ON THE SCHEME DEPENDENCE OF GRAVITATIONAL BETA FUNCTIONS*

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We discuss the arbitrariness in the choice of cutoff scheme in calculations of beta functions. We define a class of "pure" cutoff schemes, in which the cutoff is completely independent of the parameters that appear in the action. In a sense they are at the opposite extreme of the "spectrally adjusted" cutoffs, which depend on all the parameters that appear in the action. We compare the results for the beta functions of Newton's constant and of the cosmological constant obtained with a typical cutoff and with a pure cutoff, keeping all else fixed. We find that the dependence of the fixed point on an arbitrary parameter in the pure cutoff is rather mild. We then show in general that if a spectrally adjusted cutoff produces a fixed point, there is a corresponding pure cutoff that will give a fixed point in the same position.

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1. Introduction

In the last ten years much work has been done on the calculation of the beta functions of gravity, using Wilsonian Renormalization Group methods. Such calculations have focused first on the Hilbert action, usually including

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also a cosmological term [1–6], and have been extended to include terms quadratic in curvature [7–10] as well as higher polynomials in the Ricci scalar [11–13] or in some approximation even all orders of the derivative expansion [14]. Similar studies have also been done in the conformal reduction of gravity, where only the conformal factor is dynamical [15]. The main motivation for this work has been to try to (dis)prove the asymptotic safety of gravity, *i.e.* the existence of a fixed point with finitely many UV attractive directions [16], also [17] for reviews.

One slightly bothering aspect of these calculations is that very often one works with quantities that are scheme-dependent, *i.e.* depend on details of how one chooses to implement a cutoff. Scheme dependence is nothing new in quantum field theory. It is usually taken as a sign that the quantity one is calculating is not directly measurable. The leading approximation to the beta functions of dimensionless coupling constants, such as Yang–Mills and Yukawa couplings in four dimensions is scheme-independent. In the case of gravity the most interesting couplings, like Newton's constant and the cosmological constant are dimensionful and their beta functions are scheme dependent even in the lowest approximation. One hopes that at least the qualitative features of the Renormalization Group (RG) flow are scheme independent. For example, the existence of a Fixed Point (FP) as well as the number of attractive/repulsive directions at the FP should be scheme-independent. More quantitatively, one expects also the critical exponents at the FP to be scheme-independent. All the calculations performed so far support the existence of a FP with finitely many attractive directions. For pure gravity the number of attractive directions seems likely to be three [9,11–13], in accordance with the idea of asymptotic safety. At a quantitative level, when using a truncation, both the position of the FP and the critical exponents depend on the scheme, but there are some quantities, such as the dimensionless combination ΛG which are much less sensitive to cutoff scheme than others.

Here we add some tassels to this picture. We consider a class of cutoff schemes that have not received much attention in the literature, that we call "pure". We say that a cutoff is "pure" if it does not depend on any of the parameters (masses, wave function renormalizations, couplings ...) which appear in the action. These cutoffs are logically at the opposite extreme of another class of cutoffs that depend on all the couplings of the action and are called, for reasons that will be made clear in Section 3, "spectrally adjusted".

In this note we would like to do two things. The first is to calculate the properties of the gravitational fixed point in the Einstein–Hilbert truncation using a pure cutoff, and compare them to the results of other, more popular cutoffs. This we shall do in Section 4. The second is to show in general that

whenever one uses a pure cutoff, there is a choice of parameters that will yield the same FP as a spectrally adjusted cutoff. This will be the subject of Section 5.

For definiteness we shall discuss scheme-dependence in the context of Wetterich's exact RG equation [18], which uses an additive infrared cutoff. To limit the dimension of the parameter space that we explore we shall keep other aspects of the cutoff and gauge choice fixed. Although stimulated by research on gravity, the points we make in Sect. 5 are quite general and should apply also to other quantum field theories and to other flow equations.

2. The flow equation

As mentioned in the Introduction, most of the progress towards asymptotic safety of the last ten years has come from applying functional renormalization group methods to gravity. In this section we describe these tools, using first the example of a real scalar field and then describing some of the modifications that are needed in the application to gravity. A general idea put forward by Wilson is that the functional integration should not be performed in one single step covering all field fluctuations from the UV to the IR, weighting all fluctuations with the same bare action, but rather in a sequence of finite steps, updating the action at each step. A concrete implementation of this idea that is easily amenable to explicit calculations was given in 1993 by Wetterich [18]. We begin from a formal functional integral

$$e^{-W_k[j]} = \int (d\phi) e^{-\left(S(\phi)+\Delta S_k(\phi)+\int j\phi\right)}\,, \tag{1}$$

where j is an external source and

$$\Delta S_k(\phi) = \frac{1}{2}\int d^4q \phi(-q) R_k(q^2)\phi(q)\,. \tag{2}$$

The effect of the new term ΔS_k is simply to modify the (inverse) propagator of the theory: it replaces q^2 by

$$P_k(q^2) = q^2 + R_k(q^2)\,. \tag{3}$$

The kernel $R_k(q^2)$ is chosen so as to suppress the propagation of the modes with momenta $|q| \ll k^2$ and tends to zero for $|q| \gg k^2$ so that high momentum modes are integrated out without any suppression. We shall discuss the properties of this function in greater detail in Section 3. One then defines a scale-dependent effective action functional $\Gamma_k(\phi)$, as the Legendre transform of W_k, minus the term ΔS_k that we introduced in the beginning:

$$\Gamma_k[\phi] = W_k[j] - \int j\phi - \Delta S_k(\phi)\,, \tag{4}$$

where ϕ is now to be interpreted as a shorthand for $\langle\phi\rangle$, the variable conjugated to j. If the functional integral is defined by an UV cutoff, then when k tends to this cutoff the average effective action is related by a nontrivial transformation to the bare action [20]. For $k \to 0$, $\Delta S_k \to 0$ and one recovers the standard definition of the effective action (the generating function of one-particle-irreducible Green functions). It is not exactly the Wilsonian action but its definition is similar in spirit and it is somewhat easier to work with. If one evaluates this functional at one loop, it is

$$\Gamma_k^{(1)} = \frac{1}{2}\mathrm{STr}\log\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right) \tag{5}$$

and its scale dependence is given by

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{STr}\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right)^{-1} k\frac{dR_k}{dk}\ . \tag{6}$$

Here STr is a trace that includes a factor -1 for fermionic fields and a factor 2 for complex fields. It can be shown that the "renormalization group improvement" of this equation, which consists in replacing S by Γ_k in the r.h.s., leads actually to an exact equation often called the Functional Renormalization Group Equation (FRGE) [18]:

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{STr}\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} k\frac{dR_k}{dk}\ . \tag{7}$$

It is important to observe that the last term in Eq. (7) suppresses the contribution of high momentum modes so that the trace is ultraviolet finite: there is no need to use any ultraviolet regularization. In fact, once the equation has been derived, it is actually not necessary to refer to the functional integral anymore. The FRGE defines a flow in the space of all theories and if we start from any point and we follow the flow in the limit $k \to 0$, then we find the effective action, from which in principle we can derive everything we may want to know about the theory. Conversely, by following the flow towards higher energy we can establish whether the theory has a FP with the desired properties.

The application of this equation to gravity has been discussed first in [1]. Since gravity is a gauge theory, one has to take into account the complications due to the gauge fixing and ghost terms. So far the best way to deal with these complications is to use the background field method. Let $\bar{g}_{\mu\nu}$ be a fixed but otherwise arbitrary metric. We can write $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. It is not implied that h is small. We choose a gauge-fixing condition

$$S_{\mathrm{GF}}(\bar{g}, h) = \frac{1}{2}\int d^4x\,\sqrt{\bar{g}}\,\chi_\mu Y^{\mu\nu}\chi_\nu\,, \tag{8}$$

where $\chi_\nu = \nabla^\mu h_{\mu\nu} + \beta\nabla_\nu h$, $h = h^\mu_\mu$ and Y is some operator, which in the simplest cases is just equal to $\bar{g}_{\mu\nu}$. The standard formal manipulations in the path integral give rise to a ghost term

$$S_c = \int d^4x \sqrt{\bar{g}}\, \bar{C}_\nu (\Delta_{gh})^\nu_\mu C^\mu \ , \tag{9}$$

and, if Y contains derivatives, also a "third ghost" term [12, 19]

$$S_b = \frac{1}{2} \int d^4x \sqrt{\bar{g}}\, b_\mu Y^{\mu\nu} b_\nu \ . \tag{10}$$

Also the cutoff term ΔS_k is written in terms of the background metric

$$\Delta S_k(\bar{g}) = \int d^4x \sqrt{\bar{g}}\, h_{\mu\nu} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} R_k(\bar{\Delta}) h_{\rho\sigma} \ , \tag{11}$$

where $\bar{\Delta}$ is some differential operator constructed with the background metric. In this way one constructs a generating functional $W(j^{\mu\nu}, \bar{g}_{\mu\nu})$ depending on sources that couple linearly to $h_{\mu\nu}$, and on the background metric. Applying the definition Eq. (4) one obtains a functional $\Gamma_k(h_{\mu\nu}, \bar{g}_{\mu\nu})$ where $h_{\mu\nu}$ is now a shorthand for $\langle h_{\mu\nu} \rangle$, the Legendre conjugate of $j^{\mu\nu}$. One can also think of Γ_k as a functional of two metrics, namely $\langle g_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + \langle h_{\mu\nu} \rangle$ and the background metric. In the limit $k \to 0$ this functional becomes the ordinary gravitational effective action in the background gauge. The functional $\Gamma_k(g, \bar{g})$ is invariant under simultaneous coordinate transformations of g and $\bar{g}$, the so-called background gauge transformations. We will restrict our attention to the functional $\Gamma_k(g) = \Gamma_k(g, g)$ obtained by the identification of the background field (which hitherto remained completely unspecified) and the vacuum expectation value g. By construction this functional has the same gauge invariance as the original action and it contains the information about the familiar terms such as the Einstein–Hilbert action. The functional $\Gamma_k(g, \bar{g})$ contains in addition the information about the k-dependence of the gauge-fixing terms and other genuinely bimetric terms in the action [21]. In the following we will ignore the RG flow of these terms. The functional $\Gamma_k(g, \bar{g})$ obeys an FRGE that has the same form as in Eq. (7), where ϕ now stands for $h_{\mu\nu}$, C_μ, $\bar{C}_\mu$ and b.

3. Cutoff schemes

There is a large arbitrariness in the choice of the cutoff ΔS_k. Let us begin by discussing the simplest example: a scalar theory in flat space with the action truncated to:

$$S = \int dx \left(\tfrac{1}{2} (\partial\phi)^2 + V(\phi^2) \right) , \tag{12}$$

where V is an even potential. The cutoff is usually written in momentum space as in Eq. (2) where R_k is a function that is only required to satisfy the boundary conditions that are appropriate to a cutoff, namely:

- it tends rapidly to zero when $q^2 > k^2$;
- it tends to k^2 when q^2 tends to zero;
- it is monotonically decreasing in q^2;
- it is monotonically decreasing in k, uniformly in q^2.

The second condition is a kind of normalization and is not strictly necessary: it is possible to allow R_k to tend to some other finite value, or even to infinity, when $k \to 0$. However, for reasons that will become clear shortly, it is convenient to keep fixed the value of $R_k(0)$. Aside from this, the shape of the function is arbitrary. Inserting the ansatz Eq. (12) for Γ_k, the FRGE Eq. (7) becomes

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\frac{1}{(4\pi)^2}\int dzz\,\frac{\frac{\partial R_k(z)}{\partial t}}{P_k(z) + 2V' + 4\phi^2 V''}\,, \tag{13}$$

where $z = q^2$, $t = \log(k/k_0)$ and the modified inverse propagator P_k is defined as in Eq. (2).

From here, taking derivatives with respect to ϕ^2, one can extract the beta functions of the coefficients of the Taylor expansion of the potential. These beta functions will depend in general on the choice of R_k. One notable exception, in four dimensions, is the beta function of the quartic coupling, which in the limit where the mass and the higher couplings are negligible, turns out to be independent of R_k. The way in which this happens is as follows: the trace in the r.h.s. of the FRGE involves an integration over q^2. In the specific case, the function to be integrated turns out to be a total derivative and therefore depends only on the boundary values. Since these have been fixed, it is completely determined. See Appendix A in [12] for details.

The discussion can be refined by taking into account a wave function renormalization constant Z in front of the kinetic term. One then usually modifies the definition Eq. (2) by including a factor Z:

$$\Delta S_k = \frac{Z}{2}\int dq\,\phi(-q)\,R_k(q^2)\,\phi(q)\,. \tag{14}$$

The reason why this is convenient is that the inverse propagator Zq^2 and the cutoff $ZR_k(q^2)$ then neatly combine into a modified inverse propagator

$ZP_k(q^2)$ and the overall factors of Z cancel between numerator and denominator of the FRGE, which takes the form

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\frac{1}{(4\pi)^2}\int dz z\,\frac{\frac{\partial R_k(z)}{\partial t} + \eta R_k(z)}{P_k(z) + \frac{2V' + 4\phi^2 V''}{Z}}\,, \tag{15}$$

where we have defined $\eta = \frac{d\log Z}{d\log k}$.

The choice of the shape of the function R_k is an aspect of the cutoff scheme which involves infinitely many parameters. When one considers more complicated theories, further "discrete" choices present themselves. For example, in the case of gravity truncated to the Hilbert action, the second variation of the action has the following schematic form:

$$\Gamma_k^{(2)} = aZ[-\Box + bR + c\Lambda]\,, \tag{16}$$

where $Z = 1/(16\pi G)$, a, b and c are constants, $\Box = \nabla^\mu\nabla_\mu$ and the curvature term R are computed with the background metric. Tensor structures are omitted, since they are not essential for the argument. Since in curved spacetime one cannot use momentum space methods, in the definition of the cutoff one has to decompose the fields on a basis of eigenvectors of some selfadjoint differential operator. The cutoff then suppresses the modes with eigenvalues $\lambda < k^2$ (if the operator is of second order). One then has a choice of what operator to use. In [12] we called a cutoff of type I, II or III, respectively, if it is defined using the spectrum of $-\Box$, $-\Box + bR$ or $-\Box + bR + c\Lambda$.

In the literature, an overall factor of $Z = 1/(16\pi G)$ is inserted in the definition of the cutoff as in Eq. (14). In the case of gravity there are some differences. Unlike in the scalar case, the coefficient Z is dimensionful; even more important, one can show that Z is now an essential coupling constant in the sense that it cannot be eliminated by rescaling the metric without changing the unit of mass k [22]. In spite of these differences, it is still true that Z does not appear in the reference operator, so in the case of type I and II cutoffs the spectrum of the operator does not depend on k. In the type III cutoff the reference operator contains the running coupling Λ. As a consequence, also the spectrum of the reference operator changes with k. For this reason this is called a "spectrally adjusted cutoff". It has been argued in [23] that such cutoffs give improved results.

In this paper we would like to explore in the opposite direction and consider a class of cutoff functions that do not depend on *any* coupling, not even on Z. Quite generally, we will refer to a cutoff that does not contain any parameter that appears in the action as a "pure" cutoff.

4. Gravitational beta functions

We will now discuss the example of gravity in the Einstein–Hilbert truncation, where one retains only the terms with up to two derivatives of the metric:

$$\Gamma_k = \int dx\sqrt{g}\,(2\Lambda Z - ZR) + S_{\rm GF} + S_{\rm ghost}\ , \tag{17}$$

where $Z = 1/(16\pi G)$, $S_{\rm GF}$ is a gauge-fixing term and $S_{\rm ghost}$ is the ghost action. We decompose the metric into $g_{\mu\nu} = g^{(\rm B)}_{\mu\nu} + h_{\mu\nu}$ where $g^{(\rm B)}_{\mu\nu}$ is a background. We consider a de Donder background gauge:

$$S_{\rm GF}\left(g^{(\rm B)}, h\right) = \frac{Z}{2\alpha}\int dx\sqrt{g^{(\rm B)}}\chi_\mu g^{(\rm B)\mu\nu}\chi_\nu\,, \tag{18}$$

where

$$\chi_\nu = \nabla^\mu h_{\mu\nu} - \tfrac{1}{2}\nabla_\nu h\,.$$

All covariant derivatives are with respect to the background metric. In the following all metrics will be background metrics, and we will omit the superscript (B) for notational simplicity. In order to restrict the number of parameters, we will fix $\alpha = 1$ which leads to considerable simplifications.

After calculating the second variation of the action, we decompose $h_{\mu\nu}$ into its different spin components according to

$$h_{\mu\nu} = h^{\rm T}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu + \nabla_\mu\nabla_\nu\sigma - \frac{1}{d}g_{\mu\nu}\Box\sigma + \frac{1}{d}g_{\mu\nu}h \tag{19}$$

and

$$C^\mu = C^{{\rm T}\mu} + \nabla^\mu C\,, \qquad \bar{C}_\mu = \bar{C}^{\rm T}_\mu + \nabla_\mu\bar{C}\,, \tag{20}$$

where $h^{\rm T}_{\mu\nu}$ is transverse and traceless, ξ is a transverse vector, σ and h are scalars, $C^{\rm T}$ and $\bar{C}^{\rm T}$ are transverse vectors, and C and $\bar{C}$ are scalars. These fields are subject to the following differential constraints:

$$h^{{\rm T}\mu}_\mu = 0\,, \qquad \nabla^\nu h^{\rm T}_{\mu\nu} = 0\,, \qquad \nabla^\nu\xi_\nu = 0\,, \qquad \nabla^\mu\bar{C}^{\rm T}_\mu = 0\,, \qquad \nabla_\mu C^{{\rm T}\mu} = 0\,.$$

We further redefine

$$\xi_\mu \ = \ \sqrt{-\Box - \frac{R}{d}}\,\hat{\xi}_\mu\,, \qquad \sigma = \sqrt{-\Box}\sqrt{-\Box - \frac{R}{d-1}}\hat{\sigma} \tag{21}$$

which removes some powers of $-\Box$ from the second variation and furthermore cancels some Jacobian determinants that arise in the functional integral when one performs the decomposition Eq. (19). We then drop the hat from ξ and σ.

In order to extract the beta functions of Λ and Z it is enough to calculate the r.h.s. of the FRGE on the background of a sphere. Then the inverse propagator diagonalizes

$$\begin{aligned}
\Gamma^{(2)}_{h^{\mathrm{T}}_{\mu\nu}h^{\mathrm{T}}_{\rho\sigma}} &= \frac{Z}{2}\left[-\Box+\frac{2}{3}R-2\Lambda\right]\frac{1}{2}(g^{\mu\rho}\,g^{\nu\sigma}+g^{\mu\sigma}\,g^{\nu\rho})\,,\\
\Gamma^{(2)}_{\xi_\mu\xi_\nu} &= Z\left[-\Box+\frac{1}{4}R-2\Lambda\right]g^{\mu\nu}\,,\\
\Gamma^{(2)}_{hh} &= -\frac{Z}{8}\left[-\Box-2\Lambda\right]\,,\\
\Gamma^{(2)}_{\sigma\sigma} &= \frac{3Z}{8}\left[-\Box-2\Lambda\right]\,,\\
\Gamma^{(2)}_{\bar{C}^{\mathrm{T}}_\mu C^{\mathrm{T}}_\nu} &= \left[\Box+\frac{1}{4}R\right]g^{\mu\nu}\,,\\
\Gamma^{(2)}_{\bar{C}C} &= \left[\Box+\frac{1}{2}R\right]\,.
\end{aligned}\tag{22}$$

Now we define the cutoff. We require that after adding the cutoff, the modified inverse propagator has the same form as in Eq. (22) except for the replacement of $-\Box$ by $P_k(-\Box)$, where P_k is defined as in Eq. (3). This means that when the inverse propagator has the form Eq. (16), the cutoff is

$$\mathcal{R}_k = aZR_k(-\Box)\,.\tag{23}$$

This type of cutoff was introduced in [2] and studied in greater generality in [4]. With this cutoff the r.h.s. of the FRGE takes the form

$$\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}_{(2)}\frac{\partial_t R_k+\eta R_k}{P_k+\frac{2}{3}R-2\Lambda}+\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k+\eta R_k}{P_k+\frac{1}{4}R-2\Lambda}+\frac{1}{2}\mathrm{Tr}_{(0)}\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}\\
&+\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}-\mathrm{Tr}_{(1)}\frac{\partial_t R_k}{P_k-\frac{R}{4}}-\mathrm{Tr}'_{(0)}\frac{\partial_t R_k}{P_k-\frac{R}{2}}\,,
\end{aligned}\tag{24}$$

where $\eta=\frac{d\log Z}{d\log k}$. The first term comes from the spin-2, transverse traceless components, the second from the spin-1 transverse vector, the third and fourth from the scalars h and σ. The last two contributions come from the transverse and longitudinal components of the ghosts. A prime or a double prime indicate that the first or the first and second eigenvalues have to be omitted from the trace. The reason for this is explained in the references quoted above.

In order to be able to perform the calculation in closed form we choose the optimized cutoff $R_k(z)=(k^2-z)\theta(k^2-z)$ [24]. In this way one finds

the following beta functions for the dimensionless couplings $\tilde{Z} = Z/k^2$, $\tilde{\Lambda} = \Lambda/k^2$:

$$\begin{aligned}
\partial_t \tilde{Z} &= -2\tilde{Z} + \frac{373 - 654\tilde{\Lambda} + 600\tilde{\Lambda}^2}{1152\pi^2(1-2\tilde{\Lambda})^2} + \frac{\partial_t \tilde{Z}}{\tilde{Z}} \frac{29 - 9\tilde{\Lambda}}{1152\pi^2(1-2\tilde{\Lambda})^2}\,, \\
\partial_t(\tilde{Z}\tilde{\Lambda}) &= -4\tilde{Z}\tilde{\Lambda} + \frac{1+3\tilde{\Lambda}}{12\pi^2(1-2\tilde{\Lambda})} + \frac{\partial_t \tilde{Z}}{\tilde{Z}} \frac{5}{192\pi^2(1-2\tilde{\Lambda})}
\end{aligned} \tag{25}$$

which admit a FP at $\tilde{G}_* = 0.7012$, $\tilde{\Lambda}_* = 0.1715$, with critical exponents $\theta = 1.689 \pm 2.486i$.

As anticipated, we would like to examine a different type of cutoff, not depending on any of the parameters that are present in the action. The cutoff Eq. (23) depends on the parameter Z, so to define a pure cutoff we replace Z by γk^2 where γ is an arbitrary number:

$$\mathcal{R}_k = a\gamma k^2 R_k(-\Box)\ . \tag{26}$$

The FRGE now reads

$$\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}_{(2)} \frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box + \frac{2}{3}R - 2\Lambda\right) + R_k(-\Box)} \\
&\quad + \frac{1}{2}\mathrm{Tr}'_{(1)} \frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box + \frac{1}{4}R - 2\Lambda\right) + R_k(-\Box)} \\
&\quad + \frac{1}{2}\mathrm{Tr}_{(0)} \frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box - 2\Lambda\right) + R_k(-\Box)} \\
&\quad + \frac{1}{2}\mathrm{Tr}''_{(0)} \frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box - 2\Lambda\right) + R_k(-\Box)} \\
&\quad - \mathrm{Tr}_{(1)} \frac{\partial_t R_k}{P_k - \frac{R}{4}} - \mathrm{Tr}'_{(0)} \frac{\partial_t R_k}{P_k - \frac{R}{2}}\,.
\end{aligned} \tag{27}$$

This leads to the following beta functions

$$\begin{aligned}
\partial_t \tilde{Z} &= -2\tilde{Z} + \frac{49\gamma(\gamma - \tilde{Z}) + (1-2\tilde{\Lambda})(25\tilde{Z}^2 - 151\tilde{Z}\gamma + 28\gamma^2)}{192\pi^2(\gamma - \tilde{Z})^2(1-2\tilde{\Lambda})} \\
&\quad - \frac{\gamma\left[3(\gamma - \tilde{Z})^2 + \tilde{Z}(1-2\tilde{\Lambda})(101\tilde{Z} - 3\gamma)\right]}{192\pi^2(\gamma - \tilde{Z})^3} X\,, \\
\partial_t(\tilde{Z}\tilde{\Lambda}) &= -4\tilde{Z}\tilde{\Lambda} - \frac{9\gamma^2 + 4\tilde{Z}^2 - \gamma\tilde{Z}(23 - 20\tilde{\Lambda})}{32\pi^2(\gamma - \tilde{Z})^2} \\
&\quad - \frac{5\gamma\left[\gamma^2 - 2\gamma\tilde{Z} + 4\tilde{Z}^2\tilde{\Lambda}(1-\tilde{\Lambda})\right]}{16\pi^2(\gamma - \tilde{Z})^3} X\,,
\end{aligned} \tag{28}$$

where

$$X = \ln\left(\frac{\tilde{Z}(1-2\tilde{\Lambda})}{\gamma - 2\tilde{Z}\tilde{\Lambda}}\right).$$

The appearance of the logarithms is due to the mismatch between the coefficients of $-\Box$ and $R_k(-\Box)$, which leaves some explicit terms with $z = -\Box$ to be integrated over. The FP now depends on the arbitrary parameter γ, which is part of the freedom in the definition of the cutoff. This reflects itself in the position of the fixed point, as shown in figure 1. We give separately

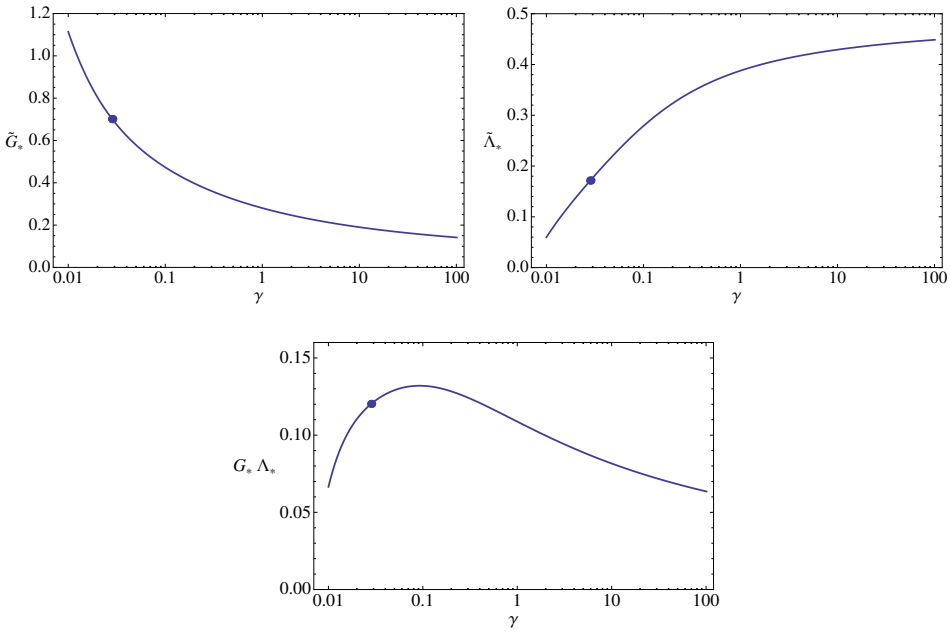


Fig. 1. Value of $\tilde{G}_*$ (left panel), $\tilde{\Lambda}_*$ (right panel) and $\Lambda_* G_*$ (lower panel) as functions of γ with a pure cutoff. The dot indicates the values for the type Ib cutoff.

the dependence of $\tilde{G}_*$, $\tilde{\Lambda}_*$ and of the dimensionless product $\Lambda_* G_*$. We see that as γ varies over four orders of magnitude, $\tilde{G}_*$, $\tilde{\Lambda}_*$ each vary by less than one order of magnitude, and $\Lambda_* G_*$ changes just by a factor smaller than 2. It had been observed before that the dimensionless product ΛG has a beta function that is gauge independent in lowest order in an expansion in $\tilde{\Lambda}$ [2]; also its value at the FP was found to be quite insensitive to the choice of gauge and cutoff. Our findings confirm this picture also for the dependence on the parameter γ. In figure 2 we also give the critical exponents as functions of γ. As with other cutoffs, they form a complex conjugate pair, but for large γ the imaginary part of the eigenvalue goes to zero and for $\gamma > 60$

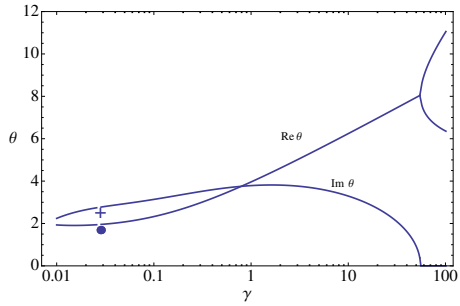


Fig. 2. Real and imaginary parts of the critical exponents as functions of γ. The dot and the cross indicate the real and imaginary part of the critical exponent for the type Ib cutoff.

we find two real eigenvalues. Clearly for very large or very small γ the properties of the FP are significantly affected, but there is a wide range of values for which the properties of the FP are quite stable.

We observe that the curves in figure 1 pass through the position of the fixed point in the type Ib cutoff examined previously, which is marked by a dot in the graphs. In other words, there is a value of γ for which the pure cutoff gives a FP in the same position as the type Ib cutoff. The corresponding value is precisely $\gamma = \tilde{Z}_* = 0.0284$. It would seem from (28) that the beta functions become singular when $\gamma = \tilde{Z}$ but if we put $\gamma = \tilde{Z} + \epsilon$ and expand in powers of ϵ, the coefficient of the negative powers of ϵ cancel out. Furthermore, one finds that the leading (ϵ-independent) terms in the expansion coincide with the first two terms on the r.h.s. of (25).

There is clearly something special happening when γ has the value of $\tilde{Z}_*$ in the type Ib cutoff. In the next section we will explain the origin of this coincidence and we will see that it holds in much greater generality.

5. Fixed points with pure and spectrally adjusted cutoffs

We have shown in the previous section, in the special case of gravity in the Einstein Hilbert truncation, that there is a value of the parameter γ for which a pure cutoff reproduces the results of a type I cutoff. In this section we argue that this property is quite general. Since we are going to move to a much more general setting we will have to make some choices in order to somehow circumscribe the scope of the argument.

We assume that Γ_k admits a derivative expansion of the form

$$\Gamma_k = \sum_i g_i \mathcal{O}_i \,, \tag{29}$$

where $\mathcal{O}_i$ are operators and g_i are numerical parameters depending on k.

The operators $\mathcal{O}_i$ are integrals of the form

$$\mathcal{O}_i = \int d^d x \sqrt{g} \Omega_i \,, \tag{30}$$

where Ω_i are (possibly nonpolynomial) functions of the fields and their derivatives, respecting all the symmetries that the theory is supposed to possess. In gauge theories, Ω_i are constructed with covariant derivatives and curvatures. The number of derivatives increases with i, but the precise correspondence need not be spelled out here.

Some of the parameters appearing in the expansion may be eliminated by field redefinitions. This is the case, for example, for the wave function renormalization constants. Such parameters are said to be "redundant" or "inessential" [25]. We assume that the theory has been parametrized in such a way that a certain subset of the g_i is redundant, while the remaining ones are "essential".

The fields do not have any scale dependence, so that

$$k \frac{d\Gamma_k}{dk} = \sum_i \beta_i \mathcal{O}_i \,, \tag{31}$$

where $\beta_i(g_j, k) = k \frac{dg_i}{dk}$ are the beta functions. In general they depend on all the g_i and also explicitly on k. Note that we call "beta functions" the derivatives of the parameters appearing in the action whether they are essential or not. One sometimes prefers to call "anomalous dimensions" the (logarithmic) derivatives of irrelevant parameters such as the wave function renormalization constants. We will not need to make this terminological distinction here. We will call β (without subscript i) the "beta functional" on the r.h.s. of the FRGE

$$\beta = \sum_i \beta_i \mathcal{O}_i \,.$$

If the operator Ω_i has dimension α_i, $\mathcal{O}_i$ has dimension $\alpha_i - d$ and g_i has dimension $d_i = d - \alpha_i$. One can now define dimensionless couplings $\tilde{g}_i$ and dimensionless operators $\tilde{\mathcal{O}}_i$ by $g_i = k^{d_i} \tilde{g}_i$ and $\mathcal{O}_i = k^{-d_i} \tilde{\mathcal{O}}_i$, so that Eq. (29) can also be written as $\Gamma_k = \sum_i \tilde{g}_i \tilde{\mathcal{O}}_i$. The condition that has to be satisfied by a FP is

$$k \frac{d\tilde{g}_i}{dk} = 0 \,, \tag{32}$$

for all essential couplings g_i. We can rewrite this as follows. From the definition of $\tilde{g}_i$ we obtain $\partial_t g_i = d_i g_i + k^{d_i} \partial_t \tilde{g}_i$. Then we can rewrite Eq. (31) as

$$k \frac{d\Gamma_k}{dk} = \sum_i d_i \tilde{g}_i \tilde{\mathcal{O}}_i + \sum_i \partial_t \tilde{g}_i \tilde{\mathcal{O}}_i \,.$$

Then the FP equation can be written compactly as

$$\left(-\sum_i d_i \tilde{g}_i \tilde{\mathcal{O}}_i + \beta\right)\Big|_{\text{essential}} = 0\,, \tag{33}$$

where the subscript "essential" means that the equation has to be projected on the subspace of essential couplings. The individual equations Eq. (32) can be obtained from the functional equation by extracting the coefficient of the operator $\tilde{\mathcal{O}}_i$. We will now compare the functional form of this equation for two classes of cutoffs.

For definiteness we start by choosing a "spectrally adjusted" cutoff, defined as follows. The second variation of the action is a differential operator

$$\Delta(g_i) = \frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} = \sum_i g_i \frac{\delta^2 \mathcal{O}_i}{\delta\phi\delta\phi}\,. \tag{34}$$

By this notation we emphasize that the operator depends on all the parameters g_i. In the case of gauge theories the operator Δ is constructed with the covariant derivative ∇_μ. We choose the cutoff $\mathcal{R}$ to be a function of the full operator Δ: $\mathcal{R}_k = R_k(\Delta(g_i))$, where R_k is one of the functions that were discussed in Section 3. Then the modified inverse propagator is

$$\Delta(g_i) + R_k(\Delta(g_i)) = P_k(\Delta(g_i))\,, \tag{35}$$

where P_k is defined as in Eq. (3). This is a cutoff of "type III", in the terminology of [12]. It is spectrally adjusted because it manifestly depends on all the couplings, masses and wave function renormalizations.

The r.h.s. of the FRGE can now be written

$$\beta = \frac{1}{2}\mathrm{STr}\left(\Delta(g_i)+R_k(\Delta(g_i))\right)^{-1}\left(\frac{\partial R_k(\Delta(g_i))}{\partial t} + R_k'(\Delta(g_i))\frac{\partial \Delta}{\partial g_i}\beta_i\right)\,, \tag{36}$$

where a prime indicates the derivative of the function with respect to its argument. In the first term one derives only the explicit dependence of the cutoff on k and in the second the dependence that comes from the flow of the g_i. From here the beta functions β_i can be obtained in a two step procedure. First one has to extract from Eq. (36) the coefficient of $\mathcal{O}_i$. Formally, we can write

$$\beta_i = \frac{\delta\beta}{\delta\mathcal{O}_i}\,.$$

This is usually the most labor-intensive part of the calculation, but still it does not immediately give the beta function, because the r.h.s. is itself a linear combination of the beta functions, of the form

$$\frac{\delta\beta}{\delta\mathcal{O}_i} = B_i + A_{ij}\beta_j\,,$$

where B_i are the one loop beta functions and A_{ij} are calculable coefficients. The beta functions can be obtained by solving this linear system:

$$\beta_i = (\mathbf{1} - A)^{-1}_{ij} B_j \,.$$

If one is only interested in the location of the FP, one can avoid this step by the following trick [26]. Since at a FP $g_i = \tilde{g}_{i*} k^{d_i}$, for some constants $\tilde{g}_{i*}$, we obtain an equivalent set of FP equations if in the beta functional we replace β_i by $d_i g_i = d_i \tilde{g}_i k^{d_i}$. This modified beta functional is

$$\bar{\beta} = \frac{1}{2}\mathrm{STr}\left(\Delta(g_i) + R_k(\Delta(g_i))\right)^{-1}\left(\frac{\partial R_k(\Delta(g_i))}{\partial t} + R_k'(\Delta(g_i))\frac{\partial \Delta}{\partial g_i} d_i \tilde{g}_i k^{d_i}\right) . \tag{37}$$

If we define

$$\bar{\beta}_i = \frac{\delta \bar{\beta}}{\delta \mathcal{O}_i}$$

these expressions do not contain the β functions anymore, and so they can be plugged directly in the FP equation. The FP equations obtained from the modified beta functions $\bar{\beta}_i$ have the same FP solutions as the ones obtained from the true beta functions β_i. We observe that the second term on the r.h.s. of Eq. (37) is not just a function of Δ. In general it is a complicated operator that will not commute with Δ itself. We actually do not have the mathematical tools to extract beta functions from such complicated traces involving functions of several noncommuting operators. However, calculability is not required here, so we can proceed formally. We now simply assume that the FP equations determined in this way have a solution at $\tilde{g}_i = \tilde{g}_{i*}$.

The source of g_i-dependence in the cutoff definition given above is the operator Δ. We can turn the cutoff into a pure cutoff if we replace all the couplings appearing in Δ by arbitrary constants, multiplied by suitable powers of k to preserve the correct dimensionalities. The cutoff is then $R_k(\Delta(\gamma_i k^{d_i}))$. With this cutoff the r.h.s. of the FRGE reads

$$\begin{aligned} \beta \;=\; & \frac{1}{2}\mathrm{STr}\left(\Delta(g_i) + R_k\left(\Delta\left(\gamma_i k^{d_i}\right)\right)\right)^{-1} \\ & \times \left(\frac{\partial R_k\left(\Delta\left(\gamma_i k^{d_i}\right)\right)}{\partial t} + R_k'\left(\Delta\left(\gamma_i k^{d_i}\right)\right)\frac{\partial \Delta}{\partial g_i} d_i \gamma_i k^{d_i}\right) . \end{aligned} \tag{38}$$

From here one can extract beta functions

$$\beta_i(g_i, \gamma_i) = \frac{\delta \beta}{\delta \mathcal{O}_i}$$

that can be used to write FP equations. These FP equations depend parametrically on the arbitrary numbers γ_i. Recalling that $g_i = \tilde{g}_i k^{d_i}$ and comparing Eq. (38) to Eq. (37) we see that the only difference lies in the replacement of $\tilde{g}_i$ by γ_i in certain functional dependences.

It is clear that since the FP equation for the spectrally adjusted cutoff has a zero when we replace everywhere $\tilde{g}_i$ by the numbers $\tilde{g}_{i*}$, then the FP equation for the pure cutoff will also have a zero when we replace all the γ_i and all the $\tilde{g}_i$ by $\tilde{g}_{i*}$. Therefore, with the particular choice of parameters $\gamma_i = \tilde{g}_{i*}$, the pure cutoff produces a FP in the same position as the spectrally adjusted type III cutoff.

This result has been derived using what we call a "type III" cutoff, because the argument is easier to make independently of the form of the action, but we believe that it holds more generally, also for other cutoffs. To illustrate this consider a generalization of what was called a "type I" cutoff in [12]. In a gauge theory the second variation defined in Eq. (34) is a differential operator constructed with the covariant derivative ∇_μ. Let us assume that the truncation of the theory is such that Δ depends on ∇_μ only through the combination $-\Box = -\nabla_\mu \nabla^\mu$. To make this explicit let us write it as $\Delta(-\Box, g_i)$. A generalized type I cutoff can be defined by the requirement that the modified inverse propagator has the same form as the original one except for the replacement of $-\Box$ by $P_k(-\Box)$:

$$\mathcal{R}_k(-\Box, g_i) = \Delta(P_k(-\Box), g_i) - \Delta(-\Box, g_i)\,. \tag{39}$$

The beta functional that one obtains with this cutoff has the form

$$\begin{aligned}\beta \;=\; & \frac{1}{2}\mathrm{STr}\left(\Delta(P_k(-\Box), g_i)\right)^{-1} \\ & \times \left(\frac{\partial \Delta}{\partial(-\Box)}\frac{\partial P_k(-\Box)}{\partial t} + \frac{\partial}{\partial g_i}\left(\Delta(P_k(-\Box), g_i) - \Delta(-\Box, g_i)\right)\beta_i\right)\end{aligned} \tag{40}$$

which upon use of the trick explained above yields equivalent FP equations as

$$\begin{aligned}\bar{\beta} = & \frac{1}{2}\mathrm{STr}\left(\Delta(P_k(-\Box), g_i)\right)^{-1} \\ & \times \left(\frac{\partial \Delta}{\partial(-\Box)}\frac{\partial P_k(-\Box)}{\partial t} + \frac{\partial}{\partial g_i}\left(\Delta(P_k(-\Box), g_i) - \Delta(-\Box, g_i)\right) d_i \tilde{g}_i k^{d_i}\right).\end{aligned} \tag{41}$$

Again we can define a pure cutoff of generalized type I by replacing g_i by $\gamma_i k^{d_i}$ in $\mathcal{R}_k$

$$\mathcal{R}_k\left(-\Box, \gamma_i k^{d_i}\right) = \Delta\left(P_k(-\Box), \gamma_i k^{d_i}\right) - \Delta\left(-\Box, \gamma_i k^{d_i}\right)\,. \tag{42}$$

The beta functional that one obtains with this cutoff has the form

$$\beta = \frac{1}{2}\mathrm{STr}\left(\Delta\left(P_k(-\Box), \gamma_i k^{d_i}\right) + \Delta\left(-\Box, g_i\right) - \Delta(-\Box, \gamma k^{d_i})\right)^{-1} \times \left(\frac{\partial\Delta}{\partial(-\Box)}\frac{\partial\mathcal{R}_k(-\Box)}{\partial t} + \left(\frac{\partial\Delta}{\partial g_i}\left(P_k(-\Box), \gamma_i k^{d_i}\right) - \frac{\partial\Delta}{\partial g_i}\left(-\Box, \gamma_i k^{d_i}\right)\right) d_i \gamma_i k^{d_i}\right). \tag{43}$$

Again we see that the two beta functionals have the same form except for the replacement of $\tilde{g}_i$ by γ_i in certain functional dependences; additional terms in the first factor cancel for $\gamma_i = \tilde{g}_i$. Therefore, the argument given above shows that if we set $\gamma_i = \tilde{g}_{i*}$ the pure cutoff will have a FP in the same position as the generalized type I cutoff.

The reason why this discussion is less general than the previous one is that this type of cutoff could only be defined if the inverse propagator has a specific form. It may be possible to generalize this argument, for example defining the cutoff by the rule

$$\nabla_\mu \;\mapsto\; \sqrt{\frac{P_k(-\Box)}{-\Box}}\,\nabla_\mu\,.$$

We will not pursue this further. The discussion of the type III cutoff is sufficient to make the point in generality. Furthermore, the type III cutoff is "ideologically" at the opposite extreme of a pure cutoff, being always fully dependent on all couplings. This is also supported by the numerical results, which show that type III cutoffs yield fixed points at the extreme end of the range of variation [12]. So it is somewhat reassuring that one can reproduce at least the FP position of a spectrally adjusted, type III cutoff by a pure cutoff.

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