GRAVITY AS AN EFFECTIVE THEORY*

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Using as inspiration the well known chiral effective Lagrangian describing the interactions of pions at low energies, in these lectures we review the quantization procedure of Einstein gravity in the spirit of effective field theories. As has been emphasized by several authors, quantum corrections to observables in gravity are, by naive power counting, very small. While some quantities are not predictable (they require local counterterms of higher dimensionality) others, non local, are. A notable example is the calculation of quantum corrections to Newton's law. Albeit tiny these corrections are of considerable theoretical importance, perhaps providing information on the ultraviolet properties of gravity. We then try to search for a situation where these non local corrections may be observable in a cosmological context in the early universe. Having seen that gravity admits an effective treatment similar to the one of pions, we pursue this analogy and propose a two-dimensional toy model where a dynamical zwei-bein is generated from a theory without any metric at all.

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1. Introduction and outline

This paper summarizes the contents of a set of lectures that were delivered in the 49th Zakopane School on Theoretical Physics on the subject of treating Einstein theory of gravitation as an effective theory and the testable consequences of this procedure, and the possibility that gravitons emerge as Goldstone states after some sort of symmetry breaking mechanism. The contents can be basically divided into two parts. The first one describes the

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treatment of effective theories taking the chiral Lagrangian of strong interactions as a starting point and proceeding to study the gravity case in parallel to the way one sets out to quantize the pion Lagrangian. This part is not original and we have freely drawn material from the works of Donoghue [1], Bjerrum-Bohr [2] and Khriplovich [3] in particular.

The second part contains original work made in collaboration with Alfaro [4], Cabrer [5], Multamäki [6] and Vagenas [6]. Some results are presented in published form for the first time here. In the lectures the subject of explicit Lorentz breaking was treated briefly, including some potential applications to astroparticle physics [7], but this part is omitted in these written notes for the sake of homogeneity and consistency of the presentation

We shall start with a succinct presentation of the pion chiral Lagrangian and the chiral counting rules. We shall move to the gravity case after that, proceeding to quantize the theory. An analogous power counting can be implemented in this case too. The power counting turns out to be more subtle when matter fields are present, as we shall see.

Next we will argue why non-local effects are necessarily present and, in fact, that they provide the only unique and non-ambiguous predictions of quantum gravity at the one-loop level. These predictions are finite and contribute in a distinctively different way to physical observables. This shall be exemplified by studying the first quantum corrections to Newton's law and also by analyzing how these corrections may affect the evolution of a de Sitter universe (inflation).

Finally, we shall give some credence to the idea that gravitons might be Goldstone bosons of some broken symmetry. We are certainly not the first to entertain this idea [8], which, on the other hand may seem hopelessly flawed due to some in-principle long-standing restrictions [9]. We shall provide a two-dimensional toy model (that, however, can be easily extended to four dimensions) that shows that such a mechanism is possible in a model that very much parallels the mechanism of chiral symmetry breaking in QCD, and how the theoretical objections might be circumvented.

2. Chiral effective theory and chiral counting

The chiral Lagrangian is a non-renormalizable theory describing accurately pion physics at low energies. It has a long story, with the first formal studies concerning renormalizability being due mostly to Weinberg [10] and later considerably extended by Gasser and Leutwyler [11]. The chiral Lagrangian contains a (infinite) number of operators organized according to the number of derivatives

$$\begin{aligned}\mathcal{L} &= f_\pi^2 \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \alpha_1 \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger \\ &\quad + \alpha_2 \mathrm{Tr}\, \partial_\mu U \partial_\nu U^\dagger \partial^\mu U \partial^\nu U^\dagger + \dots\,, \qquad (1)\\ \mathcal{L} &= \mathcal{O}(p^2) + \mathcal{O}(p^4) + \mathcal{O}(p^6) + \dots\,, \qquad (2)\end{aligned}$$

$$U \equiv \exp i\tilde{\pi}/f_\pi\,, \qquad \tilde{\pi} \equiv \pi^a \tau^a/2\,. \qquad (3)$$

Pions are the Goldstone bosons associated to the (global) symmetry breaking pattern of QCD

$$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \to \mathrm{SU}(2)_\mathrm{V}\,. \qquad (4)$$

The above Lagrangian is the most general one compatible with the symmetries of QCD and their breaking. Locality, symmetry and relevance (in the renormalization group sense) are the only guiding principles to construct $\mathcal{L}$. Renormalizability is not. In fact if we cut-off the derivative expansion at a given order the theory requires contertemps beyond that order no matter how large.

Note that, although the symmetry has been spontaneously broken, the effective Lagrangian still has the *full* symmetry

$$U \to L U R^\dagger \qquad (5)$$

i.e. the underlying symmetry is not lost in spite of the (partial) breaking.

Next let us see how a simple power counting in derivatives can be established at the level of quantum corrections. Let A_{N^π} be the amplitude for the scattering of N^π pions. At lowest order in the derivative expansion it will be of the form

$$A_{N^\pi} \sim \frac{p^2}{f_\pi^2}\,, \qquad (6)$$

where p^2 represents a generic kinematic invariant constructed with external momenta. At the next order

$$A_{N^\pi}(p_i) \sim \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{f_\pi}\right)^{N^\pi} (k^2)^{N_\mathrm{V}} \left(\frac{1}{k^2}\right)^{N_\mathrm{P}}\,, \qquad (7)$$

where N_V and N_P are the number of vertices and propagators, respectively. Consider *e.g.* $\pi\pi \to \pi\pi$ scattering. Then $N^\pi = 4$, $N_\mathrm{V} = 2$ and $N_\mathrm{P} = 2$. The integral is divergent and it yields a result of the form

$$A_{N^\pi} \sim \frac{1}{16\pi^2 f_\pi^2} p^4 \times \frac{1}{\varepsilon}\,. \qquad (8)$$

Dimensional regularization has been assumed. The divergence can thus be absorbed by redefining the coefficient of the operators at $\mathcal{O}(p^4)$ assuming that the regularization preserves chiral invariance.

This counting works to all orders and IR divergences, that potentially could spoil it, are absent (Weinberg). At each order in perturbation theory the divergences that arise can be eliminated by redefining the coefficients in the higher order operators, *e.g.*

$$\alpha_i \to \alpha_i + \frac{c_i}{\varepsilon}\,. \tag{9}$$

Note that, in addition to the pure pole in ε, logarithmic non-local terms necessarily appear. For instance in a two-point function they appear in the combination

$$\frac{1}{\varepsilon} + \log\frac{-p^2}{\mu^2}\,. \tag{10}$$

This comes about because pions are strictly massless in the chiral limit and thus a combination of momenta must necessarily normalize the μ^2 that appears for dimensional consistency in dimensional regularization.

The cut provided by the log is absolutely required by unitarity. Let us split the scattering matrix S in the usual way

$$S = I + iT\,. \tag{11}$$

The identity corresponds, obviously, to having no interaction at all.

Unitarity implies

$$\begin{aligned} S^\dagger S &= I = I + i(T - T^\dagger) + T^\dagger T\,, \\ i(T - T^\dagger) &= -T^\dagger T\,. \end{aligned} \tag{12}$$

Thus T must *necessarily* have an imaginary part. Pure powers of momenta are real by construction. Thus the logs, that bring about a cut and an imaginary part, are needed. Loops are essential, even for effective theories. There is no such thing as a 'classical effective theory' in a quantum theory.

To recapitulate, the lowest-order, tree level contribution to pion–pion scattering is $\sim \frac{p^2}{f_\pi^2}$. The one-loop chiral corrections are $\sim \frac{p^4}{16\pi^2 f_\pi^4}$. Thus the counting parameter in the loop (chiral) expansion is clearly

$$\frac{p^2}{16\pi^2 f_\pi^2}\,. \tag{13}$$

Each chiral loop gives an additional power of p^2.

The counting can actually be extended to include small departures from the chiral limit, *i.e.* allowing for non-zero quark (hence pion) masses. If $\mathcal{O}(p^{2n})$ counts as p^{2n}, soft breaking terms such as

$$\mu m \mathrm{Tr}\,\left(U + U^\dagger\right) \tag{14}$$

give the pion a mass $m_\pi^2 \sim m$. Therefore m counts as p^2 too.

Note that all coefficients in the chiral Lagrangian are nominally of $\mathcal{O}(N_c)$. Loops are automatically suppressed by powers of N_c, because $f_\pi^2 \sim N_c$ appears in the denominator, but they are enhanced by logs at low momenta as we just saw.

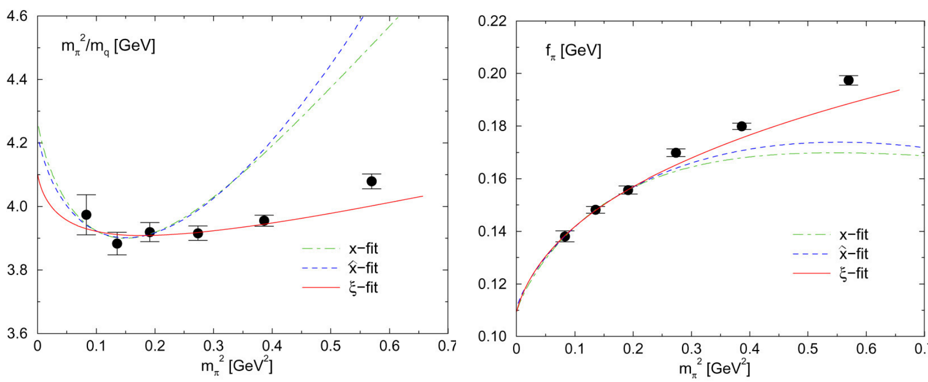

Fig. 1. Recent fits to lattice data for light masses using chiral perturbation theory at the NLO. Extracted from reference [12].

Chiral Lagrangians are extremely successful. Their application to low-energy phenomenology is nowadays standard and quite relevant. At any given order in the derivative expansion a finite number of coefficients have to be determined from experiment (or eventually lattice simulations), but then everything else is known (with the precision given by the order retained in the derivative expansion). Even without knowing these coefficients one can find combinations of observables where the unknown coefficients drop. As an illustration we show recent fits to lattice data [12] using chiral Lagrangians showing excellent agreement between their predictions and the numerical results; the point of course being that one can then use the chiral Lagrangian to extrapolate to a mass/energy regime unattainable by current numerical simulations.

3. The gravity analogy

The Einstein–Hilbert action shares several aspects with the pion chiral Lagrangian. Like the effective chiral Lagrangian it is also a non-renormalizable theory (more on this latter). It is also described, considering the most relevant operator, by a dimension two operator containing in both cases two derivatives of the dynamical variable. Both Lagrangians contain necessarily a dimensionful constant in four dimensions; the counterpart of f_π in the pion Lagrangian is the Planck mass $M_{\rm P}$. Both theories are non-linear and, finally, both describe the interactions of massless quanta. The Einstein–Hilbert action is

$$\mathcal{L} = M_{\rm P}^2\sqrt{-g}\mathcal{R} + \mathcal{L}_{\rm matter}\,, \tag{15}$$

where

$$\kappa^2 \equiv \frac{2}{M_{\rm P}^2} = 32\pi G\,. \tag{16}$$

Indeed a cursory comparison with the expressions in the previous section shows that $M_{\rm P}$ plays a role very similar to f_π.

As just mentioned $\mathcal{R}$ contains two derivatives of the dynamical variable which is the metric $g_{\mu\nu}$

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}\,, & (17)\\ \Gamma^\gamma_{\alpha\beta} &= \tfrac{1}{2}g^{\gamma\rho}\left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta}\right)\,, & (18)\\ \mathcal{R} &\sim \partial\partial g\,. & (19)\end{aligned}$$

In the chiral language, the Einstein–Hilbert action would be $\mathcal{O}(p^2)$ *i.e.* most relevant, if we omit for a second the presence of the cosmological constant which accompanies the identity operator.

Arguably, locality, symmetry and relevance in the RG sense (and not renormalizability) are the ones that single out Einstein–Hilbert action in front of *e.g.* $\mathcal{R}^2$.

Unlike the chiral Lagrangian, the Einstein–Hilbert Lagrangian (or extensions thereof including higher derivatives) has a local gauge symmetry. Indeed, gravity can be (somewhat loosely) described as the result of promoting a global symmetry (Lorentz)

$$\begin{aligned} x'^a &= \Lambda^a{}_b x^b\,, & (20)\\ \eta_{ab} &= \Lambda^c{}_a\Lambda^d{}_b\eta_{cd}\,, & (21)\end{aligned}$$

to a local one

$$\begin{aligned} x'^\mu = x'^\mu(x) \;\rightarrow\; dx'^\mu &= \Lambda^\mu{}_\nu(x)dx^\nu\,, & (22)\\ \bar{\Lambda}_\mu{}^\nu(x) &\equiv \left[\Lambda^\mu{}_\nu(x)\right]^{-1}\,, & (23)\\ \Lambda^\mu{}_\nu\bar{\Lambda}_\rho{}^\nu &= \delta^\mu_\rho\,. & (24)\end{aligned}$$

This can be accomplished if the basic field, the metric, is allowed to be a coordinate dependent field transforming as

$$g'_{\mu\nu}(x') \;=\; \bar{\Lambda}_\mu{}^\alpha \bar{\Lambda}_\nu{}^\beta g_{\alpha\beta}(x)\,, \tag{25}$$

$$d\tau^2 \;=\; g'_{\mu\nu}(x')dx'^\mu dx'^\nu = g_{\alpha\beta}(x)dx^\alpha dx^\beta\,. \tag{26}$$

Fields transform as scalars, vectors, *etc.*, under this change

$$\begin{aligned} \phi'(x') &= \phi(x)\,, \\ A'^\mu(x') &= \Lambda^\mu{}_\nu(x)A^\nu(x)\,. \end{aligned} \tag{27}$$

This means that the gauge symmetry that is present in gravity, unlike in the chiral Lagrangian, will in practice reduce the number of degrees of freedom that are observable at low energies for two reasons. One of the reasons of course is the very existence of the gauge symmetry itself. For instance, describing a spin one particle (such as a massive photon) with a four-vector is redundant; one of the four degrees of freedom completely decouples.

The other reason is easily understood just remembering what happens in the Standard Model of electroweak interactions where the global symmetry is spontaneously broken down to $\mathrm{U}(1)_{\mathrm{em}}$, but because of the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ gauge invariance originally present, all Goldstone bosons disappear yielding, in turn, some massive modes that were previously massless. The natural value for such masses is the Fermi scale ($\sim$ 250 GeV), but in gravity it would undoubtedly be the Planck mass, disappearing in practice from the low energy dynamics.

Einstein–Hilbert action has thus all the ingredients for being an effective theory describing the long distance properties of some unknown dynamics.

It is also natural to go one step further and ask whether gravitons are just Goldstone bosons of some broken symmetry. We will have more to say about this possibility in the coming sections.

3.1. Quantizing gravity

Quantum corrections in gravity are analogous to the weak field expansion in pion physics

$$U = I + i\frac{\tilde{\pi}}{f_\pi} + \dots\,. \tag{28}$$

One writes

$$g_{\mu\nu} \;\equiv\; \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{29}$$

$$g^{\mu\nu} \;=\; \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda}h_\lambda{}^\nu + \dots\,, \tag{30}$$

so in fact κ plays the same role as f_π^{-1}.

The curvatures can likewise be expanded around a given background, say $g_{\mu\nu} = \eta_{\mu\nu}$,

$$\mathcal{R}_{\mu\nu} = \frac{\kappa}{2}\left[\partial_\mu\partial_\nu h^\lambda{}_\lambda + \partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial_\lambda h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu\right] + \mathcal{O}(h^2)\,, \quad (31)$$

$$\mathcal{R} = \kappa\left[\Box h^\lambda{}_\lambda - \partial_\mu\partial_\nu h^{\mu\nu}\right] + \mathcal{O}(h^2)\,. \quad (32)$$

Indices are raised and lowered with $\eta_{\mu\nu}$. This can be done around any fixed background space time metric.

Green's functions do not exist without a gauge choice and it is most convenient to use the so-called harmonic gauge where the Green functions obey Poisson-like equations

$$\partial^\lambda h_{\mu\lambda} = \tfrac{1}{2}\partial_\mu h^\lambda{}_\lambda\,. \quad (33)$$

The well-known field equations

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} = -8\pi G T_{\mu\nu}\,, \qquad \sqrt{g}T^{\mu\nu} \equiv -2\frac{\delta}{\delta g_{\mu\nu}}\left(\sqrt{g}\mathcal{L}_m\right) \quad (34)$$

reduce in this gauge to

$$\Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda\right). \quad (35)$$

The momentum space propagator is relatively simple in this gauge. Around Minkowski space-time we obtain

$$iD_{\mu\nu\alpha\beta} = \frac{i}{q^2 + i\varepsilon}P_{\mu\nu,\alpha\beta}\,, \qquad P_{\mu\nu,\alpha\beta} \equiv \tfrac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]. \quad (36)$$

In addition one needs to include the gauge-fixing and ghost part. Around an arbitrary background $\bar{g}_{\mu\nu}$

$$\mathcal{L}_{gf} = \sqrt{\bar{g}}\left\{\left(D^\nu h_{\mu\nu} - \tfrac{1}{2}D_\mu h^\lambda_\lambda\right)\left(D_\sigma h^{\mu\sigma} - \tfrac{1}{2}D^\mu h^\sigma{}_\sigma\right)\right\}, \quad (37)$$

$$\mathcal{L}_{gh} = \sqrt{\bar{g}}\eta^{*\mu}\left[D_\lambda D^\lambda \bar{g}_{\mu\nu} - \mathcal{R}_{\mu\nu}\right]\eta^\nu\,. \quad (38)$$

It is plain that perturbative calculations in quantum gravity are quite difficult due to the proliferation of indices.

3.2. Counterterms

The following two results are well known and often quoted. The first one is due to 't Hooft and Veltman, who computed the divergences in pure gravity at the one loop level [13]. Without making use of the equations of motion, the counterterms found by 't Hooft and Veltman in the harmonic gauge are

$$\mathcal{L}^{(\mathrm{div})}_{1\,\mathrm{loop}} = -\frac{1}{16\pi^2\varepsilon}\left\{\frac{1}{120}\mathcal{R}^2 + \frac{7}{20}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right\}. \tag{39}$$

The second one is due to Goroff and Sagnotti [14] who performed a similar calculation at two loops. After using the equations of motion

$$\mathcal{L}^{(\mathrm{div})}_{2\,\mathrm{loop}} = -\frac{209\kappa^2}{5760(16\pi^2)}\frac{1}{\varepsilon}\mathcal{R}^{\alpha\beta}{}_{\gamma\delta}\mathcal{R}^{\gamma\delta}{}_{\eta\sigma}\mathcal{R}^{\eta\sigma}{}_{\alpha\beta}\,. \tag{40}$$

It is less appreciated that the two results are on a different footing. The result of 't Hooft and Veltman is gauge dependent (it was computed in a particular gauge — the harmonic gauge — and it does not correspond to any physical observable, in particular the equations of motion have not been used). The counterterm actually vanishes when the field equations in empty space are used $\mathcal{R}_{\mu\nu} = 0$. The counterterm does give a net divergence when $T_{\mu\nu} \neq 0$ and, therefore $\mathcal{R}_{\mu\nu} \neq 0$, but the result is in principle incomplete as we will see below [15].

The one-loop counterterms computed by 't Hooft and Veltman, although historically quite relevant, are thus largely irrelevant from the point of view of effective Lagrangians because they vanish on shell.

In de Sitter space, described by the action

$$S = \frac{1}{16\pi G}\int dx\sqrt{-g}(\mathcal{R} - 2\Lambda) \tag{41}$$

the counterterm structure was computed by Christensen and Duff [16] in the 80s. A more detailed analysis was performed later in [17, 18], where the gauge dependence of the counterterms was clearly exposed

$$\Gamma^{(\mathrm{div})}_{\mathrm{eff}} = -\frac{1}{16\pi^2\varepsilon}\int dx\sqrt{-g}\left[c_1\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + c_2\Lambda^2 + c_3\mathcal{R}\Lambda + c_4\mathcal{R}^2\right]. \tag{42}$$

The constants c_i are actually gauge dependent and only a combination of them is gauge invariant.

If we are interested in observables, the on-shell condition is to be imposed on the counterterms of the effective theory (as in a derivative expansion they will appear only at tree-level, see *e.g.* [11] for a discussion on this).

Using the equations of motion (in absence of matter) $\mathcal{R}_{\mu\nu} = g_{\mu\nu}\Lambda$, the previous equation reduces to the (gauge-invariant) on-shell expression [18]

$$\Gamma_{\rm eff}^{(\rm div)} = \frac{1}{16\pi^2\varepsilon}\int dx\sqrt{-g}\frac{29}{5}\Lambda^2. \tag{43}$$

On the contrary, if we set $\Lambda = 0$ above, in (42), and particularize to the harmonic gauge, we reproduce the well-known 't Hooft and Veltman divergence (39).

Let us recapitulate. Exactly as the chiral Lagrangian, the Einstein–Hilbert action requires an infinite number of counterterms

$$\mathcal{L} = M_{\rm P}^2\sqrt{-g}\mathcal{R} + \alpha_1\sqrt{-g}\mathcal{R}^2 + \alpha_2\sqrt{-g}(\mathcal{R}_{\mu\nu})^2 + \alpha_3\sqrt{-g}(\mathcal{R}_{\mu\nu\alpha\beta})^2 + \ldots. \tag{44}$$

The divergences can be absorbed by redefining the coefficients just as done in the previous section for the pion effective Lagrangian

$$\alpha_i \to \alpha_i + \frac{c_i}{\varepsilon}\,. \tag{45}$$

Power counting in gravity appears, at least superficially, quite similar to the one that can be implemented in pion physics. Of course, the natural expansion parameter is a tiny number in normal circumstances, namely

$$\frac{p^2}{16\pi^2 M_{\rm P}^2} \quad \text{or} \quad \frac{\nabla^2}{16\pi^2 M_{\rm P}^2}\,, \quad \frac{\mathcal{R}}{16\pi^2 M_{\rm P}^2}\,. \tag{46}$$

Because of this, Donoghue has termed the effective action of gravity the most effective of all effective actions!

4. Why we need genuine loop effects and power counting

Consider the following *generic* $\mathcal{R}^2$ correction to the Einstein–Hilbert action

$$\mathcal{L} = \frac{2}{\kappa^2}\mathcal{R} + c\mathcal{R}^2 + \mathcal{L}_{\rm matter}\,. \tag{47}$$

The corresponding equation of motion for a perturbation around Minkowski is (recall that we write $g = \eta + h$)

$$\Box h + \kappa^2 c^2 \Box\Box h = 8\pi G T\,. \tag{48}$$

The Green function for this equation has the form

$$\begin{aligned} G(x) &= \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot x}}{q^2+\kappa^2 c q^4} \\ &= \int \frac{d^4q}{(2\pi)^4}\left[\frac{1}{q^2} - \frac{1}{q^2+1/\kappa^2 c}\right]e^{-iq\cdot x}. \end{aligned} \tag{49}$$

Taken at face these higher order terms would lead to a correction to Newton's law

$$V(r) = -Gm_1m_2\left[\frac{1}{r} - \frac{e^{-r/\sqrt{\kappa^2 c}}}{r}\right]. \tag{50}$$

Experimental bounds indicate $c < 10^{74}$; that is, no bound at all in practice. This is of course a consequence of the 'effectiveness' of the effective action of gravity. If c was a reasonable number there would be no effect on any observable physics at terrestrial scales. Note that if $c \sim 1, \sqrt{\kappa^2 c} \sim 10^{-35}m$. The curvature is so small that $\mathcal{R}^2$ terms are completely irrelevant at ordinary scales.

However using the full solution of the wave equation is *not* compatible with the effective Lagrangian philosophy and the power counting it embodies because higher orders in κ are sensitive to higher curvatures we have not considered.

The leading behaviour of the correction is

$$\frac{e^{-r/\sqrt{\kappa^2 c}}}{r} \to 4\pi\kappa^2 c\delta^3(\vec{r})\,. \tag{51}$$

In momentum space this translates into

$$\frac{1}{q^2 + \kappa^2 c q^4} = \frac{1}{q^2} - \kappa^2 c + \cdots\,. \tag{52}$$

Thus the 'correction' to Newton's law coming from the $\mathcal{R}^2$ correction is

$$V(r) = -Gm_1M_2\left[\frac{1}{r} + 128\pi^2 Gc\delta^3(\vec{x})\right], \tag{53}$$

which is totally unobservable, even as a matter of principle.

Of course, apart from the divergences, there are finite pieces (not universal, due to the renormalization ambiguities, choice of different substraction methods, *etc.*) and, most importantly, *non-local* pieces. Indeed in dimensional regularization we get at the one-loop level

$$\frac{1}{\varepsilon} + \log\frac{-p^2}{\mu^2} \tag{54}$$

or, in position space,

$$\frac{1}{\varepsilon} + \log\frac{\nabla^2}{\mu^2}\,, \tag{55}$$

where ∇ has to be the covariant derivative on symmetry grounds, ∇^2 reducing to $-p^2$ in flat space-time. These non-localities are due to the propagation of strictly massless non-conformal modes, such as the graviton itself. Therefore they are unavoidable in quantum gravity. Notice that the coefficient is predictable; it depends entirely on the infrared properties of gravity.

5. Quantum corrections to Newton law

Let us use the 'chiral counting' arguments to derive the relevant quantum corrections to Newton's law (up to a constant). The propagator at tree level, that we symbolically write as

$$\frac{1}{p^2}\,, \tag{56}$$

gets modified by the one-loop 'chiral-like' corrections to

$$\frac{1}{p^2}\left(1 + A\frac{p^2}{M_{\rm P}^2} + B\frac{p^2}{M_{\rm P}^2}\log p^2\right)\,. \tag{57}$$

Of course the last expression is also symbolic.

Consider now the interaction of a point-like particle with an static source ($p^0 = 0$) and let us Fourier transform the previous expression for the loop-corrected propagator in order to get the potential in the non-relativistic limit. We use

$$\int d^3x \exp(i\vec{p}\,\vec{x})\,\frac{1}{p^2} \sim \frac{1}{r}\,, \qquad \int d^3x \exp(i\vec{p}\,\vec{x})\,1 \sim \delta(\vec{x})\,, \tag{58}$$

$$\int d^3x \exp(i\vec{p}\,\vec{x})\,\log p^2 \sim \frac{1}{r^3}\,. \tag{59}$$

Thus the quantum corrections to Newton's law are of the form

$$\frac{GMm}{r}\left(1 + C\frac{G\hbar}{r^2} + \dots\right)\,. \tag{60}$$

We have restored for a moment $\hbar$. Let us check dimensions. We note that

$$\left[\frac{Gm}{c^2}\right] = L\,, \qquad \left[\frac{G\hbar}{c^3}\right] = L^2 \tag{61}$$

so C is a pure number. In addition there are post-Newtonian (but classical) corrections that are not discussed here.

A long controversy regarding the value of C exists in the literature. Donoghue, Muzinich, Vokos, Hamber, Liu, Bellucci, Khriplovich, Kirilin, Holstein, Bjerrum-Bohr and others have contributed [3, 19, 20] to the determination of C. The result widely accepted as the correct one [2] is obtained by considering the inclusion of *quantum* matter fields (a scalar field actually) and considering all types of loops.

The relevant set of Feynman rules is

$$\tau_{\mu\nu} = -\frac{i\kappa}{2}\left(p_\mu p'_\nu + p'_\mu p_\nu - g_{\mu\nu}\left[p\cdot p' - m^2\right]\right) , \tag{62}$$

$$\begin{aligned}\tau_{\eta\lambda,\rho\sigma} = \frac{i\kappa^2}{2}\Big\{ & I_{\eta\lambda,\alpha\delta}I^{\delta}{}_{\beta,\rho\sigma}\left(p^\alpha p'^\beta + p'^\alpha p^\beta\right) \\ & -\frac{1}{2}\left(\eta_{\eta\lambda}I_{\rho\sigma,\alpha\beta} + \eta_{\rho\sigma}I_{\eta\lambda,\alpha\beta}\right)p'^\alpha p^\beta \\ & -\frac{1}{2}\left(I_{\eta\lambda,\rho\sigma} - \frac{1}{2}\eta_{\eta\lambda}\eta_{\rho\sigma}\right)\left[p\cdot p' - m^2\right]\Big\} ,\end{aligned} \tag{63}$$

with

$$I_{\mu\nu,\alpha\beta} \equiv \frac{1}{2}[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}] . \tag{64}$$

The first Feynman rule corresponds to a matter–matter–1-graviton vertex, while the second one describes the matter–matter–2-graviton interaction. Actually the interaction with matter always takes place via the energy-momentum tensor. Note that (quantum) matter does propagate inside loops. Please note that very heavy (matter) degrees of freedom do not necessarily decouple from quantum corrections as the coupling itself to gravity depends on the mass.

In addition one needs the 3-graviton interaction vertex which is described by quite a lengthy expression and shall not be given here. It can be found in [1].

Then, in a rather informal but otherwise obvious notation, the calculation of the local counterterms gives [3]

$$\mathcal{L}_{\mathcal{R}\mathcal{R}} = \frac{1}{3849\pi^3 r^3}(42\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2) , \tag{65}$$

$$\mathcal{L}_{\mathcal{R}T} = -\frac{\kappa}{8\pi^2 r^3}(3\mathcal{R}_{\mu\nu}T^{\mu\nu} - 2\mathcal{R}T) , \tag{66}$$

$$\mathcal{L}_{TT} = \frac{\kappa^2}{60\pi r^3}T^2 . \tag{67}$$

At this point one can make use of the lowest order equations of motion to simplify the counterterm structure

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -8\pi G T_{\mu\nu} \tag{68}$$

$$\Rightarrow \quad \mathcal{L}_{\text{total}} = -\frac{\kappa^2}{60\pi r^3}(138T_{\mu\nu}T^{\mu\nu} - 31T^2) . \tag{69}$$

Particularizing now to the case of a point-like mass, we get the final result for C, which is positive in sign: gravity is more attractive at long distances than predicted by Newton's law (although the difference is of course extremely tiny)

$$C = \frac{41}{10\pi} \,. \tag{70}$$

What happens for *classical* matter, *e.g.* a cloud of dust, is in our view still an open problem.

There are in the literature definitions of an "effective" or "running" Newton constant [21,22]. A class of diagrams is identified that dresses up G and turns it into a distance (or energy)-dependent constant $G(r)$. Unfortunately it is not clear to us that these definitions are gauge invariant; only physical observables (such as a scattering matrix) are guaranteed to be. So caution should be adopted here, although the renormalization-group analysis derived from this "running" coupling constant are, of course, very interesting.

5.1. Power counting in gravity

Let us try to establish a counting analogous to the one we did for the pion chiral Lagrangian. Some of the counting rules are obvious, others require a little thought. Let us indicate them, again symbolically

— 3-graviton coupling: $\sim \kappa q^2$;

— 4-graviton coupling: $\sim \kappa^2 q^2$;

— (On-shell) matter– 1-graviton coupling: $\sim \kappa m^2$;

— (On-shell) matter– 2-graviton coupling: $\sim \kappa^2 m^2$;

— Graviton propagator: $\sim \frac{1}{q^2}$;

— Matter propagator $\sim \frac{1}{q^2 - m^2}$.

If we iterate, for example, the 4-graviton vertex to produce a one loop diagram we shall obtain (p_i are external momenta and $q = p_1 + p_2$)

$$\mathcal{M}_{\mathrm{loop}} \sim \kappa^4 \int \frac{d^4 l}{(2\pi)^4} \frac{(l - p_1)^2 (l - p_2^2)^2}{l^2 (l - q)^2} \,. \tag{71}$$

If this loop integral is regularized dimensionally, which does not introduce powers of any new scale, the integral will be represented in terms of the exchanged momentum to the appropriate power. Thus we have

$$\mathcal{M}_{\mathrm{loop}} \sim \kappa^4 \, q^4 \,. \tag{72}$$

When matter fields are included in loops the situation is more subtle, in particular for large masses in the non-relativistic limit. Let us see why. If we compute the tree level result for matter–matter scattering the result is

$$\mathcal{M}_{\text{tree}} = \kappa^2 \frac{m_1^2 m_2^2}{q^2} \,. \tag{73}$$

Note that this is not yet the potential, hence the unfamiliar power of the masses in the numerator. Iterating this expression to form a loop one encounters internal lines where a matter field propagates. This propagator has a denominator of the form $(k-q)^2 - m^2$ that on shell and for large masses in the non-relativistic limit will behave as mq. Therefore one gets

$$\mathcal{M}_{\text{loop}} \sim \kappa^4 m_1^4 m_2^4 \int d^4l \frac{1}{m_1(l+p)} \times \frac{1}{m_2(l+p')} \times \frac{1}{(l+q')^2} \times \frac{1}{(l+q)^2} \tag{74}$$

which by the same reasoning as before is

$$\mathcal{M}_{\text{loop}} \sim \kappa^4 \frac{m_1^3 m_2^3}{q^2} \sim \kappa^2 \frac{m_1^2 m_2^2}{q^2} \times \kappa^2 m_1 m_2 \,. \tag{75}$$

Here the expansion parameter appears to be $\kappa^2 m^2$ that does not seem compatible with the 'chiral' expansion arguments.

This issue has been studied by some detail by Donoghue and Torma [23] who concluded that

$$\mathcal{M}_{(N_{\text{E}}^m, N_{\text{E}}^g)} \sim q^D \,, \tag{76}$$

where

$$D = 2 - \frac{N_{\text{E}}^m}{2} + 2N_{\text{L}} - N_{\text{V}}^m + \sum_n (n-2) N_{\text{V}}^g[n] + \sum_l l \cdot N_{\text{V}}^m[l] \,, \tag{77}$$

being N_{E}, N_{L} and N_{V} the number of external fields, loops and vertices, respectively, and the superindex referring to whether they are matter or gravity fields. If we disregard matter vertices this is identical to Weinberg's result for chiral theories [10], who concluded that the power counting expansion is sound for the pion effective Lagrangian.

However the negative N_{V}^m term appearing in D is potentially dangerous. Although no general proof exists yet, Donoghue has been able to prove cancellation of the dangerous terms at the one-loop level except for the terms leading to $1/r$ corrections (classical, non-linear). The issue is, to our knowledge, still not fully solved.

We conclude with a final comment concerning the use of the equations of motion. In chiral Lagrangians they allow us to get rid of redundant operators. For instance, taking into account that from the lowest order Lagrangian results the following Euler–Lagrange equation

$$U\Box U^\dagger - (\Box U)U^\dagger = 0 \tag{78}$$

we can set, at the next order in the chiral expansion,

$$\mathrm{Tr}\, U\Box U^\dagger \to 0\,. \tag{79}$$

However, note that in gravity, the equation of motion mixes terms of different 'chiral' order

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} = -8\pi G T_{\mu\nu} - g_{\mu\nu}\Lambda\,. \tag{80}$$

For instance, it is incorrect to use

$$\mathcal{R}_{\mu\nu} = g_{\mu\nu}\Lambda \tag{81}$$

in 't Hooft and Veltman calculation, even if Λ is generated by the v.e.v. of some scalar field (as long as is spatially constant and does not vary with time) which is induced by some (dimension four) matter sector. It just does not reproduce the de Sitter result.

6. Cosmological implications

The quantum corrections to Newton's law emerge from the universal non-local corrections to the effective action. They constitute a direct test of the quantum nature of gravitation, putting this theory on an equal footing to other quantum field theories. They are thus conceptually extremely important, but it is hard to imagine how one could measure such a tiny effect. Can these non-local quantum corrections be relevant, or at least observable, in a cosmological setting?

We are concerned here about universal non-local quantum corrections to the Einstein–Hilbert Lagrangian that take the form (again symbolically)

$$\frac{1}{16\pi^2 M_{\mathrm{P}}^2}\mathcal{R}\left[\log\nabla^2\right]\mathcal{R}\,. \tag{82}$$

There are two reasons why such apparently hopelessly small corrections might be relevant in a cosmological setting

— Curvature was much larger at early stages of the universe: in a de Sitter universe $\mathcal{R}\sim H^2$, $H^2 = 8\pi G V_0/3$, $H \le 10^{13}$ GeV (present value is 10^{-42} GeV).

— Logarithmic non local term corresponds to an interaction between geometries that is long-range in time, an effect that does not have an easy classical interpretation.

Please note that the above non-local contributions are totally unrelated to the so-called $f(\mathcal{R})$ models. They are present and unambiguously calculable in the quantum theory. It should be mentioned here too that somewhat related non-localities (but at the two loop level) were studied by Tsamis and Woodard long ago [24]. They turn out to slow down the rate of inflation.

For the purpose of the present discussion let us spell out our conventions

$$S = \frac{1}{16\pi G}\int dx\sqrt{-g}(\mathcal{R}-2\Lambda)+S_{\text{matter}}\,, \quad \mathcal{R}_{\mu\nu}-\tfrac{1}{2}\mathcal{R}g_{\mu\nu} = -8\pi G T_{\mu\nu}-\Lambda g_{\mu\nu}\,. \tag{83}$$

Quantum corrections to the Einstein–Hilbert action were originally computed by 't Hooft and Veltman in the case of vanishing cosmological constant [13], and by Chistensen and Duff for a de Sitter background [16]. The key ingredient we shall need is the divergent part of the one-loop effective action. Setting $d = 4 + 2\varepsilon$

$$\Gamma_{\text{eff}}^{\text{div}} = -\frac{1}{16\pi^2\varepsilon}\int dx\sqrt{-g}\left[c_1\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + c_2\Lambda^2 + c_3\mathcal{R}\Lambda + c_4\mathcal{R}^2\right]. \tag{84}$$

The constants c_i are actually gauge dependent as has already been mentioned and only a combination of them is gauge invariant. This is clearly discussed in [17, 18].

Using the equations of motion (in absence of matter) $\mathcal{R}_{\mu\nu} = g_{\mu\nu}\Lambda$, the previous equation reduces to the (gauge-invariant) on-shell expression

$$\Gamma_{\text{eff}}^{\text{div}} = \frac{1}{16\pi^2\varepsilon}\int dx\sqrt{-g}\,\frac{29}{5}\Lambda^2\,. \tag{85}$$

If we set $\Lambda = 0$ above, we get the well-known 't Hooft and Veltman divergence, that in the so-called minimal gauge is

$$\Gamma_{\text{eff}}^{\text{div}} = -\frac{1}{16\pi^2\varepsilon}\int dx\sqrt{-g}\left[\frac{7}{20}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{120}\mathcal{R}^2\right]. \tag{86}$$

If the equations of motion are used in the absence of matter this divergence is absent.

Let us now try to investigate to what extent the non-local quantum corrections to the effective action, represented by (82) can modify the evolution of the cosmological scale factor in a Friedman–Robertson–Walker universe.

In what follows we summarize the results presented in [5,6]. For the sake of discussion, we shall begin by considering here a simplified effective action that includes only terms containing the scalar

$$S = \kappa^2 \left(\int dx \sqrt{-g} \mathcal{R} + \tilde{\alpha} \int dx \sqrt{-g} \mathcal{R} \ln \left(\nabla^2/\mu^2 \right) \mathcal{R} + \tilde{\beta} \int dx \sqrt{-g} \mathcal{R}^2 \right)$$
$$\equiv \kappa^2 \left(S_1 + \tilde{\alpha} S_2 + \tilde{\beta} S_3 \right), \tag{87}$$

where $\kappa^2 = M_{\rm P}^2/16\pi = 1/16\pi G$ and μ is the subtraction scale. The coupling $\tilde{\beta}$ is μ dependent in such a way that the total action S is μ-independent.

Note that

— The value of $\tilde{\beta}$ is actually dependent on the UV structure of the theory (it contains information on all the modes — massive or not — that have been integrated out).

— The value of $\tilde{\alpha}$ is unambiguous: it depends only on the IR structure of gravity (described by the Einstein–Hilbert Lagrangian) and the massless (nonconformal) modes.

In conformal time

$$g_{\mu\nu} = a^2(\tau) \eta_{\mu\nu} , \qquad \mathcal{R} = 6 \frac{a''(\tau)}{a^3(\tau)} , \qquad \sqrt{-g} = a^4(\tau) . \tag{88}$$

We first obtain the variation of the local part

$$\frac{\delta S_1}{\delta a(\tau)} = 12 a'', \quad \frac{\delta S_3}{\delta a(\tau)} = 72 \left(-3 \frac{(a'')^2}{a^3} - 4 \frac{a' a'''}{a^3} + 6 \frac{(a')^2 a''}{a^4} + \frac{a^{(4)}}{a^2} \right) . \tag{89}$$

In order to obtain the variation of the non-local (logarithmic piece) we need to compute

$$\langle x | \log \nabla^2 | y \rangle , \tag{90}$$

where in conformal coordinates

$$\nabla^2 = a^{-3} \Box \, a + \tfrac{1}{6} \mathcal{R} . \tag{91}$$

To the order we are computing we can neglect the $\mathcal{R}$ term in the previous equation and commute the scale factor a with the flat d'Alembertian

$$\nabla^2 = \left(\frac{a}{a_0} \right)^{-2} \Box , \tag{92}$$

where $a_0 = a(0)$. With this rescaling (absorbable in $\tilde{\beta}$), at $\tau = 0$ the d'Alembertian in conformal space matches with the Minkowskian one.

We can now separate S_2 in turn into a local and a genuinely non-local piece

$$S_2 = \int dx\sqrt{-g}\ \left(-2\mathcal{R}\ln(a)\mathcal{R} + \mathcal{R}\ln(\Box/\mu^2)\mathcal{R}\right) \equiv S_2^{\mathrm{I}} + S_2^{\mathrm{II}}. \tag{93}$$

$$\begin{aligned}\frac{\delta S_2^{\mathrm{I}}}{\delta a(\tau)} =& -72\left\{\frac{(a')^2a''}{a^4}\left[12\ln a - 10\right]\right. \\ &\left.+\frac{a'a'''}{a^3}\left[-8\ln a + 4\right] + \frac{(a'')^2}{a^3}\left[-6\ln a + 2\right] + \frac{a^{(4)}}{a^2}2\ln a\right\}.\end{aligned} \tag{94}$$

Finally we have to compute

$$\langle x|\ln\Box|y\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\langle x|\Box^{\varepsilon}|y\rangle - \frac{1}{\varepsilon}\langle x|y\rangle\,. \tag{95}$$

The (covariant) delta function is in one-to-one correspondence with the counterterm. The Green's function we are interested will be

$$\sim \frac{1}{|x-y|^{4+2\varepsilon}}\,. \tag{96}$$

After integration of $\vec{x} - \vec{y}$ we get

$$\sim \frac{1}{|t-t'|^{1+2\varepsilon}}\,. \tag{97}$$

So

$$S_2^{\mathrm{II}} = 36\int d\tau\frac{a''(\tau)}{a(\tau)}\int_0^{\tau} d\tau'\frac{1}{\tau-\tau'}\frac{a''(\tau')}{a(\tau')}\,. \tag{98}$$

Note the limits of integration ensuring causality. Technically speaking we are using here the in–in effective action and not the in–out one that would be appropriate for a scattering process.

The variation of S_2^{II} is

$$\begin{aligned}\frac{\delta S_2^{\mathrm{II}}}{\delta a(\tau)} =\ & 36\left\{\left[2a^{-3}(\tau)\left(a'(\tau)\right)^2 - 2a^{-2}(\tau)a''(\tau)\right]\int_0^{\tau} d\tau'\frac{1}{\tau-\tau'}\frac{a''(\tau')}{a(\tau')}\right. \\ &-2a^{-2}(\tau)a'(\tau)\frac{\partial}{\partial\tau}\left(\int_0^{\tau} d\tau'\frac{1}{\tau-\tau'}\frac{a''(\tau')}{a(\tau')}\right) \\ &\left.+a^{-1}(\tau)\frac{\partial^2}{\partial\tau^2}\left(\int_0^{\tau} d\tau'\frac{1}{\tau-\tau'}\frac{a''(\tau')}{a(\tau')}\right)\right\}.\end{aligned} \tag{99}$$

In the spirit of effective Lagrangians we would obtain first the lowest order equation of motion from S_1 and plug it in $\tilde{\alpha}(S_2^{\mathrm{I}}+S_2^{\mathrm{II}})+\tilde{\beta}S_3$. As can be seen by inspection, quantum corrections act as an external driving force superimposed to Einstein equations.

In a FRW universe without matter and with zero cosmological constant the non-local pieces are actually zero (*i.e.* there are no log terms) when one considers physical observables and the equations of motion are used. Therefore the toy model we have considered is not realistic, but it has served us to develop our tools.

Let us now move to the more physically relevant case of a de Sitter universe. The relevant one-loop corrected effective action is

$$\begin{aligned} S &= \frac{1}{16\pi G}\int dx\sqrt{-g}(\mathcal{R}-2\Lambda)+\frac{1}{16\pi^2}\int dx\sqrt{-g}\frac{29}{5}\Lambda\ln\frac{\nabla^2}{\mu^2}\Lambda \\ &\quad + \text{local terms of} \quad \mathcal{O}(p^4)\,. \end{aligned} \tag{100}$$

We write S as

$$S\equiv\kappa^2\left(\int dx\sqrt{-g}(\mathcal{R}-2\Lambda)+\tilde{\alpha}S_2\right), \tag{101}$$

with

$$\tilde{\alpha}=\frac{G}{\pi}\times\frac{29}{5}\,. \tag{102}$$

We split S_2 in two parts

$$S_2^{\mathrm{I}}=-2\int dx\sqrt{-g}\Lambda^2\ln(a)\,,\qquad S_2^{\mathrm{II}}=\int dx\sqrt{-g}\Lambda\ln(\Box/\mu^2)\Lambda\,, \tag{103}$$

and obtain the corresponding variations following the method outlined previously

$$\frac{\delta S_2^{\mathrm{I}}}{\delta a(\tau)} = -2\Lambda^2a^3(\tau)\left[4\ln(a(\tau))+1\right], \tag{104}$$

$$\frac{\delta S_2^{\mathrm{II}}}{\delta a(\tau)} = 2\Lambda^2a(\tau)\int_0^\tau d\tau' a^2(\tau')\frac{\mu^{-2\varepsilon}}{|\tau-\tau'|^{1+2\varepsilon}}\,. \tag{105}$$

The equation of motion will be

$$12a''(\tau)-8\Lambda a^3(\tau)+\tilde{\alpha}\frac{\delta S_2}{\delta a(\tau)}=0 \tag{106}$$

which at lowest order is just

$$12a''(\tau)-24H^2a^3(\tau)=0\,,\qquad H^2=\frac{\Lambda}{3}\,. \tag{107}$$

The lowest order solution (with $a(0) = 1$) is

$$a_{\mathrm{I}}(\tau) = \frac{1}{1 - H\tau}\,. \tag{108}$$

The final step is to plug the 0-th order solution $a_{\mathrm{I}}(\tau)$ into the variation of S_2 and recalculate the solution for $a(\tau)$. Note that we use a perturbative procedure is of course only valid as long as the correction is small compared to the unperturbed solutions.

We introduce a variable s defined $a_{\mathrm{I}}(\tau) = e^s$. Then s counts the number of e-folds

$$\frac{\delta S_2^{\mathrm{I}}}{\delta a(\tau)} = -2\Lambda^2 e^{3s}\left[4s+1\right], \qquad \frac{\delta S_2^{\mathrm{II}}}{\delta a(\tau)} = 2\Lambda^2 e^s I(s) \tag{109}$$

and the equation of motion reads

$$a''(s) + a'(s) - 2e^{-2s}a^3(s) = \tfrac{3}{2}\tilde{\alpha}H^2\left(-e^s(1+4s) + e^{-s}I(s)\right), \tag{110}$$

where I is

$$I(s) = \ln\left(\frac{\mu}{H}(1-e^{-s})\right)e^{2s} + e^s(1 - e^s - se^s)\,, \tag{111}$$

and the equation to solve is

$$a''(s) + a'(s) - 2e^{-2s}a^3(s) = \tfrac{3}{2}\tilde{\alpha}H^2\left[-(5s+2)e^{ss} + 1 + e^s\ln\left(\tfrac{\mu}{H}(1-e^{-s})\right)\right]. \tag{112}$$

Note that $\tilde{\alpha}$ appears only in the combination $\tilde{\alpha}H^2$. Since there are H large uncertainties in H in practice only the sign of $\tilde{\alpha}$ is relevant. In addition, there is some ambiguity associated to the choice of the renormalization scale that appears in the combination $\ln(\mu/H)$. This is shown in figure 2.

Let us now assume that $a = a(\tau, \vec{x})$; *i.e.* we allow for some space inhomogeneities. Then

$$\frac{\delta S_2^{\mathrm{II}}}{\delta a(\tau,\vec{x})} \sim \Lambda^2 a(\tau,\vec{x})\int_0^\tau d\tau' d^3\vec{y}\, a^2(\tau',\vec{y})\frac{\mu^{-2\varepsilon}}{|x-y|^{4+2\varepsilon}}\,. \tag{113}$$

This corresponds to new correlations of a quantum nature between different points. The consequences of this term have not been fully investigated yet.

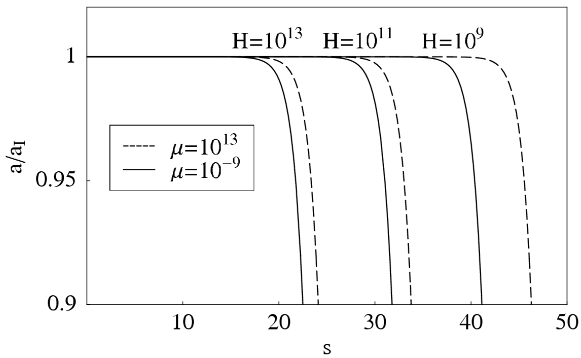

Fig. 2. The scale factor relative to the inflationary expansion for different values of μ and H (all units are GeV). We can see that the curves present a very similar behaviour for the different values shown, though a higher value of H leads earlier to deviations from the usual inflationary expansion. Higher values of μ also have this effect, which is larger as H increases. In fact, if we considered values of μ/H large enough (but not relevant physically), the logarithm term would become dominant and the deviation would be positive.

7. Gravity as a Goldstone phenomenon

We have given in the previous sections arguments why the Einstein–Hilbert action could be viewed as the most relevant term, in the sense of the renormalization-group, of an effective theory.

Let us review them:

— Dimensionful coupling constant ($M_{\mathrm{P}} \sim f_\pi$).

— Derivative couplings ($\sqrt{-g}\mathcal{R} \sim g\partial\partial g$).

— Choice of action based on RG criteria of relevance, not on renormalizability (unlike Yang–Mills).

— Power counting analogous to ChPT.

— Massless quanta ($\pi \leftrightarrow g_{\mu\nu}$).

— Existence of a global symmetry to be broken (see below).

Here we want to pursue this line of thought further. As an entertainment, without making any particularly strong claim of relevance, we shall investigate a formulation inspired as much as possible in the chiral symmetry breaking of QCD. It has the following characteristics:

— No *a priori* metric, only affine connection is needed (parallelism).
— Lagrangian is manifestly independent of the metric.
— Breaking is triggered by a fermion condensate.

A different model along these lines was considered some time ago by Russo and coworkers [8].

We seek inspiration in the effective Lagrangians of QCD at long distances. A successful model for QCD is the so-called chiral quark model. Consider the matter part Lagrangian of QCD with massless quarks (2 flavours)

$$\mathcal{L} = i\bar{\psi}\ \partial\!\!\!/ \psi = i\bar{\psi}_{\mathrm{L}}\ \partial\!\!\!/ \psi_{\mathrm{L}} + i\bar{\psi}_{\mathrm{R}}\ \partial\!\!\!/ \psi_{\mathrm{R}}\,. \tag{114}$$

This theory has a global SU(2) $\times$ SU(2) symmetry that forbids a mass term M.

However after chiral symmetry breaking pions appear and they must be included in the effective theory. Then it is possible to add the following term

$$-M\bar{\psi}_{\mathrm{L}}U\psi_{\mathrm{R}} - M\bar{\psi}_{\mathrm{R}}U^{\dagger}\psi_{\mathrm{L}}\,, \tag{115}$$

that is invariant under the full global symmetry

$$\psi_{\mathrm{L}} \to L\psi_{\mathrm{L}}\,, \qquad \psi_{\mathrm{R}} \to R\psi_{\mathrm{R}}\,, \qquad U \to LUR^{\dagger}\,. \tag{116}$$

Chiral symmetry breaking is also characterized by the presence of a fermion condensate

$$\langle \bar{\psi}\psi \rangle \neq 0\,. \tag{117}$$

In order to determine whether the condensate is zero or not one is to solve a 'gap'-like equation in some modelization of QCD, or on the lattice. The final step is to integrate out the fermions using the self-generated effective mass as an infrared regulator. This reproduces the chiral effective Lagrangian discussed in the beginning of the lectures, although the low-energy constants α_i obtained in this way are not necessarily the real ones, as the chiral quark model is only a simplification of QCD and not the real thing.

There is only one possible term bilinear in fermions that is invariant under Lorentz $\times$ Diff

$$\bar{\psi}_a\gamma^a\nabla_\mu\psi^\mu\,. \tag{118}$$

To define ∇ we only need an affine connection

$$\nabla_\mu\psi^\mu = \partial_\mu\psi^\mu + i\omega_\mu^{ab}\sigma_{ab}\psi^\mu + \Gamma^\nu_{\mu\nu}\psi^\mu\,. \tag{119}$$

Note that no metric is needed at all to define the action if we assume that ψ^μ behaves as a contravariant spinorial vector density under Diff. Then, $\Gamma^\mu_{\nu\rho}$ does not enter, only the spin connection. If we keep this spin connection fixed, *i.e.* we do not consider it to be a dynamical field for the time being, there is no invariance under general coordinate transformations, but only under the global group $\mathrm{SO}(d) \times \mathrm{GL}(d)$ (assuming an Euclidean signature)[1].

Eventually we would like to find a non trivial condensate such as

$$\left\langle \bar{\psi}_a \psi^\mu \right\rangle \sim e_a^\mu \,. \tag{120}$$

In the absence of the (so far) external connection, we expect a constant value for e_a^μ (note that the constant of proportionality has dimensions of mass if we take e_a^μ to be dimensionless). It is of course irrelevant in which direction it points; all the vacua will be equivalent. If the condensate appears one can always choose $e_a^\mu = \delta_a^\mu$ without loss of generality. We shall interpret e_a^μ as the (inverse) n-bein. Note that once a dynamical value for e_a^μ is generated we can write terms such as $M\bar{\psi}_a e_\mu^a \psi^\mu$, where e_μ^a (the n-bein) is defined by $e_\mu^a e_b^\mu = \delta_b^a$. Of course one can introduce quantities such as $g^{\mu\nu} = e_a^\mu e_b^\nu \delta_{ab}$ and its inverse $g_{\mu\nu}$ defined by $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$.

Note that a large number of Goldstone bosons are produced. The original symmetry group $\mathrm{G} = \mathrm{SO}(d) \times \mathrm{GL}(d)$ has $\frac{d(d-1)}{2} + d^2$ generators. After the breaking $\mathrm{G} \to \mathrm{H}$, with $\mathrm{H} = \mathrm{SO}(d)$, which has a total of $\frac{d(d-1)}{2}$ generators, leaving d^2 broken generators, as expected. It remains to be seen how many of those actually couple to physical states.

In order to trigger the appearance of a vacuum expectation value we have to include some dynamics to induce the symmetry breaking. The model we propose is to add the interaction piece

$$S_{\mathrm{I}} = \int d^4x ((\bar{\psi}_a \psi^\mu + \bar{\psi}^\mu \psi_a) B_\mu^a + c \det(B_\mu^a)) \,. \tag{121}$$

Note that the interaction term also behaves as a density thanks to the covariant Levi–Civita symbol hidden in the determinant of B_μ^a. If we consider the equation of motion for the auxiliary field B_μ^a we get

$$\left\langle \bar{\psi}_a \psi^\mu \right\rangle = 2c \varepsilon^{\mu\nu} \varepsilon_{ab} B_\nu^b \,. \tag{122}$$

So the vacuum expectation value of the field B would correspond to the value of the n-bein, up to a (dimensional) constant.

In what follows we shall consider the above model for $D = 2$ for simplicity. Note the peculiar 'free' kinetic term $\gamma^a \otimes k_\mu$. We write explicitly in two dimensions the bilinear operator acting on the fermion fields. Note that

[1] We recommend the reader to follow the discussion presented by Percacci [25] in these proceedings.

indices $a, b, \ldots$ can be raised and lowered freely in Euclidean space.

$$M = \begin{pmatrix} B_{11} & k_1 & B_{12} & k_2 \\ k_1 & B_{11} & k_2 & B_{12} \\ B_{21} & -ik_1 & B_{22} & -ik_2 \\ ik_1 & B_{21} & ik_2 & B_{22} \end{pmatrix} \tag{123}$$

and we also define

$$\varDelta^{ab} \equiv MM^\dagger \equiv \sum_\mu iD_\mu^a \cdot iD_\mu^b \,, \tag{124}$$

where

$$D_\mu^a = \gamma^a(\partial_\mu + iw_\mu\sigma_3) - iB_\mu^a \,. \tag{125}$$

We want to compute the effective action after integration of the fermion degrees of freedom using the heat kernel method. Then

$$W = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}\,\langle x|e^{-t\varDelta}|x\rangle \,, \tag{126}$$

$$\langle x|e^{-t\varDelta}|x\rangle = \frac{1}{t^{D/2}}\int \frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left[e^{-k^2\gamma^a\gamma^b + i\sqrt{t}(\gamma^a D_\mu^b k_\mu + D_\mu^a k_\mu \gamma^b) + tD_\mu^a D_\mu^b}\right] \,, \tag{127}$$

where $\varDelta$ has been defined above. Note that the exponent is a matrix in Lorentz and Dirac indices (the latter not explicitly written). Once we know $W(w, B)$ we can differentiate with respect B_μ^a and obtain the relation between the 'n-bein' and the spin connection using a logic similar to the one defined by the Palatini formalism [26].

Note that

$$e^{-k^2\gamma^a\gamma^b} = \delta^{ab} - \frac{1}{D}\gamma^a\gamma^b + \frac{1}{D}\gamma^a\gamma^b e^{-Dk^2} \equiv P^{ab} + \frac{1}{D}\gamma^a\gamma^b e^{-Dk^2} \,. \tag{128}$$

Thus the exponential, considered as a matrix, has zero modes and therefore the heat kernel calculation is non-standard and quite laborious.

Here we shall limit ourselves to the case where there is no connection at all and then indicate how one could proceed beyond that (rather trivial) limit, to include a non-zero spin connection. We refer the interested reader to [4] for more details.

If $w = 0$ then one can use homogeneity and isotropy arguments to look for constant solutions of the gap equation associated to the following effective potential

$$V_{\mathrm{eff}} = c\det(B_\mu^a) + 2\int \frac{d^nk}{(2\pi)^n}\mathrm{Tr}\,(\log(-\gamma^a k_\mu + B_\mu^a)) \,. \tag{129}$$

The extremum of $V_{\rm eff}$ are found from

$$c n \varepsilon_{a a_2 \ldots . a_n} \varepsilon^{\mu \mu_2 \ldots . \mu_n} B^{a_2}_{\mu_2} \ldots . B^{a_n}_{\mu_n} + 2 \mathrm{Tr} \int \frac{d^n k}{(2\pi)^n} (-\gamma \otimes k + B)^{-1} \Big|^{\mu}_{a} = 0 \,. \tag{130}$$

Notice that the equations are invariant under the permutation

$$B_{ij} \to B_{\sigma(i)\sigma(j)} \,, \qquad k_i \to k_{\sigma(i)} \,, \qquad \sigma \varepsilon S_2 \,. \tag{131}$$

The 'gap equation' to solve for constant values of B_{ij} is

$$c B_{ij} - \frac{1}{16\pi} B_{ij} \log \frac{\det B}{\mu} = 0 \,. \tag{132}$$

A logarithmic divergence has been absorbed in c. This equation has a non-trivial solution that we can always choose, as indicated before, to be $B^a_\mu \sim \delta^\mu_a$.

The next step is to consider $w_\mu(x) \neq 0$. It is technically convenient to consider the heat kernel for the operator $M^\dagger M$ rather than $M M^\dagger$, although of course the determinants are identical. It is also important to maintain a covariant appearance as long as possible (note that there is no 'metric' so far and no way of lowering or raising indices). The final result has to be of course covariant, since our starting point is.

In conclusion, this leads us to the evaluation of the effective action

$$W = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \mathrm{Tr} \left\langle x | e^{-t\Delta} | x \right\rangle \,, \tag{133}$$

where now

$$\Delta \equiv \mathcal{M}^\dagger \mathcal{M} \,, \tag{134}$$

with

$$\mathcal{M} = i \mathcal{D}^b_\mu \,, \qquad \mathcal{M}^\dagger = i \mathcal{D}_{\nu b} \tag{135}$$

and

$$\begin{aligned} \mathcal{D}^b_\mu &= \xi^{\dagger\ b}_{La} \gamma^a (\partial_\rho + i w_\rho \sigma_3) \xi^\rho_{R\ \mu} - i \bar{B}^b_\mu \,, \\ \mathcal{D}_{\nu b} &= \xi^{\dagger\ \sigma}_{R\nu} (\partial_\sigma + i w_\sigma \sigma_3) \gamma_a \xi^a_{L\ b} - i \bar{B}_{\nu b} \,. \end{aligned} \tag{136}$$

Δ now has coordinate (and Dirac) indices. In the previous expressions we have decomposed

$$B^a_\mu = \xi^a_{L\ b} \bar{B}^b_\nu \xi^{-1\nu}_{R\ \mu} \,; \qquad \bar{B}^b_\nu = \xi^{\dagger\ b}_{La} B^a_\mu \xi^\mu_{R\ \nu} \,; \qquad \bar{B}_{\nu b} = \xi^{\dagger\ \mu}_{R\nu} B_{\mu a} \xi^a_{L\ b} \,, \tag{137}$$

where $\bar{B}^b_\mu = M\delta^b_\mu$ is the background which we can take the role of a mass term in the integration over t in the heat kernel. Note that we have redefined the fermion fields to absorb the matrices $\xi_{\rm L}$ and $\xi_{\rm R}$.

This way of doing things ensures the formal covariance of the heat kernel expansion. It is not too difficult to see that the lowest non-trivial order gives

$$W = \frac{\mu^2 e^{\tilde{c}}}{16\pi} \int d^2x \sqrt{\mathrm{Det}[(\xi^\sigma_{R\,\mu}\xi^{\dagger\rho}_{R\,\mu})^{-1}]}\,, \tag{138}$$

where a summation over μ is to be understood and where $M^2 = \mu^2 e^{\tilde{c}}$ with $\tilde{c} = 16\pi c - 1$. This is just the expected cosmological term with $g^{\sigma\rho} = \sum_\mu \xi^\sigma_{R\,\mu}\xi^{\dagger\rho}_{R\,\mu}$.

The next term in the heat kernel expansion should produce the relation ensuring that the metric is compatible with the spin connection. Finally, one would allow the spin connection to be a dynamical variable.

As mentioned before, there is apparently a fundamental problem in considering theories where the graviton is generated dynamically. If we refer to the original paper by Weinberg and Witten [9], the apparent pathology of these theories lies in the fact that the energy-momentum tensor has to be identically zero if particles with spin higher than one appear. Actually, at a very naive level the energy-momentum tensor of the toy model presented here *is* zero as the model contains no metric with respect to which one can derive. Probably an energy-momentum tensor could be defined in some way, but this is not totally obvious, and it is not clear to what extent the conditions assumed by Weinberg and Witten apply.

The previous two-dimensional example is all too trivial but it shows perfectly the general ideas. It seems conceivable to entertain the idea that a mechanism analogous to chiral symmetry breaking may trigger the dynamical appearance of some degrees of freedom that at the very least reproduce formally Einstein–Hilbert action. This leads to rather interesting results, for instance we expect the following relation between the Planck mass and the dynamically generated mass

$$M_{\rm P}^2 \sim \frac{M^2}{16\pi^2}\log\frac{\mu}{M}\,. \tag{139}$$

We have also seen above how a relation between the would-be cosmological constant and the parameters of the underlying theory appears.

This is probably an appropriate place to stop and we recommend the interested reader to examine the results that will be presented in [4].

8. Summary

In these lectures we review the physical consequences of treating gravity at the quantum level as an effective theory, not very different from what is done in pion physics. Because it contains massless states, non-local logarithmic terms in the effective action should then be present.

We have analyzed the relevance of the non-local quantum corrections due to the virtual exchange of gravitons and other massless modes to the evolution of the cosmological scale factor in FRW universes. The effect is largest in a de Sitter universe with a large cosmological constant. The effects are nonetheless locally absolutely tiny, but they lead to a noticeable secular effect that slows down the inflationary expansion. Although this has not been discussed in detail in these lectures, in a matter dominated universe the effect is a lot smaller, and it appears to be of the opposite sign. Quantum effects seem to enhance the expansion rate in this case. These effects have no classical analogy.

Note that the results presented here are not 'just another model'. Quantum gravity non-local loop corrections exist. They are required by unitarity if gravity is to be a consistent quantum theory. The non-localities also give rise to other consequences; for instance it would be very interesting to compute the space correlations that these logarithmic terms introduce.

In the final part we have discussed a toy model where gravitons appear as a Goldstone states. The model has originally no metric whatsoever; it is generated dynamically.

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