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NEW ASPECTS OF TWO-DIMENSIONAL QUANTUM GRAVITY*

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Causal dynamical triangulations (CDT) can be used as a regularization of quantum gravity. In two dimensions the theory can be solved anlytically, even before the cut-off is removed and one can study in detail the how to take the continuum limit. We show how the CDT theory is related to Euclidean 2d quantum gravity (Liouville quantum gravity), how it can be generalized and how this generalized CDT model has a string field theory representation as well as a matrix model representation of a new kind, and finally how it examplifies the possibility that time in quantum gravity might be the stochastic time related to the branching of space into baby universes.

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1. Introduction

Presumably quantum gravity makes sense as an ordinary effective quantum field theory at low energies. At high energies it is presently unclear how to view space-time. Is space-time an emergent low-energy structure as advocated in string theory, does it require a new concept of quantization as believed by people working on loop quantum gravity, or is quantum gravity "just" an almost standard quantum field theory with non-trivial (non-perturbative) ultraviolet behaviour? As long as we do not know the answer we have an obligation to pursue all avenues.

The theory of quantum gravity which starts by providing an ultraviolet regularization in the form of a lattice theory, the lattice link length being the (diffeomorphism invariant) UV cut-off, and where in addition the lattice respects causality, is denoted CDT (causal dynamical triangulations). It is formulated in the spirit of the last of the three approaches mentioned above: old "boring" quantum field theory with a non-trivial fixed point [1,2]. It allows a rotation to Euclidean space-time, the action used being the Euclidean Regge action for the piecewise linear geometry represented by the (now) Euclidean lattice (see [3,4] for details of the Regge action in the CDT approach). The non-perturbative path integral is performed by summing over lattices originating from Lorentzian lattices with a causal structure. Like in ordinary lattice field theories we approach the continuum by fine-tuning the bare coupling constants. The rotation to Euclidean space-time allows the use of Monte Carlo simulations of the theory and in four-dimensional space-time, which for obvious reasons has our main interest, there exists a region of coupling constant space where the infrared behaviour of the universe seen by the computer is that of (Euclidean) de Sitter space-time [5,6] (for a pedagogical review, see [7]). One might think this is a trivial result, but quite the contrary: in Euclidean space-time the Einstein action is unbounded from below and the de Sitter solution is only a saddle point. Thus we are clearly dealing with an "emergent" property of the path integral, a genuine non-perturbative effect arising from an interplay between the regularization and the path integral measure, and it can only be valid in some limited region of the (bare) coupling constant space. This is precisely what is observed. One of the main questions to be answered in such a lattice theory is what happens when the lattice spacing is taken to zero, *i.e.* when the cut-off is removed. Is it possible at all to remove the cut off, and if so what are the ultraviolet properties of the theory? Non-trivial UV properties have been observed [8], properties which have been reproduced by other "field theoretical" approaches to quantum gravity [9,10]. However, in the CDT-lattice approach it has so far been difficult to penetrate into the trans-Planckian region. Active research is ongoing to achieve precisely this.

Numerical simulations are very useful for understanding whether a non-perturbatively defined quantum field theory has a chance to make sense. Likewise it is useful for checking if certain conjectured non-perturbative features of the theory have a chance of being true, and one can even discover new, unexpected phenomena. In this way, the numerical simulations work like experiments, and this is the spirit in which the above mentioned simulations have been conducted. However, numerical simulations have their limitations in the sense that they will never provide a proof of the existence of a theory and it might be difficult in detail to follow the way the continuum limit is approached since it requires larger and larger lattices. It is thus of interest and importance to be able to study this in detail, even if only in a toy model. Two-dimensional quantum gravity is such a toy model which has a surprising rich structure. Many of the intriguing questions in quantum gravity and in lattice quantum gravity are still present in the two-dimensional theory. What is nice about the two-dimensional theory is that it can be solved analytically even at the discretized level.

In the rest of this article we will discuss the two-dimensional CDT theory. In Sec. 2 we define the "bare" model. In Sec. 3 we generalize the model, allowing for local causality violations and we discuss the generalized model's relation to a specific matrix model (Sec. 4). What is new is that the matrix model directly describes the continuum limit of the gravity theory. In Sec. 5 we show how to formulate a complete string field theory for the model and in Sec. 6 we show the equivalence to the matrix model defined in Sec. 4. The matrix model allows us to define the theory non-perturbatively and we show how to calculate non-perturbatively (*i.e.* including the summation over all topologies) certain observables. In Sec. 7 we discuss relations to other models. In Sec. 8 we show that the string field theory can be understood as a special kind of stochastic quantization of space, a phenomenon first noticed in the context of 2d Liouville quantum gravity in [11]. Stochastic quantization defines a non-perturbative Hamiltonian. This is discussed in Sec. 9.

2. The CDT formalism

CDT stands in the tradition of [12], which advocated that in a gravitational path integral with the correct, Lorentzian signature of space-time one should sum over causal geometries only. More specifically, we adopted this idea when it became clear that attempts to formulate a *Euclidean* nonperturbative quantum gravity theory run into trouble in space-time dimension d larger than two. Here we will discuss the implementation only when the space-time dimension is two.

Thus we start from Lorentzian simplicial space-times with $d = 2$ and insist that only causally well-behaved geometries appear in the (regularized) Lorentzian path integral. A crucial property of our explicit construction is that each of the configurations allows for a rotation to Euclidean signature as mentioned above. We rotate to a Euclidean regime in order to perform the sum over geometries (and rotate back again afterward if needed). We stress here that although the sum is performed over geometries with Euclidean signature, it is different from what one would obtain in a theory of quantum gravity based *ab initio* on Euclidean space-times. The reason is that not all Euclidean geometries with a given topology are included in the "causal" sum since, in general, they have no correspondence to a causal Lorentzian geometry.

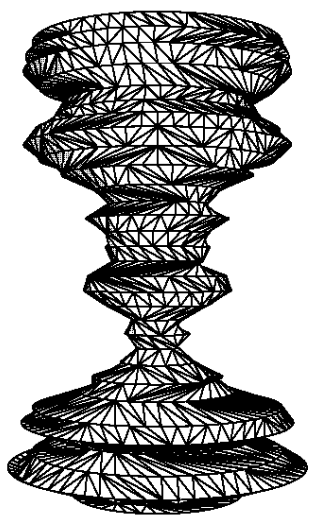

Fig. 1. Piecewise linear space-time histories 1+1 dimensional quantum gravity.

We refer to [3] for a detailed description of how to construct the class of piecewise linear geometries used in the Lorentzian path integral in higher dimensions. The most important assumption is the existence of a global proper-time foliation. This is illustrated in Fig. 1 in the case of two dimensions. We have a sum over two-geometries, "stretching" between two "one-geometries" separated a proper time t and constructed from two-dimensional building blocks. In Fig. 2 we have shown how to fill the two-dimensional space-time between the space (with topology S^1) at time t_n and time $t_{n+1} = t_n + a$ where a denotes the lattice spacing. While we in the lattice model often use units where everything is measured in lattice length (*i.e.* the lattice links have length one), we are, of course, interested in taking the limit $a \to 0$ to recover continuum physics.

In the path integral we will be summing over all possible ways to connect a given 1d "triangulation" at time t_n and a given 1d triangulation at t_{n+1} to a slab of 2d space-time as shown in Fig. 2, and in addition we will sum over all 1d "triangulations" of S^1 at times t_n. Thus we are demanding that the time-slicing is such that the topology of space does not change when space "evolves" from time t_n to time t_{n+1}.

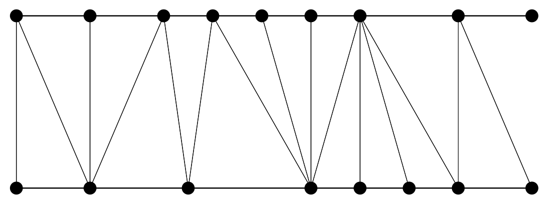

Fig. 2. The propagation of a spatial slice from time t to time $t+1$. The ends of the strip should be joined to form a band with topology $S^1 \times [0,1]$.

The Einstein–Hilbert action S^{EH} in two dimensions is almost trivial. According to the Gauss–Bonnett theorem the curvature term is a topological invariant and does not contribute to the equations of motion as long as the topology of space-time is unchanged. And even if the topology changes the change in the curvature term is just a number, the change in the Euler characteristic of the 2d surface. We will first ignore this term, since we are first not allowing topology change. Thus the (Euclidean) action simply consists of the cosmological term:

$$S_{\mathrm{E}}^{\mathrm{EH}} = \lambda \int d^2x\, \sqrt{g} \longrightarrow S_{\mathrm{E}}^{\mathrm{Regge}} = \Lambda N_2\,, \tag{1}$$

where N_2 denotes the total number of triangles in the two-dimensional triangulation. We denote the discretized action *the Regge action* since it is a trivial example of the natural action for piecewise linear geometries introduced by Regge [13]. The dimensionless lattice cosmological coupling constant Λ will be related to the continuum cosmological coupling constant λ by an additive renormalization:

$$\Lambda = \Lambda_0 + \tfrac{1}{2}\lambda\, a^2\,, \tag{2}$$

the factor $1/2$ being conventional. The path integral or partition function for the CDT version of quantum gravity is now

$$\begin{aligned} G_\lambda^{(0)}(l_1,l_2;t) &= \int \mathcal{D}[g]\, e^{-S_{\mathrm{E}}^{\mathrm{EH}}[g]} \rightarrow \\ G_\Lambda^{(0)}(L_1,L_2,T) &= \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}}\, e^{-S_{\mathrm{E}}(\mathcal{T})}\,, \end{aligned} \tag{3}$$

where the summation is over all causal triangulations $\mathcal{T}$ of the kind described above with a total of T time steps, an "entrance loop" of length $l_1 = L_1 a$ and an "exit loop" of length $l_2 = L_2 a$, and where we have dropped the superscript "Regge" on the discretized action. The factor $1/C_{\mathcal{T}}$ is a symmetry factor, given by the order of the automorphism group of the triangulation $\mathcal{T}$.

Our next task is to *evaluate* the sum over triangulations in (3), if possible, analytically. Surprisingly, it can be done [14]. One can simply count the number of triangulations in a slice like the one shown in Fig. 2 and from this the total number of triangulations in T slices. As usual, when it comes to counting, it is often convenient to introduce the generating function for the numbers one wants to count and first find this function. In our model the generating function has a direct physical interpretation. We define

$$\tilde{G}^{(0)}_{\Lambda}(X_1, X_2; t) = \sum_{L_1, L_2} e^{-X_1 L_1}\, e^{-X_2 L_2} G^{(0)}_{\Lambda}(L_1, L_2; T)\,. \tag{4}$$

Thus $\tilde{G}^{(0)}_{\Lambda}(X_1, X_2; T)$ is the generating function of the numbers $G^{(0)}_{\Lambda}(L_1, L_2; T)$ if we write $Z_1 = e^{-X_1}$, $Z_2 = e^{-X_2}$. But we can also view X as a (bare) dimensionless boundary cosmological constant, such that a boundary cosmological term XL has been added to the action. In this way $\tilde{G}^{(0)}_{\Lambda}(X_1, X_2; T)$ represents the sum over triangulations where the length of the boundaries are allowed to fluctuate, the fluctuations controlled by the value X of the boundary cosmological constant. In general, we expect, just based on standard dimensional analysis, the boundary cosmological constants X_i to be subjected to an additive renormalization when the continuum limit is approached. Like (2) we expect

$$X = X_c + xa\,, \tag{5}$$

where x then denotes the continuum boundary cosmological constant, and one, after renormalization, has the continuum boundary cosmological action $x\,l$.

We refer to [14] for the explicit combinatorial arguments which allow us to find $\tilde{G}^{(0)}_{\Lambda}(X_1, X_2; T)$. Let us just state the following results: one can derive an exact iterative equation (using notation $Z = e^{-X}$, $W = e^{-Y}$, $Q = e^{-\Lambda}$)

$$\tilde{G}^{(0)}_{\Lambda}(Z, W; T) = \frac{QZ}{1 - QZ}\, \tilde{G}^{(0)}_{\Lambda}\left(\frac{Q}{1 - QZ}, W; T - 1\right). \tag{6}$$

This equation can be iterated and the solution written as

$$\begin{aligned}&\tilde{G}^{(0)}_{\Lambda}(Z, W; T)\\ &= \frac{F^{2t}(1 - F^2)^2\, ZW}{[(1-ZF) - F^{2t+1}(F-Z)]\,[(1-ZF)(1-WF) - F^{2T}(F-Z)(F-W)]}\,,\end{aligned} \tag{7}$$

where F is

$$F = \frac{1-\sqrt{1-4Q^2}}{2Q}\,. \tag{8}$$

These equations tell us that $Q_c = 1/2$ and that $Z_c = 1$ and we can now take the continuum limit in (7) using $t = T\,a$ and find

$$\begin{aligned}\tilde{G}^{(0)}_\lambda(x,y;t) \;=\; & \frac{4\lambda\; e^{-2\sqrt{\lambda}t}}{(\sqrt{\lambda}+x)+e^{-2\sqrt{\lambda}t}(\sqrt{\lambda}-x)} \\ & \times \frac{1}{(\sqrt{\lambda}+x)(\sqrt{\lambda}+y)-e^{-2\sqrt{\lambda}t}(\sqrt{\lambda}-x)(\sqrt{\lambda}-y)}\,. \end{aligned} \tag{9}$$

Further, the continuum version of (4):

$$\tilde{G}^{(0)}_\lambda(x_1,x_2;t) = \int_0^\infty dl_1 dl_2\, e^{-x_1 l_1 - x_2 l_2} G^{(0)}_\lambda(l_1,l_2;t) \tag{10}$$

allows us to obtain:

$$G^{(0)}_\lambda(l_1,l_2;t) = \frac{e^{-[\coth\sqrt{\lambda}t]\sqrt{\lambda}(l_1+l_2)}}{\sinh\sqrt{\lambda}t}\;\frac{\sqrt{\lambda l_1 l_2}}{l_2}\;\; I_1\left(\frac{2\sqrt{\langle l_1 l_2}}{\sinh\sqrt{\lambda}t}\right)\,, \tag{11}$$

where $I_1(x)$ is a modified Bessel function of the first kind. Quite remarkable this expression was first obtained using entirely continuum reasoning by Nakayama [15].

Finally, from (2) and (5) one can now obtain the continuum limit of the iteration equation (6):

$$\frac{\partial}{\partial t}\tilde{G}^{(0)}_\lambda(x,y;t) + \frac{\partial}{\partial x}\Big[(x^2-\lambda)\tilde{G}^{(0)}_\lambda(x,y;t)\Big] = 0\,, \tag{12}$$

This is a standard first order partial differential equation which should be solved with the boundary condition

$$\tilde{G}^{(0)}_\lambda(x,y;t=0) = \frac{1}{x+y} \tag{13}$$

corresponding to

$$G^{(0)}_\lambda(l_1,l_2;t=0) = \delta(l_1-l_2)\,. \tag{14}$$

The solution is thus

$$\tilde{G}^{(0)}_\lambda(x,y;t) = \frac{\bar{x}^2(t;x)-\lambda}{x^2-\lambda}\;\frac{1}{\bar{x}(t;x)+y}\,, \tag{15}$$

where $\bar{x}(t;x)$ is the solution to the characteristic equation

$$\frac{d\bar{x}}{dt} = -(\bar{x}^2 - \lambda)\,, \qquad \bar{x}(t=0) = x\,. \tag{16}$$

It is readily seen that the solution is indeed given by (9) since we obtain

$$\bar{x}(t) = \sqrt{\lambda}\,\frac{(\sqrt{\lambda}+x) - e^{-2\sqrt{\lambda}t}(\sqrt{\lambda}-x)}{(\sqrt{\lambda}+x) + e^{-2\sqrt{\lambda}t}(\sqrt{\lambda}-x)}\,. \tag{17}$$

If we interpret the propagator $G^{(0)}_\lambda(l_1,l_2;t)$ as the matrix element between two boundary states of a Hamiltonian evolution in "time" T,

$$G^{(0)}_\lambda(l_1,l_2;t) = \left\langle l_1 \right| e^{-H_0 t} \left| l_2 \right\rangle \tag{18}$$

we can, after an inverse Laplace transformation, read off the functional form of the Hamiltonian operator H_0 from (12),

$$\begin{aligned} \tilde{H}_0(x) &= \frac{\partial}{\partial x}\left(x^2 - \lambda\right)\,, \\ H_0(l) &= -l\frac{\partial^2}{\partial l^2} + \lambda l\,. \end{aligned} \tag{19}$$

This end our short review of basic 2d CDT. We have here emphasized that all continuum results can be obtained by explicit solving the lattice model and taking the continuum limit simply by letting the lattice spacing $a \to 0$. The same will be true for the generalized CDT model described below, but to make the presentation more streamlined we will drop the explicit route via a lattice and work directly in the continuum.

3. Generalized CDT

It is natural the ask what happens if the strict requirement of "classical" causality on each geometry appearing in the path integral is relaxed. While causality is a reasonable requirement as an outcome of a sensible physical theory, there is no compelling reason to impose it on each individual geometry in the path integral, since these are not physical observables. We used it, inspired by [12], as a guiding principle for obtaining a path integral which is different from the standard Euclidean path integral, which was seemingly a necessity in higher than two space-time dimensions.

In Fig. 3 we show what happens if we allow causality to be violated locally by allowing space to split in two at a certain time t, but we never allow the "baby" universe which splits off to come back to the "parent" universe. The baby universe thus continues its life and is assumed to vanish, shrink to nothing, at some later time. We now integrate over all such configurations in the path integral. From the point of view of Euclidean space-time we are simply integrating over all space-times with the topology of a cylinder. However, returning to the original Minkowskian picture it is clear that at the point, where space splits in two the light-cone is degenerate and one is violating causality in the strict local sense that each space-time point should have a future and a past light-cone. Similarly, when the baby universe "ends" its time evolution the light-cone structure is degenerate. These points thus have a diffeomorphism invariant meaning in space-times with Lorentzian structure, and it makes sense to associated a coupling constant g_s with the process of space branching in two disconnected pieces.

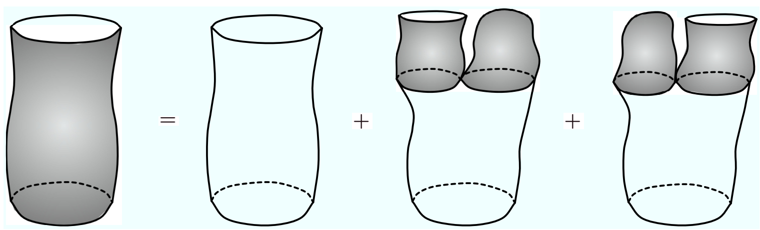

Fig. 3. In all four graphs, the geodesic distance from the final to the initial loop is given by t. Differentiating with respect to t leads to Eq. (20). Shaded parts of graphs represent the full, g_s-dependent propagator and disc amplitude, and non-shaded parts the CDT propagator.

The equation corresponding to Fig. 3 is [16]

$$\frac{\partial}{\partial t}\tilde{G}_{\lambda,g_s}(x,y;t) = -\frac{\partial}{\partial x}\Big[\Big((x^2-\lambda)+2g_s\, W_{\lambda,g_s}(x)\Big)\tilde{G}_{\lambda,g_s}(x,y;t)\Big]\,. \qquad (20)$$

$W_{\lambda,g_s}(x)$ is denoted the disk amplitude with a fixed cosmological constant x. It is related to the disk amplitude with a fixed boundary length by

$$\tilde{W}_{\lambda,g_s}(x) = \int_0^\infty dl\, e^{-xl} W_{\lambda,g_s}(l)\,. \qquad (21)$$

It describes the "propagation" of the a spatial universe until it vanishes in the vacuum. If we did not allow any spatial branching we would simply have

$$\tilde{W}^{(0)}_{\lambda}(x) = \int_0^\infty dt\; G^{(0)}_{\lambda}(x, l=0;t) = \frac{1}{x+\sqrt{\lambda}}\,, \qquad (22)$$

where $G_\lambda^{(0)}(x,l;t)$ denotes the Laplace transform of $G_\lambda^{(0)}(l',l;t)$ with respect to l'. From the composition rules for $G_{\lambda,g_s}(l_1,l_2;t)$ it follows that it has (mass) dimension 1. Thus $G_{\lambda,g_s}(x,l_2;t)$ is dimensionless and it follows that the (mass) dimension of the coupling constant g_s must be 3. In a discretized theory it will appear as the dimensionless combination $g_s a^3$, a being the lattice spacing, and one can show that the creation of more than one baby universe at a given time t is suppressed by powers of a (see [16] for details). Thus we only need to consider the process shown in Fig. 3. For a fixed cosmological constant λ and boundary cosmological constants x, y expressions like $\tilde{G}_{\lambda,g_s}(x,y;t)$ and $\tilde{W}_{\lambda,g_s}(x)$ will have a power series expansion in the dimensionless variable

$$\kappa = \frac{g_s}{\lambda^{3/2}} \tag{23}$$

and the radius of convergence is of the order of one. Thus the coupling constant g_s indeed acts to tame the creation of baby universes and if g_s exceeds this critical value Eq. (20) breaks down and is replaced by another equation corresponding to Liouville quantum gravity with central change $c = 0$ (see [16] for a detailed discussion).

Differentiating the integral equation corresponding to Fig. 3 with respect to the time t one obtains (20). The disc amplitude $W_{\lambda,g_s}(x)$ is at this stage unknown. However, one has graphical representation for the disc amplitude shown in Fig. 4. It translates into the equation [16]

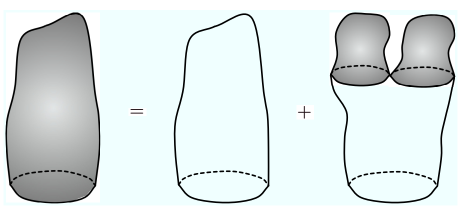

Fig. 4. Graphical illustration of Eq. (65). Shaded parts represent the full disc amplitude, unshaded parts the CDT disc amplitude and the CDT propagator.

$$\begin{aligned}\tilde{W}_{\lambda,g_s}(x) &= \tilde{W}_\lambda^{(0)}(x) \\ &+ g_s \int_0^\infty dt \int_0^\infty dl_1 dl_2\, (l_1+l_2) G_\lambda^{(0)}(x, l_1+l_2;t) W_{\lambda,g_s}(l_1) W_{\lambda,g_s}(l_2)\,. \end{aligned} \tag{24}$$

The superscript (0) indicates the CDT amplitudes without baby universe branching, calculated above. We assume

$$\tilde{W}_{\lambda,g_s=0}(x) = \tilde{W}_\lambda^{(0)}(x)\,, \tag{25}$$

and similarly for $G^{(0)}_{\lambda,g_s}$. The integrations in (24) can be performed, yielding

$$\tilde{W}_{\lambda,g_s}(x) = \frac{1}{x+\sqrt{\lambda}} + \frac{g_s}{x^2-\lambda}\Big(\tilde{W}^2_{\lambda,g_s}(\sqrt{\lambda}) - \tilde{W}^2_{\lambda,g_s}(x)\Big) . \tag{26}$$

Solving for $\tilde{W}_{\lambda,g_s}(x)$ we find

$$\tilde{W}_{\lambda,g_s}(x) = \frac{-(x^2-\lambda)+\hat{W}_{\lambda,g_s}(x)}{2g_s} , \tag{27}$$

where we have defined

$$\hat{W}_{\lambda,g_s}(x) = \sqrt{(x^2-\lambda)^2 + 4g_s\Big(g_s\tilde{W}^2_{\lambda,g_s}(\sqrt{\lambda}) + x - \sqrt{\lambda}\Big)} . \tag{28}$$

$\tilde{W}_{\lambda,g_s}(x)$ is determined up to the value $\tilde{W}_{\lambda,g_s}(\sqrt{\lambda})$. We will now show that this value is fixed by consistency requirements of the quantum geometry. If we insert the solution (27) into Eq. (20) we obtain

$$\frac{\partial}{\partial t}\tilde{G}_{\lambda,g_s}(x,y;t) = -\frac{\partial}{\partial x}\Big[\hat{W}_{\lambda,g_s}(x)\,\tilde{G}_{\lambda,g_s}(x,y;t)\Big] . \tag{29}$$

In analogy with (12) and (15), this is solved by

$$\tilde{G}_{\lambda,g_s}(x,y;t) = \frac{\hat{W}_{\lambda,g_s}(\bar{x}(t,x))}{\hat{W}_{\lambda,g_s}(x)}\,\frac{1}{\bar{x}(t,x)+y} , \tag{30}$$

where $\bar{x}(t,x)$ is the solution of the characteristic equation for (29), the generalization of Eq. (16):

$$\frac{d\bar{x}}{dt} = -\hat{W}_{\lambda,g_s}(\bar{x}) , \qquad \bar{x}(0,x) = x , \tag{31}$$

such that

$$t = \int_{\bar{x}(t)}^{x} \frac{dy}{\hat{W}_{\lambda,g_s}(y)} . \tag{32}$$

Physically, we require that t can take values from 0 to ∞, as opposed to just in a finite interval. From expression (32) for t this is only possible if the polynomial under the square root in the defining equation (28) has a double zero, which fixes the function $\hat{W}_{\lambda,g_s}(x)$ to

$$\hat{W}_{\lambda,g_s}(x) = (x-\alpha)\sqrt{(x+\alpha)^2-2g_s/\alpha} = \lambda(\tilde{x}-u)\sqrt{(\tilde{x}+u)^2-2\kappa} , \tag{33}$$

where

$$x = \tilde{x}\sqrt{\lambda}\,, \qquad \alpha = u\sqrt{\lambda}\,, \qquad u^3 - u + \kappa = 0\,. \tag{34}$$

In order to have a physically acceptable $\tilde{W}_{\lambda,g_s}(x)$, one has to choose the solution to the third-order equation which is closest to 1 and the above statements about the expansion of $\tilde{W}_{\lambda,g_s}(x)$ in a power series in κ follows from (27), (33) and (34).

4. The matrix model representation

The formulas (33) and (27) are standard formulas for the resolvent of a Hermitean matrix model, calculated to leading order in N, the size of the matrix. In fact the following matrix model

$$Z(\lambda, g_s) = \int d\phi\, e^{-N\,\mathrm{Tr}\,V(\phi)}\,, \qquad V(\phi) = \frac{1}{g_s}\Big(\lambda\phi - \frac{1}{3}\phi^3\Big) \tag{35}$$

has a resolvent

$$\left\langle \frac{1}{N}\ \mathrm{Tr}\left(\frac{1}{x-\phi}\right)\right\rangle = W_{\lambda,g_s(x)}(x) + O\left(\frac{1}{N^2}\right)\,, \tag{36}$$

where $W_{\lambda,g_s}(x)$ is given by (27), and where the expectation value of a matrix expression $\mathcal{O}(\phi)$ is defined as

$$\langle \mathcal{O}(\phi)\rangle = \frac{1}{Z(\lambda,g_s)}\int d\phi\, e^{-N\,\mathrm{Tr}\,V(\phi)}\ \mathcal{O}(\phi)\,. \tag{37}$$

What is surprising here, compared to "old" matrix model approaches to 2d Euclidean quantum gravity, is that the large N limit reproduces directly the continuum theory. No scaling limit has to be taken. The situation is more like in the Kontsevich matrix model, which directly describes continuum 2d gravity aspects. In fact the cubic potential is "almost" like the cubic potential in the Kontsevich matrix model, but the wold-sheet interpretation is different.

Can the above correspondence be made systematic in an large N expansion and can the matrix model representation help us to a non-perturbative definition of generalized 2d CDT gravity? The answer is yes [17].

First we have to formulate the CDT model *from first principles* such that we allow for baby universes to join the "parent" universe again, *i.e.* we have to allow for topology changes of the 2d universe, and next we have to check if this generalization is correctly captured by the matrix model (35) [18].

5. CDT string field theory

In quantum field theory particles can be created and annihilated if the process does not violate any conservation law of the theory. In string field theories one operates in the same way with operators which can create and annihilate strings. From the 2d quantum gravity point of view we thus have a third-quantization of gravity: one-dimensional universes can be created and destroyed. In [19] such a formalism was developed for non-critical strings (or 2d Euclidean quantum gravity). In [18] the formalism was applied to 2d CDT gravity leading to a string field theory or third quantization for CDT which allows us in principle to calculate any amplitude involving creation and annihilation of universes.

Let us briefly review this formalism. The starting point is the assumption of a vacuum from which universes can be created. We denote this state $|0\rangle$ and define creation and annihilation operators:

$$\begin{aligned} [\Psi(l), \Psi^\dagger(l')] &= l\delta(l-l')\,, \\ \Psi(l)|0\rangle &= \langle 0|\Psi^\dagger(l) = 0\,. \end{aligned} \tag{38}$$

The factor l multiplying the delta-function is introduced for convenience, see [18] for a discussion.

Associated with the spatial universe we have a Hilbert space on the positive half-line, and a corresponding scalar product (making $H_0(l)$ defined in Eq. (19) Hermitian):

$$\langle \psi_1|\psi_2\rangle = \int \frac{dl}{l}\; \psi_1^*(l)\psi_2(l)\,. \tag{39}$$

The introduction of the operators $\Psi(l)$ and $\Psi^\dagger(l)$ in (38) can be thought of as analogous to the standard second quantization in many-body theory. The single particle Hamiltonian H_0 defined by (19) becomes in our case the “single universe” Hamiltonian. It has eigenfunctions $\psi_n(l)$ with corresponding eigenvalues $e_n = 2n\sqrt{\lambda}$, $n = 1, 2, \ldots$:

$$\psi_n(l) = l\, e^{-\sqrt{\lambda}l} p_{n-1}(l)\,, \qquad H_0(l)\psi_n(l) = e_n\psi_n(l)\,, \tag{40}$$

where $p_{n-1}(l)$ is a polynomial of the order of $n-1$. Note that the disk amplitude $W_\lambda^{(0)}(l)$, which is obtained from (22), formally corresponds to $n = 0$ in (40):

$$W_\lambda^{(0)}(l) = e^{-\sqrt{\lambda}l}\,, \qquad H_0(l)W_\lambda^{(0)}(l) = 0\,. \tag{41}$$

This last equation can be viewed as a kind of Wheeler–deWitt equation if we view the disk function as the Hartle–Hawking wave function. However, $W_\lambda^{(0)}(l)$ does not belong to the spectrum of $H_0(l)$ since it is not normalizable when one uses the measure (39).

We now introduce creation and annihilation operators $a_n^\dagger$ and a_n corresponding to these states, acting on the Fock-vacuum $|0\rangle$ and satisfying $[a_n, a_m^\dagger] = \delta_{n,m}$. We define

$$\Psi(l) = \sum_n a_n \psi_n(l)\,, \qquad \Psi^\dagger(l) = \sum_n a_n^\dagger \psi_n^*(l)\,, \tag{42}$$

and from the orthonormality of the eigenfunctions with respect to the measure dl/l we recover (38). The "second-quantized" Hamiltonian is

$$\hat{H}_0 = \int_0^\infty \frac{dl}{l}\, \Psi^\dagger(l) H_0(l) \Psi(l)\,, \tag{43}$$

and the propagator $\tilde{G}_\lambda(l_1, l_2; t)$ is now obtained as

$$\tilde{G}_\lambda^{(0)}(l_1, l_2; t) = \langle 0|\Psi(l_2) e^{-t\hat{H}_0} \Psi^\dagger(l_1)|0\rangle\,. \tag{44}$$

While this is trivial, the advantage of the formalism is that it automatically takes care of symmetry factors (like in the many-body applications in statistical field theory) both when many spatial universes are at play and when they are joining and splitting. We can follow [19] and define the following Hamiltonian, describing the interaction between spatial universes:

$$\begin{aligned}\hat{H} \;=\; & \hat{H}_0 - g_s \int dl_1 \int dl_2 \Psi^\dagger(l_1)\Psi^\dagger(l_2)\Psi(l_1+l_2) \\ & - \alpha g_s \int dl_1 \int dl_2 \Psi^\dagger(l_1+l_2)\Psi(l_2)\Psi(l_1) - \int \frac{dl}{l}\, \rho(l)\Psi(l)\,,\end{aligned} \tag{45}$$

where the different terms of the Hamiltonian are illustrated in Fig. 5. Here g_s is the coupling constant we have already encountered in Sec. 3 of mass dimension 3. The factor α is just inserted to be able to identify the action

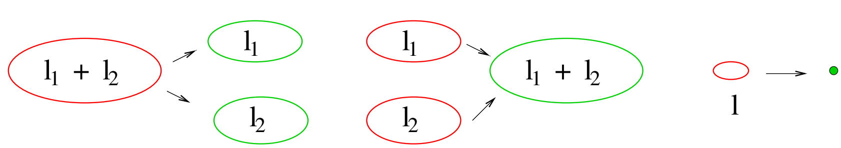

Fig. 5. Graphical illustration of the various terms in Eq. (45).

of the two g_s-terms in (45) when expanding in powers of g_s. We will think of $\alpha = 1$ unless explicitly stated differently. When $\alpha = 1$, $\hat{H}$ is hermitian except for the presence of the tadpole term. It tells us that universes can vanish, but cannot be created from nothing. The meaning of the two interaction terms is as follows: the first term replaces a universe of length $l_1 + l_2$ with two universes of length l_1 and l_2. This is one of the processes shown in Fig. 5. The second term represents the opposite process, where two spatial universes merge into one, *i.e.* the time-reversed picture. The coupling constant g_s clearly appears as a kind of string coupling constant: one factor g_s for splitting spatial universes, one factor g_s for merging spatial universes and thus a factor g_s^2 when the space-time topology changes, but there are also factors for branching alone. This is only compatible with an Euclidean sft-picture if we associate a puncture (and thus a topology change) with the vanishing of a baby universe. As discussed above, this is indeed not unnatural from a Lorentzian point of view. From this point of view the appearance of a tadpole term is more natural in the CDT framework than in the original Euclidean framework in [19]. The tadpole term is a formal realization of this puncture "process", where the light-cone becomes degenerate.

In principle, we can now calculate the process, where we start out with m spatial universes at time 0 and end with n universes at time t, represented as

$$G_{\lambda,g_s}(l_1,..,l_m;l_1',..,l_n';t) = \langle 0|\Psi(l_1')\dots\Psi(l_n')\, e^{-t\hat{H}}\Psi^\dagger(l_1)\dots\Psi^\dagger(l_m)|0\rangle\,. \tag{46}$$

5.1. Dyson–Schwinger equations

The disk amplitude is one of a set of functions for which it is possible to derive Dyson–Schwinger equations (DSE). The disk amplitude is characterized by the fact that at $t = 0$ we have a spatial universe of some length, and at some point it vanishes in the "vacuum". Let us consider the more general situation, where a set of spatial universes of some lengths l_i exists at time $t = 0$, and where the universes vanish at later times. Define the generating function:

$$Z(J) = \lim_{t\to\infty} \langle 0|e^{-t\hat{H}}\, e^{\int dl\, J(l)\Psi^\dagger(l)}|0\rangle\,. \tag{47}$$

Notice that if the tadpole term had not been present in $\hat{H}$ $Z(J)$ would have trivially been equal 1. We have

$$\lim_{t\to\infty}\langle 0|e^{-t\hat{H}}\,\Psi^\dagger(l_1)\cdots\Psi^\dagger(l_n)|0\rangle = \left.\frac{\delta^n Z(J)}{\delta J(l_1)\cdots\delta J(l_n)}\right|_{J=0}\,. \tag{48}$$

$Z(J)$ is the generating functional for universes that disappear in the vacuum. We now have

$$0 = \lim_{t\to\infty}\left[\frac{\partial}{\partial t}\,\langle 0|e^{-t\hat{H}}\, e^{\int dl\, J(l)\Psi^\dagger(l)}|0\rangle = -\langle 0|e^{-t\hat{H}}\;\hat{H}\, e^{\int dl\, J(l)\Psi^\dagger(l)}|0\rangle\right]\,. \tag{49}$$

Commuting the $\Psi(l)$'s in $\hat{H}$ past the source term effectively replaces these operators by $lJ(l)$, after which they can be moved to the left of any $\Psi^\dagger(l)$ and outside $\langle 0|$. After that the remaining $\Psi^\dagger(l)$'s in $\hat{H}$ can be replaced by $\delta/\delta J(l)$ and also moved outside $\langle 0|$, leaving us with an integro-differential operator acting on $Z(J)$:

$$0 = \int_0^\infty dl J(l)\, O\left(l, J, \frac{\delta}{\delta J}\right) Z(J)\,, \tag{50}$$

where

$$\begin{aligned} O\left(l, J, \frac{\delta}{\delta J}\right) &= H_0(l)\frac{\delta}{\delta J(l)} - \delta(l) - g_s l \int_0^l dl' \frac{\delta^2}{\delta J(l')\delta J(l-l')} \\ &\quad - \alpha g_s l \int_0^\infty dl' l' J(l') \frac{\delta}{\delta J(l+l')}\,. \end{aligned} \tag{51}$$

$Z(J)$ is a generating functional which also includes totally disconnected universes which never "interact" with each other. The generating functional for connected universes is obtained in the standard way from field theory by taking the logarithm of $Z(J)$. Thus we write:

$$F(J) = \log Z(J)\,, \tag{52}$$

and we have

$$\lim_{t\to\infty} \langle 0|e^{-t\hat{H}}\Psi^\dagger(l_1)\cdots\Psi^\dagger(l_n)|0\rangle_{\rm con} = \left.\frac{\delta^n F(J)}{\delta J(l_1)\cdots\delta J(l_n)}\right|_{J=0}\,, \tag{53}$$

and we can readily transfer the DSE (50)–(51) into an equation for the connected functional $F(J)$:

$$\begin{aligned} 0 = \int_0^\infty dl\, J(l) \Bigg\{ & H_0(l)\,\frac{\delta F(J)}{\delta J(l)} - \delta(l) - g_s l \int_0^l dl'\, \frac{\delta^2 F(J)}{\delta J(l')\delta J(l-l')} \\ & -g_s l \int_0^l dl' \frac{\delta F(J)}{\delta J(l')}\,\frac{\delta F(J)}{\delta J(l-l')} - \alpha g_s l \int_0^\infty dl' l' J(l') \frac{\delta F(J)}{\delta J(l+l')} \Bigg\}\,. \end{aligned} \tag{54}$$

From Eq. (54) one obtains the DSE by differentiating (54) after $J(l)$ a number of times and then taking $J(l) = 0$.

5.2. Application of the DSE

Let us introduce the notation

$$w(l_1,\ldots,l_n) \equiv \left.\frac{\delta^n F(J)}{\delta J(l_1)\cdots\delta J(l_n)}\right|_{J=0} \tag{55}$$

as well as the Laplace transform $w(x_1,\ldots,x_n)$ (like in (10)). Let us differentiate Eq. (54) after $J(l)$ once and two twice, then take $J(l)=0$ and Laplace transform the obtained equations. We obtain the following equations (where $H_0(x)f(x)=\partial_x[(x^2-\lambda)f(x)]$):

$$0 = H_0(x)w(x) - 1 + g_s\partial_x\left(w(x,x)+w(x)w(x)\right), \tag{56}$$

$$\begin{aligned} 0 = {} & (H_0(x)+H_0(y))w(x,y) + g_s\partial_x w(x,x,y) + g_s\partial_y w(x,y,y) \\ & + 2g_s\left(\partial_x[w(x)w(x,y)] + \partial_y[w(y)w(x,y)]\right) \\ & + 2\alpha g_s\partial_x\partial_y\Big(\frac{w(x)-w(y)}{x-y}\Big). \end{aligned} \tag{57}$$

The structure of the DSE for an increasing number of arguments is hopefully clear (see [18] for details).

We can solve the DSE iteratively. For this purpose let us introduce the expansion of $w(x_1,\ldots,x_n)$ in terms of the coupling constants g_s and α:

$$w(x_1,\ldots,x_n) = \sum_{k=n-1}^{\infty}\alpha^k \sum_{m=k-1}^{\infty} g_s^m\, w(x_1,\ldots,x_n;m,k). \tag{58}$$

The amplitude $w(x_1,\ldots,x_n)$ starts with the power $(\alpha g_s)^{n-1}$ since we have to perform n mergings during the time evolution in order to create a connected geometry if we begin with n separated spatial loops. Thus one can find the lowest order contribution to $w(x_1)$ from (56), use that to find the lowest order contribution to $w(x_1,x_2)$ from (57), *etc.* Returning to Eq. (56) we can use the lowest order expression for $w(x_1,x_2)$ to find the next order correction to $w(x_1)$, *etc.*

As mentioned above the amplitude $w(x_1,\ldots,x_n)$ starts with the power $(\alpha g_s)^{n-1}$ coming from merging the n disconnected spatial universes. The rest of the powers of αg_s will result in a topology change of the resulting, connected worldsheet. From an Euclidean point of view it is thus more appropriate to reorganize the series as follows

$$w(x_1,\ldots,x_n) = (\alpha g_s)^{n-1}\sum_{h=0}^{\infty}(\alpha g_s^2)^h w_h(x_1,\ldots,x_n), \tag{59}$$

$$w_h(x_1,\ldots,x_n) = \sum_{j=0}^{\infty} g_s^j w(x_1,\ldots,x_n;n-1+2h+j,n-1+h) \tag{60}$$

and aim for a topological expansion in αg_s^2, at each order solving for all possible baby-universe creations which at some point will vanish into the vacuum. Thus $w_h(x_1, \ldots, x_n)$ will be a function of g_s although we do not write it explicitly. The DSE allows us to obtain the topological expansion iteratively, much the same way we already did as a power expansion in g_s.

6. The matrix model, once again

Let us consider our $N \times N$ Hermitian matrix with the cubic potential (35) and define the observable

$$W(x_1, \ldots, x_n)_d = N^{n-2} \left\langle (\mathrm{Tr}\left(\frac{1}{x_1 - M}\right) \cdots \left(\mathrm{tr}\frac{1}{x_1 - M}\right)\right\rangle , \tag{61}$$

where the subscript d refers to the fact that the correlator will contain disconnected parts. We denote the connected part of the correlator by $W(x_1 \ldots, x_n)$. It is standard matrix model technology to find the matrix model DSEs for $W(x_1 \ldots, x_n)$. We refer to [20–23] for details. *One obtains precisely the same set of coupled equations as (56)–(57) if we identify*:

$$\alpha = \frac{1}{N^2} , \tag{62}$$

and the discussion surrounding the expansion (59) is nothing but the standard discussion of the large N expansion of the multi-loop correlators (see for instance [23] or the more recent papers [24–26]). Thus we conclude that there is a perturbative agreement between the matrix model (35) and the CDT SFT in the sense that perturbatively:

$$W(x_1, \ldots, x_n) = w(x_1, \ldots, x_n) . \tag{63}$$

In practice the SFT *is* only defined perturbatively, although in principle we have available the string field Hamiltonian. However, we can now use the matrix model to extract non-pertubative information. The identification of the matrix model and the CDT SFT DSEs were based on (62), but in the SFT we are interested in $\alpha = 1$, *i.e.*, formally in $N = 1$, in which case the matrix integrals reduce to ordinary integrals. This means that we will consider the entire sum over topologies "in one go":

$$Z(g, \lambda) = \int dm \, \exp\left[-\frac{1}{g_s}\left(\lambda m - \frac{1}{3}\, m^3\right)\right] , \tag{64}$$

while the observables (61) can be written as

$$W_d(x_1, \ldots, x_n) = \frac{1}{Z(g_s, \lambda)} \int dm \, \frac{\exp\left[-\frac{1}{g_s}\left(\lambda m - \frac{1}{3}\, m^3\right)\right]}{(x_1 - m) \cdots (x_n - m)} . \tag{65}$$

These integrals should be understood as formal power series in the dimensionless variable κ defined by Eq. (23). Any choice of an integration contour which makes the integral well defined and reproduces the formal power series is a potential nonperturbative definition of these observables. However, different contours might produce different nonperturbative contributions (*i.e.* which cannot be expanded in powers of t), and there may even be nonperturbative contributions which are not captured by any choice of integration contour. As usual in such situations, additional physics input is needed to fix these contributions.

To illustrate the point, let us start by evaluating the partition function given in (64). We have to decide on an integration path in the complex plane in order to define the integral. One possibility is to take a path along the negative axis and then along either the positive or the negative imaginary axis. The corresponding integrals are

$$\begin{aligned} Z(g_s, \lambda) &= \sqrt{\lambda}\kappa^{1/3} F_{\pm}\left(\kappa^{-2/3}\right) , \\ F_{\pm}(\kappa^{-2/3}) &= 2\pi\, e^{\pm i\pi/6} \mathrm{Ai}\left(\kappa^{-2/3}\, e^{\pm 2\pi i/3}\right) , \end{aligned} \tag{66}$$

where Ai denotes the Airy function. Both $F_{\pm}$ have the same asymptotic expansion in κ, with positive coefficients. Had we chosen the integration path entirely along the imaginary axis we would have obtained ($2\pi i$ times) $\mathrm{Ai}(\kappa^{-2/3})$, but this has an asymptotic expansion in κ with coefficients of oscillating sign, which is at odds with its interpretation as a probability amplitude. In the notation of [27] we have

$$F_{\pm}(z) = \pi\Big(\mathrm{Bi}(z) \pm i\mathrm{Ai}(z)\Big) , \tag{67}$$

from which one deduces immediately that the functions $F_{\pm}(\kappa^{-2/3})$ are not real. However, since $\mathrm{Bi}(\kappa^{-2/3})$ grows like $e^{2/(3\kappa)}$ for small κ while $\mathrm{Ai}(\kappa^{-2/3})$ falls off like $e^{-2/(3\kappa)}$, their imaginary parts are exponentially small in $1/\kappa$ compared to the real part, and therefore, do not contribute to the asymptotic expansion in κ. An obvious way to *define* a partition function which is real and shares the same asymptotic expansion is by symmetrization,

$$\tfrac{1}{2}\left(F_+ + F_-\right) \equiv \pi \mathrm{Bi}\,. \tag{68}$$

The situation parallels the one encountered in the double scaling limit of the "old" matrix model [30], and discussed in detail in [32], but is less complicated. We will return to a discussion of this in the next section.

Presently, let us collectively denote by $F(z)$ any of the functions $F_{\pm}(z)$ or $\pi\mathrm{Bi}(z)$, leading to the tentative identification

$$Z(g_s, \lambda) = \sqrt{\lambda}\,\kappa^{1/3}\, F\Big(\kappa^{-2/3}\Big) , \qquad F''(z) = zF(z) , \tag{69}$$

where we have included the differential equation satisfied by the Airy functions for later reference. In preparation for the computation of the observables $\tilde{W}_d(x_1,\ldots,x_n)$ we introduce the dimensionless variables

$$x = \tilde{x}\,\sqrt{\lambda}, \quad m = g_s^{1/3}\beta\,, \qquad \tilde{W}_d(x_1,\ldots,x_n) = \lambda^{-n/2}\tilde{w}_d(\tilde{x}_1,\ldots,\tilde{x}_n)\,. \quad (70)$$

Assuming $\tilde{x}_k > 0$, we can write

$$\frac{1}{\tilde{x} - \kappa^{1/3}\,\beta} = \int_0^\infty d\alpha\ \exp\left[-\left(\tilde{x} - \kappa^{1/3}\beta\right)\ \alpha\right]. \quad (71)$$

We can use this identity to re-express the pole terms in Eq. (65) to obtain the integral representation

$$\tilde{w}_d(\tilde{x}_1,\ldots,\tilde{x}_n) = \int_0^\infty \prod_{i=1}^n d\alpha_i\, e^{-(\tilde{x}_1\alpha_1+\cdots+\tilde{x}_n\alpha_n)}\ \frac{F\left(\kappa^{-\frac{2}{3}} - \kappa^{\frac{1}{3}}\sum_{i=1}^n \alpha_i\right)}{F\left(\kappa^{-\frac{2}{3}}\right)} \quad (72)$$

for the amplitude with dimensionless arguments. By an inverse Laplace transformation we thus obtain:

$$W_d(l_1,\ldots,l_n) = \frac{F(\kappa^{-2/3} - \kappa^{1/3}\sqrt{\lambda}\,(l_1+\cdots+l_n))}{F(\kappa^{-2/3})}\,. \quad (73)$$

For the special case $n = 1$ we find

$$W(l) = \frac{F(\kappa^{-2/3} - \kappa^{1/3}\sqrt{\lambda}\,l)}{F(\kappa^{-2/3})} \quad (74)$$

for the disc amplitude, together with the remarkable relation

$$W_d(l_1,\ldots,l_n) = W(l_1+\cdots+l_n)\,. \quad (75)$$

By Laplace transformation this formula implies the relation

$$\tilde{W}_d(x_1,\ldots,x_n) = \sum_{i=1}^n \frac{\tilde{W}(x_i)}{\prod_{j\neq i}^n (x_j - x_i)}\,. \quad (76)$$

From $\tilde{W}_d(x_1,\ldots,x_n)$ we can construct the connected multiloop functions $\tilde{W}(x_1,\ldots,x_n)$ using standard field theory. Let us remark that the asymptotic expansion in κ of $\tilde{W}(x_1,\ldots,x_n)$, of course, agrees with that obtained by recursively solving the CDT Dyson–Schwinger equations.

7. Relation with other models

Can we identify a know continuum conformal field theory coupled to 2d gravity, which leads to the functions we have here calculated? The answer is yes. As a starting point note that for the disk amplitude we have

$$\tilde{W}^{(0)}_{\lambda}(x) = \frac{1}{x+\sqrt{\lambda}} = \frac{1}{x} - \frac{\sqrt{\lambda}}{x^2} + \cdots \,, \tag{77}$$

from which we conclude that the susceptibility exponent $\gamma = 1/2$ (the lowest non-analytic power of λ in the disk amplitude is $\lambda^{1-\gamma}$). $\gamma = 1/2$ is the generic value of the susceptibility exponent for the so-called branched polymers and the way branched polymers enter into the game of 2d gravity coupled to conformal field theories is as follows: recall that the $(2, 2m-1)$, $m = 2, 3, \ldots$, minimal conformal field theories coupled to 2d Euclidean quantum gravity can be described as double-scaling limits of one-matrix models with certain fine-tuned matrix potentials of the order of at least $m+1$. Formally, the case $m = 1$, which corresponds to a somewhat degenerate $(2, 1)$ conformal field theory with central charge $c = -2$ (which when coupled to 2d gravity is called *topological gravity*), is then described by a special double-scaling limit of the purely Gaussian matrix model (see the review [33]). In this double-scaling limit one obtains for the so-called FZZT brane precisely the Airy function, see [32, 34–36] for recent discussions. For this model $\gamma = -1$. While it is possible to describe 2d topological quantum gravity by a double-scaling limit of the Gaussian matrix model, the most natural geometric interpretation of the Gaussian matrix model is in terms of branched polymers, in the sense that the integral

$$\frac{\int dM \ \mathrm{tr}\, M^{2n}\, e^{-\frac{1}{2}\mathrm{tr} M^2}}{\int dM\, e^{-\frac{1}{2}\mathrm{tr} M^2}} \tag{78}$$

can be thought of as the gluing of a boundary of length n into a double-line branched polymer of length n. Since the branched polymers are also allowed to form closed loops, their partition function contains a sum over topologies "en miniature", and one can indeed define a double-scaling limit of the model. When solving for the partition function in this limit, one obtains precisely our $Z(g_s, \lambda)$ of Eq. (69)! (see [37], where this remarkable result was first proved, for details). It does not imply that the generalized CDT model is just branched polymers, it is much richer since it has a 2d surface representation and many more observables, but the non-perturbative branching process is clearly that of branched polymers. The relation between the generalized CDT model and the $c = -2$ model is as follows: we have the "right" branch of Liouville theory with susceptibility exponent $\gamma_-(= -1)$ and a "wrong" branch with susceptibility exponent γ_+, related by

$$\gamma_+ = -\frac{\gamma_-}{1-\gamma_-} \,. \tag{79}$$

The interpretation of this γ_+ in relation to γ_- in terms of geometry can be found in [38–40], and for earlier related work see [41, 42]. It is all related to dominance or non-dominance of branching of baby universes. The simplest example is precisely given by $c = -2$: topological quantum gravity has $\gamma_- = -1$ whose dual is the "wrong" $\gamma_+ = 1/2$, which happens to be the value occurring generically in the theory of branched polymers (see, *e.g.*, [43–45] for a discussion of why branched polymers and baby universes are generic and even dominant in many situations in non-critical string theory and even in higher dimensional quantum gravity).

8. Stochastic quantization

It is a most remarkable fact that the above mentioned result can all be understood as a result of stochastic quantization of *space*. In this picture time becomes the *stochastic time* related with the branching of space into baby universes and the original CDT model described in Sec. 2 becomes the classical limit where no stochastic processes are present [46].

Recall the Langevin stochastic differential equation for a single variable x (see, for example, [47, 48]).

$$\dot{x}^{(\nu)}(t) = -f\left(x^{(\nu)}(t)\right) + \sqrt{\Omega}\,\nu(t)\,, \tag{80}$$

where the dot denotes differentiation with respect to stochastic time t, $\nu(t)$ is a Gaussian white-noise term of unit width and $f(x)$ a dissipative drift force:

$$f(x) = \frac{\partial S(x)}{\partial x} \,. \tag{81}$$

The noise term creates a probability distribution of $x(t)$, reflecting the assumed stochastic nature of the noise term, with an associated probability distribution

$$P(x, x_0; t) = \left\langle \delta\left(x - x^{(\nu)}(t; x_0)\right)\right\rangle_\nu \,, \tag{82}$$

where the expectation value refers to an average over the Gaussian noise. $P(x, x_0; t)$ satisfies the Fokker–Planck equation

$$\frac{\partial P(x, x_0; t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}\,\Omega\,\frac{\partial P(x, x_0; t)}{\partial x} + f(x)P(x, x_0; t)\right) \,. \tag{83}$$

This is an imaginary-time Schrödinger equation, with $\sqrt{\Omega}$ playing a role similar to $\hbar$. It enables us to write P as a propagator for a Hamiltonian operator $\hat{H}$,

$$P(x, x_0; t) = \langle x | e^{-t\hat{H}} | x_0 \rangle \,, \qquad \hat{H} = \tfrac{1}{2}\, \Omega \hat{p}^2 + i\hat{p} f(\hat{x}) \,, \tag{84}$$

with initial condition $x(t = 0) = x_0$, and $\hat{p} = -i\partial_x$. It follows that by defining

$$\tilde{G}(x_0, x; t) \equiv \frac{\partial}{\partial x_0} P(x, x_0; t) \tag{85}$$

the function $\tilde{G}(x_0, x; t)$ satisfies the differential equation

$$\frac{\partial \tilde{G}(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \left(\frac{1}{2}\, \Omega\, \frac{\partial \tilde{G}(x_0, x; t)}{\partial x_0} - f(x_0)\, \tilde{G}(x_0, x; t) \right) . \tag{86}$$

Omitting the noise term corresponds to taking the limit $\Omega \to 0$. One can then drop the functional average over the noise in (82) to obtain

$$P_{\rm cl}(x, x_0; t) = \delta(x - x(t, x_0)) \,, \qquad \tilde{G}_{\rm cl}(x_0, x; t) = \frac{\partial}{\partial x_0} \delta(x - x(t, x_0)) \,. \tag{87}$$

It is readily verified that these functions satisfy Eqs. (83) and (86) with $\Omega = 0$. Thus we have for $S(x) = -\lambda x + x^3/3$:

$$\frac{\partial \tilde{G}_{\rm cl}(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \Big((\lambda - x_0^2)\, \tilde{G}_{\rm cl}(x_0, x; t) \Big) \,. \tag{88}$$

Comparing now Eqs. (12) and (88), we see that we can formally reinterpret $\tilde{G}^{(0)}_\lambda(x_0, x; t)$ — an amplitude obtained by nonperturbatively quantizing Lorentzian pure gravity in two dimensions — as the "*classical* probability" $\tilde{G}_{\rm cl}(x_0, x; t)$ corresponding to the action $S(x) = -\lambda x + x^3/3$ of a zero-dimensional system in the context of stochastic quantization, only is the boundary condition different, since in the case of CDT x is not an ordinary real variable, but the cosmological constant. The correct boundary conditions are thus the ones stated in Eqs. (13), (14).

Stochastic quantization of the system amounts to replacing

$$\tilde{G}^{(0)}_\lambda(x_0, x; t) \to \tilde{G}(x_0, x; t) \,, \tag{89}$$

where $\tilde{G}(x_0, x; t)$ satisfies the differential equation corresponding to Eq. (86), namely,

$$\frac{\partial \tilde{G}(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \left(g_s \frac{\partial}{\partial x_0} + \lambda - x_0^2 \right) \tilde{G}(x_0, x; t) \,. \tag{90}$$

We have introduced the parameter $g_s := \Omega/2$, which will allow us to reproduce the matrix model and SFT results reported above.

A neat geometric interpretation of how stochastic quantization can capture topologically nontrivial amplitudes has been given in [49]. Applied to the present case, we can view the propagation in stochastic time t for a given noise term $\nu(t)$ as classical in the sense that solving the Langevin equation (80) for $x^{(\nu)}(t)$ iteratively gives precisely the tree diagrams with one external leg corresponding to the action $S(x)$ (and including the derivative term $\dot{x}^{(\nu)}(t)$), with the noise term acting as a source term. Performing the functional integration over the Gaussian noise term corresponds to integrating out the sources and creating loops, or, if we have several independent trees, to merging these trees and creating diagrams with several external legs. If the dynamics of the quantum states of the spatial universe takes place via the strictly causal CDT-propagator $\hat{G}_0 = e^{-t\hat{H}_0}$, a single spatial universe of length l cannot split into two spatial universes. Similarly, no two spatial universes are allowed to merge as a function of stochastic time. However, introducing the noise term *and* subsequently performing a functional integration over it makes these processes possible. This explains how the stochastic quantization can automatically generate the amplitudes which are introduced by hand in a string field theory, be it of Euclidean character as described in [49], or within the framework of CDT.

What is new in the CDT string field theory considered here is that we can use the corresponding stochastic field theory to solve the model. Since we arrive at closed equations valid to all orders in the genus expansion let us translate equations (90) to l-space

$$\frac{\partial G(l_0, l; t)}{\partial t} = -H(l_0)\, G(l_0, l; t)\,, \tag{91}$$

where the *extended* Hamiltonian

$$H(l) = -l\frac{\partial^2}{\partial l^2} + \lambda l - g_s l^2 = H_0(l) - g_s l^2 \tag{92}$$

now has an extra potential term coming from the inclusion of branching points compared to the Hamiltonian $H_0(l)$ defined in (19). It is truly remarkable that all branching and joining is contained in this simple extra term. Formally $H(l)$ is a well-defined Hermitian operator with respect to the measure (39) (we will discuss some subtleties in the next section).

We can now write down the generalization of Wheeler–deWitt equation (39) for the disk amplitude

$$\hat{H}(l)W(l) = 0\,. \tag{93}$$

Contrary to $W_\lambda^{(0)}(l)$ appearing in (39), $W(l)$ contains all branchings and all topology changes, and the solution is precisely (74)! This justifies the choice $g_s = \Omega/2$ mentioned above. Recall that $E = 0$ does not belong to the spectrum of $H_0(l)$ since $W_0(l)$ is not integrable at zero with respect to the measure (39). Exactly the same is true for the extended Hamiltonian $H(l)$ and the corresponding Hartle–Hawking amplitude $W(l)$.

We have also as a generalization of (18) that

$$G(l_0, l; t) = \langle l|e^{-tH(l)}|l_0\rangle \tag{94}$$

describes the nonperturbative propagation of a spatial loop of length l_0 to a spatial loop of length l in proper (or stochastic) time t, now including the summation over all genera.

9. The extended Hamiltonian

In order to analyze the spectrum of $H(l)$, it is convenient to put the differential operator into standard form. After a change of variables

$$l = \tfrac{1}{2}z^2\,, \qquad \psi(l) = \sqrt{z}\phi(z)\,, \tag{95}$$

the eigenvalue equation becomes

$$H(z)\phi(z) = E\phi(z)\,, \qquad H(z) = -\frac{1}{2}\frac{d^2}{dz^2} + \frac{1}{2}\lambda z^2 + \frac{3}{8z^2} - \frac{g_s}{4}z^4\,. \tag{96}$$

This shows that the potential is unbounded from below, but such that the eigenvalue spectrum is still discrete: whenever the potential is unbounded below with fall-off faster than $-z^2$, the spectrum is discrete, reflecting the fact that the classical escape time to infinity is finite (see [50] for a detailed discussion relevant to the present situation). For small g_s, there is a large barrier of height $\lambda^2/(2g_s)$ separating the unbounded region for $l > \lambda/g_s$ from the region $0 \le l \le \lambda/(2g_s)$ where the potential grows. This situation is perfectly suited to applying a standard WKB analysis. For energies lower than $\lambda^2/(2g_s)$, the eigenfunctions (40) of $H_0(l)$ will be good approximations to those of $\hat{H}(l)$. However, when $l > \lambda/g_s$ the exponential fall-off of $\psi_n^{(0)}(l)$ will be replaced by an oscillatory behaviour, with the wave function falling off only like $1/l^{1/4}$. The corresponding $\psi_n(l)$ is still square-integrable since we have to use the measure (39). For energies larger than $\lambda^2/(2g_s)$, the solutions will be entirely oscillatory, but still square-integrable.

Thus a somewhat drastic change has occurred in the quantum behaviour of the one-dimensional universe as a consequence of allowing topology changes. In the original, strictly causal quantum gravity model an eigenstate $\psi_n^{(0)}(l)$ of the spatial universe had an average size of the order of $1/\sqrt{\lambda}$. However, allowing for branching and topology change, the average size of the universe is now infinite!

As discussed in [50], Hamiltonians with unbounded potentials like (96) have a one-parameter family of self adjoint extensions and we still have to choose one of those such that the spectrum of $H(l)$ can be determined unambiguously. One way of doing this is to appeal again to stochastic quantization, following the strategy used by Greensite and Halpern [51], which was applied to the double-scaling limit of matrix models in [50, 52, 53]. The Hamiltonian (84) corresponding to the Fokker–Planck equation (90), namely,

$$\begin{aligned} H(x)\psi(x) &= -g_s \frac{d^2\psi(x)}{dx^2} + \frac{d}{dx}\left(\frac{dS(x)}{dx}\,\psi(x)\right), \\ S(x) &= \tfrac{x^3}{3} - \lambda\, x\,, \end{aligned} \tag{97}$$

is not Hermitian if we view x as an ordinary real variable and wave functions $\psi(x)$ as endowed with the standard scalar product on the real line. However, by a similarity transformation one can transform $H(x)$ to a new operator

$$\tilde{H}(x) = e^{-S(x)/2g_s} H(x)\, e^{S(x)/2g_s}\,, \qquad \tilde{\psi}(x) = e^{-S(x)/2g_s}\psi(x)\,, \tag{98}$$

which *is* Hermitian on $L^2(R, dx)$. We have

$$\tilde{H}(x) = -g_s \frac{d^2}{dx^2} + \left(\frac{1}{4g_s}\left(\frac{dS(x)}{dx}\right)^2 + \frac{1}{2}\frac{d^2S(x)}{dx^2}\right), \tag{99}$$

which after substitution of the explicit form of the action becomes

$$\tilde{H}(x) = -g_s \frac{d^2}{dx^2} + V(x)\,, \qquad V(x) = \frac{1}{4g_s}\left(\lambda - x^2\right)^2 + x\,. \tag{100}$$

The fact that one can write

$$\tilde{H}(x) = R^\dagger R\,, \qquad R = -\sqrt{g_s}\frac{d}{dx} + \frac{1}{2\sqrt{g_s}}\frac{dS(x)}{dx} \tag{101}$$

implies that the spectrum of $\tilde{H}(x)$ is positive, discrete and unambiguous. We conclude that the formalism of stochastic quantization has provided us with a nonperturbative definition of the CDT string field theory.

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