, \frac{q^{\nu_1} q^{\nu_2} \cdots q^{\nu_r}}{\prod_{j\in\mathcal{D}} \left[(q+p_j)^2 - m_j^2\right]} \,. \qquad (9)$$

Substituting this identity into Eq. (8), we find recursive relations for the coefficients $\mathcal{G}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$. The only extra ingredient needed are the relations dictating how the vertices and the numerators of the propagators from the field theory under consideration decompose under momentum shift. In our example of gluonic QCD the 4-point vertex and the numerator of the propagator in the Feynman gauge do not depend on momenta at all, and the 3-point vertex satisfies

$$V^{abc}_{\mu\nu\rho}(q+p_1,p_2) = V^{abc}_{\mu\nu\rho}(p_1,p_2) + \mathrm{i}g_{\mathrm{s}} f^{abc}[g_{\mu\sigma}g_{\nu\rho} + g_{\rho\mu}g_{\nu\sigma} - 2g_{\nu\mu}g_{\rho\sigma}]\, q^{\sigma} \,. \qquad (10)$$

A similar relation also allows us to express the coefficient $\mathcal{H}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$ from Eq. (7) in terms of the coefficients $\mathcal{G}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$.

4. Results

The scheme presented above has been applied to the calculation of color-ordered multi-gluon one-loop amplitudes, or so-called *primitive amplitudes*. Details of the particular application are described in [17]. A `Fortran77` program has been written which reproduces all the numeric results for such amplitudes given in [2] up to 10 gluons to at least 4 decimals precision. Furthermore, recursive equations for quark loops have been derived and implemented and proven to produce gauge-invariant results. Average cputimes for one amplitude evaluation are comparable to those in [2] up to 10 external gluons. For higher numbers they become clearly worse.

In order to assess the numerical accuracy, the results for the $1/\epsilon$-pole of the amplitude, obtained by replacing the finite part of the scalar basis functions by their coefficient of the $1/\epsilon$-pole at the beginning of the recursion for the tensor integrals, is compared to the result obtained with the analytic formula from [18]. The left of Fig. 1 shows the distribution of $\log_{10}|1 - \text{Num}/\text{Ana}|$ for a number of 6, 8 and 10 external gluons. "Num" refers to the result for the $1/\epsilon$-pole of the amplitude obtained with the presented program, "Ana" refers to the result obtained with the analytic formula. The distributions are obtained from large samples of uniformly distributed phase space points, satisfying the same kinematical cuts as in [2].

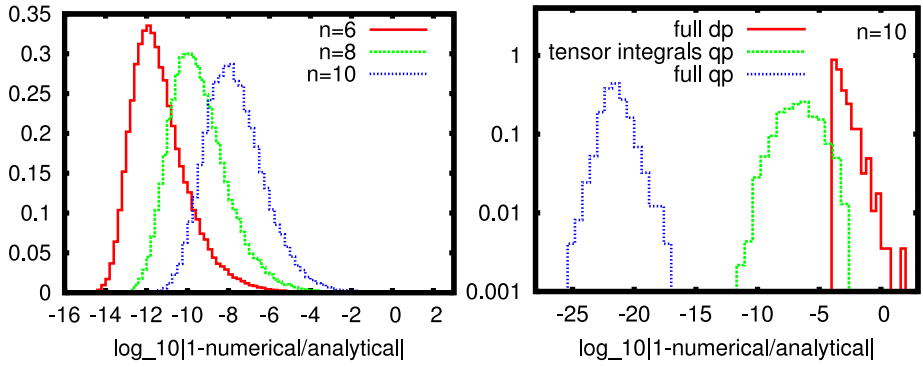


Fig. 1. Distribution of the numerical accuracy.

The right of Fig. 1 shows the tail of the distribution for $n = 10$ in a log-scale for the y-axis. Furthermore depicted are the distributions obtained when, for the phase space points in this tail, only the tensor integrals are calculated at quadruple machine precision (qp), and when the full amplitude is calculated at qp. The graphs show that unacceptable low precision in the, relatively small, tail can be cured within a staged scenario of increased machine precision.

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