BLACKFOLD APPROACH FOR HIGHER-DIMENSIONAL BLACK HOLES*

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In this article we review the blackfold approach, a recently developed effective worldvolume description of higher-dimensional black holes, that captures the long wavelength dynamics of black holes with horizons exhibiting two characteristic lengths of very different size. In this regime the black hole is regarded as a black brane curved into a submanifold of a background spacetime, and can be formulated in terms of an effective fluid that lives on a dynamical worldvolume. We discuss the resulting blackfold equations, which separate into a set of intrinsic and extrinsic equations. The general solution of the intrinsic fluid equations for stationary configurations is presented along with a class of novel stationary black hole solutions. We also comment on how the formalism can be used to study time evolution and stability of blackfolds.

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1. Introduction

It has been realized in recent years that the dynamics of black holes in spacetimes of dimension $D \geq 5$ is much richer than in four dimensions. The origin of this rich variety was recently identified as residing in the possibility of having horizons that are much longer along some directions than in others [1,2]. The exploration of the phases and properties of higher-dimensional black holes is of intrinsic interest in gravity, where the spacetime dimension can be viewed as a tunable parameter. At the same time, there is a wealth of applications in string theory, such as the microscopic understanding of black hole entropy and dual interpretations in the context of the gauge/gravity correspondence, including the recently proposed fluid/AdS-gravity correspondence. See *e.g.* the reviews [3–9] for various aspects of the phase structure of black objects in higher-dimensional gravity and more elaborate exposition of the motivation.

We review here the effective worldvolume theory for higher-dimensional black holes, proposed in Ref. [1] and further developed and generalized in Ref. [2]. This effective theory is based on the observation that the main feature behind novel properties of higher-dimensional black holes, compared to four-dimensional ones, is that their horizons can have two characteristic lengths of very different size. We restrict ourselves here, for simplicity, to the case of neutral, vacuum black holes, where the two length scales are associated with the mass and angular momentum

$$\ell_M \sim (GM)^{1/D-3}\,, \qquad \ell_J \sim \frac{J}{M}\,. \tag{1}$$

In the four-dimensional case these length scales are of the same order, due to the Kerr-bound $J \leq GM^2$. In higher-dimensional spacetimes, however, it is known that black hole solutions exist for which the angular momentum for given mass can be arbitrarily high, so that $\ell_J \gg \ell_M$. In particular, this is observed in five dimensions for ultraspinning black rings [7, 10] and in six and higher dimensions for ultraspinning Myers–Perry black holes [11, 12]. The ability to have these two separated lengths is the origin of essentially all novel features of these higher-dimensional black holes, and suggests therefore an organization according to a hierarchy of scales:

1. $\ell_J \lesssim \ell_M$: black holes behave qualitatively similarly to the four-dimensional Kerr black hole.
2. $\ell_J \approx \ell_M$: threshold of new black hole dynamics.
3. $\ell_J \gg \ell_M$: the separation of scales enables an effective description of long-wavelength dynamics in terms of blackfolds.

The first and second regimes involve the full non-linearities of General Relativity, but in the first case there are no hints of any qualitatively new properties of black holes compared to four-dimensional ones. However, in the second regime, when the two scales begin to diverge, there is ample evidence of the onset of new phenomena: horizon instabilities, inhomogeneous ('pinched') phases, non-spherical horizon topologies, and absence of uniqueness [10, 12–14]. This regime seems hard to investigate by means of exact analytical techniques, but the presence of only one scale in the problem is actually convenient for numerical investigation, The focus in this lecture is on the third regime, where the existence of a small parameter ℓ_M/ℓ_J allows the introduction of efficient approximate analytical methods, which we call the "blackfold approach".

The effective description of black holes in the third regime is based on the idea that in the limit $\ell_M/\ell_J \to 0$, keeping the horizon size finite, the black hole geometry approaches that of an infinitely extended black brane, possibly boosted along some of its worldvolume directions. This has been observed in all presently known exact solutions that admit an ultraspinning limit. In particular, in five dimensions the black ring becomes thin in the ultraspinning limit and the geometry approaches that of a boosted black string [5, 15]. In particular, the thickness r_0 of the ring is much smaller than its radius R in this limit, *i.e.* $\ell_M/\ell_J \sim r_0/R \to 0$. In six and higher dimensions Myers–Perry black holes pancake along the plane of rotation in the ultraspinning limit and the geometry approaches that of a black brane [12]. In this case this means that the thickness r_0 of the pancake is much smaller than its radius R.

The connection to (boosted) black branes observed in these examples gives the crucial hint to identify the variables for a general effective description of higher-dimensional black holes in this regime. Namely, the black hole is regarded as *black* brane whose worldvolume spans a curved submani*fold* of a background spacetime, referred to as a *blackfold* [1, 2]. This generalizes the simplest example of constructing a thin black ring in any dimension by considering a circular boosted black string, as considered earlier in Ref. [13] and expanded upon in [16, 17]. But the ultraspinning limit of Myers–Perry black holes is also correctly captured by the approach, along with a rich bestiary of new stationary black holes [1,18]. Moreover, the blackfold formalism allows to study time evolution and stability of black holes and branes [2].

To leading order in the expansion in ℓ_M/ℓ_J one can neglect the backreaction of the blackfold on the geometry, so that it is a 'test' blackfold. Corrections to the geometry can then be systematically obtained using the method of matched asymptotic expansion. This was used in *e.g.* Ref. [13] (following [19,20]) to compute the first corrections to the geometry and physical quantities for higher-dimensional rings. In this lecture, we describe the zeroth order construction, which already reveals many new types of black objects including their physical properties and allows for the study of time-dependence and stability.

The blackfold approach draws heavily from the beautiful theory of classical brane dynamics developed by Carter in [21], which generalizes the dynamics of a particle traversing a worldline in a background spacetime to that of branes tracing out embedded worldvolumes. Another central ingredient of the approach, regarded as a long-wavelength effective theory, is that part of its dynamics takes the form of the dynamics of a fluid that lives on a dynamical worldvolume. In this context, black branes correspond to specific types of fluid, which are to leading order of the perfect fluid form[1]. The blackfold equations then reduce to a set of intrinsic equations, the Euler equations of the fluid, and a set of extrinsic equations for the worldvolume embedding, which is a generalized geodesic equation for the motion of branes. The effective theory of black holes when $\ell_M/\ell_J \ll 1$ is thus a theory that describes how to bend the worldvolume of a black brane in a background spacetime. In this regard we treat black branes in a manner similar to other familiar extended objects such as cosmic strings or D-branes. The main novelty is that black branes possess black hole horizons, so when their worldvolume is spatially compact we obtain a black hole with finite horizon area.

The outline of this lecture is as follows. We first present in Section 2 the conceptual basis that underlies the blackfold approach as a worldvolume theory of dynamics of black branes. Then we discuss in Section 3 the blackfold equations, which comprise a set of coupled non-linear differential equations for the collective coordinates of a neutral black brane. As an important special case we subsequently consider in Section 4 stationary blackfolds, including physical quantities, horizon topology, the possibility of boundaries and an action principle. We continue in Section 5 by briefly exhibiting some new solutions as well as application of the approach to describe ultra-spinning Myers–Perry black holes. Finally, we briefly discuss in Section 6 the stability of blackfolds, including a simply way to see the Gregory–Laflamme instability of black branes. We close in Section 7 with a future outlook.

2. Effective worldvolume theory

We present here the effective worldvolume theory for black holes that results by integrating out the gravitational short-distance degrees of freedom. Note that since the extended objects in the construction are curved black branes they possess an event horizon.

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[1] A closely related recent development in which black hole dynamics is mapped to fluid dynamics is the 'fluid/AdS-gravity correspondence' [22]. As explained in Ref. [2], this correspondence can be seen as contained in the blackfold approach. There are also suggestive connections to the membrane paradigm [23, 24].

2.1. Collective coordinates for a black brane

We start by schematically splitting the degrees of freedom of General Relativity into long and short wavelength components,

$$g_{\mu\nu} = \left\{ g_{\mu\nu}^{(\mathrm{long})}, g_{\mu\nu}^{(\mathrm{short})} \right\} , \tag{2}$$

where $\mu, \nu = 0, \ldots, D-1$ are spacetime indices. The Einstein–Hilbert action is then approximated as

$$I_{\mathrm{EH}} = \frac{1}{16\pi G} \int d^D x \sqrt{-g} R \approx \frac{1}{16\pi G} \int d^D x \sqrt{-g^{(\mathrm{long})}} R^{(\mathrm{long})} + I_{\mathrm{eff}} \left[g_{\mu\nu}^{(\mathrm{long})}, \phi \right] , \tag{3}$$

where $I_{\mathrm{eff}}[g_{\mu\nu}^{(\mathrm{long})}, \phi]$ is an effective action obtained after integrating-out the short-wavelength gravitational degrees of freedom. What is meant by this will be explained below, but we first wish to identify the 'collective coordinates' ϕ that enter this effective action, which describes the resulting coupling to the long-wavelength component of the gravitational field.

Since the limit $\ell_M/\ell_J \to 0$ of known black holes results in flat black branes, the effective theory is taken to describe the collective dynamics of a boosted black p-brane

$$ds^2_{p-\mathrm{brane}} = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b \right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega^2_{n+1} , \tag{4}$$

with r_0 the horizon radius and u^a the velocity field, satisfying $u^a u^b \eta_{ab} = -1$. Here, and in the following we set the spacetime dimension to be $D = 3+p+n$ and denote the worldvolume coordinates by $\sigma^a = (t, z^i)$, $a, b = 0, \ldots, p$.

The D collective coordinates of the black brane are then

$$\phi(\sigma^a) = \{ X^{\perp}(\sigma^a), r_0(\sigma^a), u^i(\sigma^a) \} , \tag{5}$$

where r_0 is the 'horizon thickness', u^i the p independent components of the velocity field and $X^{\perp}$ the $D-p-1$ position of the brane in directions transverse to the worldvolume. In the long-wavelength effective theory these are allowed to vary slowly along the worldvolume $\mathcal{W}_{p+1}$, over a length scale R much longer than the size-scale of the black brane,

$$R \gg r_0 . \tag{6}$$

Here, the scale R is typically set by the smallest intrinsic or extrinsic curvature radius of the worldvolume.

In order to preserve manifest diffeomorphism invariance it is convenient to introduce some gauge redundancy and enlarge the set of embedding coordinates of the worldvolume of the black brane to include all the spacetime coordinates $X^\mu(\sigma^a)$. From this embedding one can compute an induced metric

$$\gamma_{ab} = g^{(\mathrm{long})}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \,, \tag{7}$$

which is the geometry induced on the worldvolume of the brane by the far-zone ($r \gg r_0$) background metric $g^{(\mathrm{long})}_{\mu\nu}$. The near-zone ($r \ll R$) metric is the solution (4) in the strict $R \to \infty$ limit. However, for R large but finite, the collective coordinates depend on σ, so that the near-zone metric for the black brane is of the form

$$ds^2_{(\mathrm{short})} = \left(\gamma_{ab}(\sigma) + \frac{r_0^n(\sigma)}{r^n}u_a(\sigma)u_b(\sigma)\right)d\sigma^a d\sigma^b + \frac{dr^2}{1-\frac{r_0^n(\sigma)}{r^n}} + r^2 d\Omega^2_{n+1} + \ldots \tag{8}$$

which is an approximate solution to Einstein equations in the limit $\frac{r_0}{R} \ll 1$. Furthermore, the long and short degrees of freedom interact together in the 'overlap' or 'matching-zone' $r_0 \ll r \ll R$, where the metrics $g^{(\mathrm{long})}_{\mu\nu}$ and $g^{(\mathrm{short})}_{\mu\nu}$ must match.

Note that we are considering here black branes that are not rotating along the transverse $(n+1)$-sphere, nor do we include any possible deformations of it. This choice is solely made in order to simplify the analysis. As discussed in more detail in Ref. [2] internal spin can be added as a further conserved charge on the worldvolume. Other generalizations such as lumpy blackfolds can be considered as well.

2.2. Effective stress tensor

The way in which the short-distance dynamics is integrated out is by solving Einstein equations at distances $r \ll R$ and encoding the effects of the solution at distances $r \gg r_0$ in an effective stress tensor that depends only on the collective coordinates. In particular, the stress tensor is such that its effect on the long-wavelength field $g^{(\mathrm{long})}_{\mu\nu}$ is the same as that of the black brane at distances $r \gg r_0$.

If we consider the effective equations of motion resulting from (3) by varying with respect to the long-wavelength metric $g^{(\mathrm{long})}_{\mu\nu}$, one finds Einstein equations for $g^{(\mathrm{long})}_{\mu\nu}$, sourced by an effective worldvolume stress tensor

$$T^{\mathrm{eff}}_{\mu\nu} = -\frac{2}{\sqrt{-g_{(\mathrm{long})}}}\frac{\delta I_{\mathrm{eff}}}{\delta g^{\mu\nu}_{(\mathrm{long})}}\bigg|_{\mathcal{W}_{p+1}} . \tag{9}$$

The appropriate notion of this effective stress-tensor can be argued [2] to be the quasilocal stress-energy tensor introduced by Brown and York [25]. For ease of notion, we drop from now on the superscript "eff" on the stress-tensor $T_{\mu\nu}$ and "(long)" from the background metric $g_{\mu\nu}$.

The effective stress tensor is computed in the zone $r_0 \ll r \ll R$, where the gravitational field is weak and the quasilocal stress tensor $T_{ab} = \partial_a X^\mu \partial_b X^\nu T_{\mu\nu}$ is, to leading order in r_0/R, the same as the ADM stress tensor. It is not difficult to compute this for the boosted black p-brane (4), and introducing a slow variation of the collective coordinates the stress tensor becomes

$$T^{ab}(\sigma) = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n(\sigma) \left(n u^a(\sigma) u^b(\sigma) - \gamma^{ab}(\sigma) \right) + \ldots \,. \tag{10}$$

Here the dots stand for terms with gradients of $\ln r_0$, u^a, and γ_{ab}, which are taken to be small and are neglected to the order we are working in.

2.3. Fluid perspective

We note that the effective stress-tensor (10) is of the perfect fluid form

$$T^{ab} = (\varepsilon + P) u^a u^b + P \gamma^{ab} \,, \tag{11}$$

with energy density ε and pressure P

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} (n+1) r_0^n \,, \qquad P = -\frac{1}{n+1}\, \varepsilon \,, \tag{12}$$

and velocity u^a satisfying $u^a u^b \gamma_{ab} = -1$. This is in fact expected on general grounds, since the the long-wavelength effective theory for any kind of brane will take the form of a derivative expansion for an effective stress-energy tensor that satisfies the conservation equations $D_a T^{ab} = 0$ (D_a is the worldvolume covariant derivative). To lowest derivative order the stress tensor will then be that of an isotropic perfect fluid.

Thermodynamics provides the universal macroscopic description of equilibrium configurations, and fluid dynamics is the general long-wavelength description of fluctuations under the assumption of local equilibrium. The equation of state $P(\varepsilon)$ for a neutral black brane is thus given in (12) and the system in its rest frame has local entropy density and temperature

$$s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \,, \qquad \mathcal{T} = \frac{n}{4\pi r_0} \,. \tag{13}$$

These obey the laws of thermodynamics and Gibbs–Duhem equation

$$d\varepsilon = \mathcal{T} ds \,, \qquad \varepsilon + P = \mathcal{T} s \,. \tag{14}$$

Going beyond the perfect fluid approximation (11), the stress tensor will acquire dissipative terms proportional to gradients of $\ln r_0$, u^a, γ^{ab}. These are not discussed here, and are, at any rate, absent for stationary configurations.

3. Blackfold dynamics

Part of the general effective theory of blackfold dynamics can thus be formulated as a theory of a fluid on a dynamical worldvolume. We now present the full dynamical equations that this system should satisfy and apply it to the blackfold approach.

3.1. Blackfold equations

Before we present the dynamical equations for branes, it is instructive to consider the well-known case of a free point particle with mass m moving along a worldline $x^\mu(\tau)$ in a background spacetime $g_{\mu\nu}$. The induced metric on the wordline is $\gamma_{\tau\tau} = \partial_\tau x^\mu \partial_\tau x^\nu g_{\mu\nu} = u^\mu u^\nu g_{\mu\nu} = -1$. The stress-tensor on the worldline is $T^{\tau\tau} = m$ so that

$$T^{\mu\nu} = \partial_\tau x^\mu \partial_\tau x^\nu T^{\tau\tau} = m u^\mu u^\nu \,. \tag{15}$$

We demand conservation of this stress tensor under the tangential covariant derivative $\overline{\nabla}_\mu = -u_\mu u^\nu \nabla_\nu$. This gives

$$0 = \overline{\nabla}_\mu T^{\mu\nu} = m a^\nu + (D_\tau m) u^\mu \,, \tag{16}$$

where the acceleration is $a^\mu = u^\rho \nabla_\rho u^\mu \equiv \dot{u}^\mu$ and we used $u^\mu a_\mu = 0$. We therefore find two sets of equations,

$$m a^\nu \;=\; 0 \qquad \text{(geodesic equation)}\,, \tag{17}$$

$$D_\tau m \;=\; 0 \qquad \text{(mass conservation)}\,, \tag{18}$$

corresponding to the geodesic equation and conservation of mass, respectively.

The above can be generalized to the case of extended objects, branes, employing various notions about the geometry of worldvolume embeddings. In particular, using the induced metric (7) one introduces the first fundamental form of the submanifold and perpendicular projector

$$h^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \gamma^{ab} \,, \qquad \perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} \,. \tag{19}$$

The tensor $h^\mu{}_\nu$ projects onto the worldvolume $\mathcal{W}_{p+1}$, while $\perp^\mu{}_\nu$ projects along directions orthogonal. From this one computes the extrinsic curvature tensor

$$K_{\mu\nu}{}^\rho = h_\mu{}^\sigma \overline{\nabla}_\nu h_\sigma{}^\rho \,, \qquad \overline{\nabla}_\mu = h_\mu{}^\nu \nabla_\nu \,, \tag{20}$$

which is tangent to $\mathcal{W}_{p+1}$ along its (symmetric) lower indices μ, ν, and orthogonal to $\mathcal{W}_{p+1}$ along ρ. For use below, we also note that its trace is the mean curvature vector

$$K^\rho = h^{\mu\nu} K_{\mu\nu}{}^\rho = \overline{\nabla}_\mu h^{\mu\rho} \,. \tag{21}$$

We refer for further details of this analysis to Ref. [2], and the original paper of Carter [21].

Given a brane stress-energy tensor with support on the $p+1$-dimensional worldvolume $\mathcal{W}_{p+1}$ one then requires it to obey the conservation equation

$$\overline{\nabla}_\mu T^{\mu\rho} = 0 \,. \tag{22}$$

This follows from the assumption that this effective stress-energy tensor derives from an underlying conservative dynamics (in the present case, General Relativity) and that spacetime diffeomorphism invariance holds. In parallel with (17), (18) the D equations on the equal number of worldvolume field variables $\phi(\sigma^a)$, separate into $D-p-1$ equations in directions orthogonal to $\mathcal{W}_{p+1}$ and $p+1$ equations parallel to $\mathcal{W}_{p+1}$,

$$T^{\mu\nu} K_{\mu\nu}{}^\rho \;=\; 0 \qquad (\text{extrinsic equations})\,, \tag{23}$$

$$D_a T^{ab} \;=\; 0 \qquad (\text{intrinsic equations})\,. \tag{24}$$

One may now apply these equations to the generic stress-energy tensor of a perfect fluid and the results can be found in *e.g.* Ref. [2].

Here, we use the specific stress tensor (12) of a neutral black brane, in which case the equations (23), (24) become after some algebra

$$K^\rho \;=\; n \perp^\rho{}_\mu \dot{u}^\mu \,, \tag{25}$$

$$\dot{u}_a + \frac{1}{n+1} u_a D_b u^b \;=\; \partial_a \ln r_0 \,. \tag{26}$$

This set of *blackfold equations* describes the general collective dynamics of a neutral black brane. Physically, these equations show that the force that accelerates an element of the fluid is given along worldvolume directions by pressure gradients (Euler equation) and in directions transverse to the worldvolume by the extrinsic curvature.

We also note that the extrinsic equations (23) can be written explicitly in terms of the embedding $X^\mu(\sigma^a)$, yielding the form

$$T^{ab}\left(D_a \partial_b X^\rho + \Gamma^\rho_{\mu\nu} \partial_a X^\mu \partial_b X^\nu\right) = 0 \,, \tag{27}$$

which can be regarded as the generalization to p-branes of geodesic equation (17) for free particles, or more simply, of “mass $\times$ acceleration $= 0$”.

One of the differences of blackfolds as compared to other branes is that they have event horizons, which is reflected in the effective theory in the existence of an entropy and in the local thermodynamic equilibrium of the effective fluid. In the blackfold approach it is assumed that the regularity of the event horizon under long-wavelength perturbations — including those that bend the worldvolume away from the flat geometry or that excite the effective fluid away from equilibrium — is satisfied when the blackfold equations are satisfied. There is so far significant evidence that this is the case. Analysis of the perturbations of black strings that bend them into a circle [13] (and extensions thereof to branes curved into tori [18]) show that the extrinsic equations (23) are equivalent to demanding absence of singularities on, or outside the horizon. Moreover, the intrinsic, hydrodynamical perturbations of a black brane in AdS have been studied in detail in [22] and shown to be consistent with horizon regularity. A rigorous proof of the statement is, however, at present not known.

We finally note that in (2) and (3) it was assumed that the full dynamics at all wavelengths is described by vacuum General Relativity, *i.e.* the Einstein–Hilbert action with no matter nor cosmological constant. However, it can be shown [2] that the blackfold equations (25), (26) are enough to describe neutral blackfolds in any configuration that, at small distances, is dominated by the Einstein–Hilbert term. For instance, this will be the case for blackfolds in the presence of a cosmological constant as long as $r_0 \ll |\Lambda|^{-1/2}$ (see [16] for an explicit application), or for blackfolds in an external background gauge field as long as the typical length scale of the background field around the blackfold is much larger than r_0. No restriction on R other than $R \gg r_0$ needs to be imposed. On the other hand, for charged blackfolds, where the gauge field has short-wavelength components, the situation is different. The extension to this will be discussed in a future work.

4. Stationary blackfolds

We now turn to equilibrium blackfold configurations that remain stationary in time, which correspond to stationary black holes.

4.1. Solution of the intrinsic equations

Using a general result proven in [26] for stationary fluid configurations, it is possible to solve the blackfold equations explicitly for the worldvolume variables [2], namely the thickness r_0 and velocity u, so one is left only with the extrinsic equations (25) for the worldvolume embedding $X^\mu(\sigma)$.

The solution to the intrinsic equations (26) in this case can be summarized by

$$u^\mu = \frac{\boldsymbol{k}^\mu}{|\boldsymbol{k}|}\,, \qquad r_0 = \frac{n|\boldsymbol{k}|}{2\kappa}\,. \tag{28}$$

Here, $\boldsymbol{k}$ is a time-like Killing vector of the background spacetime

$$\boldsymbol{k} = \xi + \sum_i \Omega_i \chi_i\,, \tag{29}$$

where ξ is the (canonically normalized) generator of time-translations of the background space-time and χ_i are generators of angular rotations in the background space-time normalized such that the orbits have periods 2π. The Ω_i are thus the corresponding angular velocities. The constant κ in (28) is the surface gravity of the black hole horizon.

To have a stationary blackfold the Killing vector fields ξ and χ_i should correspond to symmetries of the submanifold $\mathcal{W}_{p+1}$. In a local rest frame on the blackfold we can relate ξ and χ_i to Killing vectors of the static black p-brane by

$$\xi = R_0 \frac{\partial}{\partial t}\,, \qquad \chi_i = R_i \frac{\partial}{\partial z^i}\,, \tag{30}$$

where t and z^i are the canonically normalized world-volume coordinates of a static flat black p-brane. R_0 is a red-shift factor between the background space-time and the blackfold worldvolume and R_i are the proper radii of the orbits of χ_i. With this, the norm of $\boldsymbol{k}$ is seen to be

$$|\boldsymbol{k}| = R_0\sqrt{1-V^2}\,, \qquad V^2 = \frac{1}{R_0^2}\sum_i \Omega_i^2 R_i^2\,, \tag{31}$$

where V is the velocity field of the blackfold. Thus $|\boldsymbol{k}|$ can be regarded as the relativistic Lorentz factor at a point in $\mathcal{W}_{p+1}$, with a possible local redshift, all relative to the reference frame of ξ-static observers. Plugging (31) into (28) we obtain that for given κ and Ω_i the thickness r_0 is solved in terms of the R_a as

$$r_0(\sigma) = \frac{nR_0(\sigma)}{2\kappa}\sqrt{1-V^2(\sigma)}\,. \tag{32}$$

The conditions that κ and Ω_i must remain uniform over the blackfold worldvolume were referred to in [1] as the blackness conditions. There they were imposed by requiring regularity of the black hole horizon, invoking general theorems for stationary black holes, namely the zeroth law of black hole mechanics and horizon rigidity. Alternately, as summarized above and shown in [2], one can derive them as general consequences of stationary fluid dynamics, where κ and Ω_i appear as integration constants.

We finally note that black branes (and other fluid branes) may have 'free' boundaries without any boundary stresses. In this case, it can be shown that the pressure must approach zero at the boundary $P|_{\partial\mathcal{W}_{p+1}} = 0$, which is the Young–Laplace equation for a bounded fluid when there is no surface tension that could balance the fluid pressure at the boundary. For a neutral blackfold, vanishing pressure at the boundary means

$$r_0|_{\partial\mathcal{W}_{p+1}} = 0\,. \tag{33}$$

The geometric interpretation of this is that the thickness of the horizon must approach zero size at the boundary, so that the horizon closes off at the edge of the blackfold. Using (32) we see that for stationary blackfolds this condition implies[2] that the fluid approaches the speed of light at the boundary,

$$V^2|_{\partial\mathcal{W}_{p+1}} = 1\,. \tag{34}$$

This happens for example for Myers–Perry black holes with one ultraspin, which can be shown [1, 18] to be described by a rigidly-rotating blackfold disk.

4.2. Horizon geometry, mass and angular momenta

The blackfold construction puts, on any point in the spatial section $\mathcal{B}_p$ of $\mathcal{W}_{p+1}$, a (small) transverse sphere s^{n+1} with Schwarzschild radius $r_0(\sigma)$. Thus the blackfold represents a black hole with a horizon geometry that is a product of $\mathcal{B}_p$ and s^{n+1} — the product is warped since the radius of the s^{n+1} varies along $\mathcal{B}_p$. The null generators of the horizon are proportional to the velocity field u.

If r_0 is non-zero everywhere on $\mathcal{B}_p$ then the s^{n+1} are trivially fibered on $\mathcal{B}_p$ and the horizon topology is

$$(\text{topology of } \mathcal{B}_p) \times s^{n+1}\,. \tag{35}$$

However, if $\mathcal{B}_p$ has boundaries, as discussed above, then r_0 will shrink to zero size at them, resulting in a non-trivial fibration and different topology. A simple but very relevant instance of this happens when $\mathcal{B}_p$ is a topological p-ball. Then the horizon topology can easily be seen to be $S^{p+n+1} = S^{D-2}$.

[2] It may also happen because the boundary is an infinite-redshift surface, $R_0 \to 0$.

The physical properties of stationary blackfolds can be obtained by integrating appropriate quantities over the spatial section $\mathcal{B}_p$ of the brane worldvolume. As a consequence, the mass, angular momenta and entropy are given by

$$M = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int\limits_{\mathcal{B}_p} dV_{(p)}\, R_0^{n+1}(1-V^2)^{\frac{n-2}{2}}\left(n+1-V^2\right), \quad (36)$$

$$J_i = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n n\Omega_i \int\limits_{\mathcal{B}_p} dV_{(p)}\, R_0^{n-1}(1-V^2)^{\frac{n-2}{2}} R_i^2\,, \quad (37)$$

$$S = \frac{A_H}{4G} = \frac{\Omega_{(n+1)}}{4G}\left(\frac{n}{2\kappa}\right)^{n+1} \int\limits_{\mathcal{B}_p} dV_{(p)}\, R_0^{n+1}(1-V^2)^{n/2}\,. \quad (38)$$

An immediate physical consequence that can be derived from the expressions above is that

$$a(j) \sim j^{-\frac{p}{D-3-p}}\,, \quad (39)$$

where $j \sim \ell_J/\ell_M$ and $a \sim A_H/M^{D-2}$. Here, it is assumed for simplicity that all length scales along $\mathcal{B}_p$ are of order R and that the velocities and redshift are moderate. This shows that for a given number of large non-zero angular momenta, the blackfold with smallest p is entropically preferred at fixed mass.

4.3. Action principle and first law of stationary blackfolds

Having solved the intrinsic equations (26) in the form (28), we are left with the extrinsic equations (25), where we can now eliminate the variables $r_0(\sigma)$ and $u^a(\sigma)$ in terms of the embedding functions $X^\mu(\sigma)$. The remaining extrinsic equation that determines the embedding can then be written as

$$K^\rho = n \perp^{\rho\mu} \partial_\mu \log(R_0\sqrt{1-V^2})\,, \quad (40)$$

where $\perp^{\rho\mu}$ (see (19)) projects along directions perpendicular to the worldvolume. These are thus $D-p-1$ equations, the solutions of which describe stationary blackfolds to leading order in the 'test blackfold' approximation.

The extrinsic blackfold equations (40) can equivalently be found be varying the action [2]

$$I = \beta \int\limits_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}\left(1-V^2\right)^{n/2}\,, \quad (41)$$

where $\mathcal{B}_p$ is the spatial section of $\mathcal{W}_{p+1}$ and $dV_{(p)}$ is the integration measure on $\mathcal{B}_p$. The action (41) is in many applications the most efficient way to find solutions of the stationary blackfold equations (40).

Interestingly, the action (41) is directly related to the effective action that approximates, in the blackfold regime $r_0/R \ll 1$, the gravitational Euclidean action of the black hole

$$\beta^{-1} I = G = M - \sum_i \Omega_i J_i - TS\,. \tag{42}$$

This can be easily checked using the expressions (36)–(38), and holds for any embedding, not necessarily a solution to the extrinsic equations. If we regard M, J_i and S as functionals of the embedding $X^\mu(\sigma)$, it follows that extrema of the action (42) satisfy the first law of thermodynamics $dM = TdS + \Omega_i dJ_i$. We thus see that solutions of the blackfold equations satisfy the 'equilibrium state' version of the first law of black hole mechanics. Conversely, the blackfold equations for stationary configurations can be obtained as the requirement that the first law be satisfied.

5. Explicit solutions

We discuss here some explicit solutions [1,18] of the stationary blackfold equations. The first class of these is new, and includes the thin black ring solutions of [13] as a special case. The second class shows that ultraspinning Myers–Perry black holes are correctly captured by the blackfold approach.

5.1. Odd-sphere blackfolds

Consider a single odd-sphere, $\mathcal{B}_p = S^{2k+1}$, which we embed in a $2k+2$-dimensional flat subspace of $\mathbb{R}^{D-1}$ with metric

$$d\rho^2 + \rho^2 \sum_{i=1}^{k+1} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right)\,, \qquad \sum_{i=1}^{k+1} \mu_i^2 = 1\,. \tag{43}$$

The sphere is embedded as $\rho = R$ and the worldvolume spatial coordinates can be taken to be k independent μ_i plus the $k+1$ Cartan angles ϕ_i. Then we have $R_i = R\mu_i$ and $R_0 = 1$. We furthermore assume that all the angular velocities along the ϕ_i are equal in magnitude, $|\Omega_i| = \Omega$. From (31) it then follows that the velocity $V = R\Omega$ is uniform over the blackfold, and so is the thickness r_0. Computing the action (41) for this case and extremizing one finds the equilibrium condition

$$R = \sqrt{\frac{p}{n+p}}\frac{1}{\Omega} \tag{44}$$

(for $p = 1$ we recover the result for black rings in [13]). The horizon geometry is $\mathcal{H} = S^{2k+1} \times s^{n+1}$. One may also consider non-equal angular velocities. Then the radius $\rho = R(\mu_i)$ depends non-trivially on μ_i and one must solve a second-order differential equation, which requires numerical analysis.

This is easily generalized to products of odd-spheres, $\mathcal{B}_p = \prod_{p_a \in \mathrm{odd}} S^{p_a}$, $p = \sum_a p_a$, where we take each odd-sphere to have constant radius R_a, and let each odd-sphere rotate along all its Cartan angles with angular velocities equal in magnitude to $\Omega^{(a)}$. The equations of equilibrium factorize for each sphere and are solved for

$$R_a = \sqrt{\frac{p_a}{n+p}} \frac{1}{\Omega^{(a)}} \,. \tag{45}$$

The horizon geometry is $\mathcal{H} = \prod_{p_a \in \mathrm{odd}} S^{p_a} \times s^{n+1}$, and the mass, angular momenta, and area of the blackfold are easily obtained plugging these results in the general formulas above.

5.2. Ultraspinning Myers–Perry black holes as even-ball blackfolds

It is not possible to find blackfold equations in a Minkowski background for $\mathcal{B}_p$ a topological even-sphere. The reason is that the tension at fixed-points of the rotation group cannot be counterbalanced by centrifugal forces in this case. Instead, there exist solutions where $\mathcal{B}_p$ is an ellipsoidal even-ball, with thickness r_0 vanishing at the boundary of the ball so the horizon topology is S^{D-2}. These reproduce precisely all the physical properties of a Myers–Perry black hole with $p/2$ ultra-spins. This provides a highly non-trivial check on the approach and also shows that the method remains sensible when the rotation has fixed-points, in this case at the center of the ball. They also provide an explicit example of blackfolds with varying thickness $r_0(\sigma^\alpha)$.

As an illustration, we discuss here the simplest non-trivial case of $p = 2$ given originally in [1]. The general case can be found in [18]. We start by considering a black 2-fold extending along a plane $dr^2 + r^2 d\phi^2$ in Minkowski space, which trivially solves the blackfold equations (40). We embed $\mathcal{B}_2$ as $\sigma^1 = \phi$, $\sigma^2 = r$, and introduce a rigid brane rotation Ω along the ϕ direction, so that the velocity is $V = r\Omega$. Comparing with (34), it follows that this becomes light-like at $r = 1/\Omega$, so that the horizon r_0 closes off at that radius. As a consequence, $\mathcal{B}_2$ becomes the disk $0 \le r \le \Omega^{-1}$.

Computing the physical quantities of this blackfold using (36)–(38) one finds that these reproduce exactly the values for an ultraspinning Myers–Perry black hole in $D = n + 5$ dimensions, with a single spin parameter $a = 1/\Omega$ and with horizon radial coordinate $r_+ = n/(2\kappa)$, to leading order in r_+/a [12]. The shape of the horizon is also accurately reproduced, and in both cases a is the horizon radius in the plane parallel to the rotation.

6. Instabilities

The blackfold approach can be used to study the perturbative dynamics of a black hole when the perturbation wavelength is long $\lambda \gg r_0$. Such perturbations can be either intrinsic variations in the thickness r_0 and local velocity u, or extrinsic variations in the worldvolume embedding geometry X. In general, these two kinds of perturbations are coupled, but for perturbations with wavelength

$$r_0 \ll \lambda \ll R\,, \tag{46}$$

the worldvolume looks essentially flat and the intrinsic and extrinsic perturbations decouple. A standard analysis for the general case of a perfect fluid brane, initially at rest, shows that in this regime the transverse, elastic oscillations of the brane propagate with speed

$$c_{\mathrm{T}}^2 = -\frac{P}{\varepsilon}\,, \tag{47}$$

and the longitudinal, sound-mode oscillations of the fluid propagate with speed

$$c_{\mathrm{L}}^2 = \frac{dP}{d\varepsilon}\,. \tag{48}$$

These follow from the extrinsic and intrinsic equations respectively (see *e.g.* [27, 28])

A simple consequence is that for equations of state $P = w\varepsilon$ with constant w, the fluid is unstable to either longitudinal or transverse oscillations. This applies in particular to the case of neutral blackfolds for which

$$c_{\mathrm{L}}^2 = -c_{\mathrm{T}}^2 = -\frac{1}{n+1}\,, \tag{49}$$

so these are generically unstable to longitudinal sound-mode oscillations and stable to elastic oscillations in the range of wavelengths (46). This instability is a manifestation of Gregory–Laflamme instability [6, 29] of black branes. In particular, it should be related to the long wavelength component of the horizon perturbation

$$\delta r_0 \sim e^{\Omega t + i k_i z^i}\,. \tag{50}$$

The dispersion relation predicted by (49) in the limit of small k,

$$\Omega = \frac{1}{\sqrt{n+1}}\, k\,, \tag{51}$$

is in good numerical agreement with the slope at the origin in figure 1 of [29].

We also note that using (14) one may rewrite (51) as

$$\Omega = \sqrt{\frac{s}{|c_v|}}\, k\,, \tag{52}$$

where c_v is the isovolumetric specific heat. This shows that the black brane is dynamically unstable (to long-wavelength GL modes) if and only if it is locally thermodynamically unstable, $c_v < 0$. This is precisely the content of the 'correlated stability conjecture' of Gubser and Mitra [30]. The blackfold approach thus seems a powerful and simple method to examine instability of branes and shows the correlation between dynamical and thermodynamical stability, using the thermodynamics of the effective fluid.

7. Outlook

We conclude with a brief outlook.

The formalism reviewed in this lecture resembles to some extent two different earlier effective descriptions of black hole dynamics. The extrinsic part is a generalization to p-branes of the effective worldline formalism for small black holes [31–35]. The intrinsic part is similar to other fluid-dynamical formalisms for horizon fluctuations, such as the membrane paradigm [23, 24] and the fluid/AdS-gravity correspondence [22]. In parallel with these developments, it should be possible to produce a systematic derivation of the blackfold equations, which would allow to go beyond 'perfect fluid' and 'generalized geodesic' approximation and account for dissipation and effects of internal structure and gravitational self-force. The study of higher order corrections within the blackfold approach should also shed further light on the validity of our assumption that the blackfold equations ensure regularity of the event horizon.

We have restricted here to neutral blackfolds, but the method can be readily generalized to charged blackfolds (to be discussed in future work) as well as other backgrounds (see *e.g.* [16]). The construction of charged blackfolds is bound to have many interesting application in string theory and will be useful to further elucidate the relation of the blackfold approach to the fluid/AdS-gravity correspondence. It would also be interesting to study possible connections with the extremal limit, which introduces a long length scale transverse to the horizon and allows to decouple a different sector of the physics, namely the near-horizon region [36].

We have presented some of the most simple solutions of the stationary blackfold equations, so it would be interesting to examine more complicated novel solutions (see also [18]). Finally, a more detailed analysis of the perturbations of solutions to the blackfold equations and their stability, in the regime where intrinsic and extrinsic perturbations are coupled, is expected to give new insights into the phase structure and stability of higher-dimensional black holes.

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