HAUSDORFF AND SPECTRAL DIMENSION OF INFINITE RANDOM GRAPHS*

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We give an elementary introduction to the construction of probability distributions on sets of infinite graphs, called random graphs, as limits of ensembles of finite graphs motivated by a brief discussion of the incipient infinite cluster in bond percolation on an infinite graph. The Hausdorff and spectral dimension of random graphs are introduced. For illustrational purposes some concrete examples of random combs are considered, for which it is shown that their Hausdorff and spectral dimensions equal 1 except on a critical curve on which they exhibit non-trivial behaviour. Models of so-called generic planar random trees are defined and a proof that their Hausdorff and spectral dimensions equal 2 and 4/3, respectively, is outlined. In a concluding section we describe some open problems.

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1. Introduction

Random graphs occur in many areas of physics and mathematics, either as useful technical tools or as models of physical systems involving randomness, *e.g.* in the theory of random media and in polymer physics. The random graphs we shall be interested in are not of the Rényi–Erdös type, where randomness is realised in terms of a given fixed probability p for the presence of an edge between any pair of vertices in a given vertex set. Instead, the probability distribution on graphs is defined by a local weight function, typically depending on the degree of vertices. Our main focus will be on certain large scale properties of random graphs and it turns out to be convenient for this purpose to consider ensembles of infinite graphs. We construct such ensembles by taking limits of distributions on finite graphs of fixed size N for $N \to \infty$ in a somewhat similar spirit as the construction of

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the thermodynamic limit for statistical mechanical systems. One particular motivating example for this approach is that of discretised quantum gravity. Restricting to two dimensions the basic object in such models is a triangulated surface, where the triangulation is considered as defining a discretised metric on the surface such that summing over triangulations can be viewed as a discrete approximation to a continuum integral over metrics on the surface, see [2] for details. The ultimate goal of such an approach to quantum gravity is to define integrals of appropriate observables as continuum limits of the corresponding discrete expressions and preferably in such a way that the resulting continuum quantities define a probability distribution on a suitable class of continuum surfaces. On the other hand, one expects that the large scale characteristics of the continuum limit should be detectable as universal long distance features of the discretised models. An important example of such a quantity is the dimensionality of surfaces. Two notions of dimension, the Hausdorff dimension and the spectral dimension, will be defined and discussed in some detail for concrete models of random trees and random surfaces.

The main purpose of this paper is to motivate and give an elementary introduction to the construction of probability distributions on infinite graphs as limits of ensembles of finite graphs in Sections 2 and 3, illustrated by simple examples of so-called random combs in Section 4. We also outline, in Section 5, the definition of some models of planar random trees and results on the Hausdorff and spectral dimension for these cases. In the concluding Section 6 we describe some open problems.

2. Basic concepts

We start by introducing the basic concepts entering the following discussion and at the same time fixing some notation.

2.1. Graphs

A *graph* G is specified by its *vertex* set $V(G)$ and its *edge* set $E(G)$. Vertices will be denoted by v or v_i *etc.* An edge is then an unordered pair (v, v') of different vertices. Both finite and infinite graphs will be considered, *i.e.* $V(G)$ may be finite or infinite, and all graphs will be assumed to be locally finite, *i.e.* the number σ_v of edges containing a vertex v, called the *degree* of v, is finite for all $v \in V(G)$. By the *size* of G we shall mean the number of edges in G and denote it by $|G|$, *i.e.* $|G| = \sharp E(G)$, where $\sharp M$ is used to denote the number of elements in a set M.

A *path* in G is a sequence of different edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ where v_0 and v_k are called the *end vertices*. If $v_0 = v_k$ the path is called a *circuit* originating at v_0. The graph G is called *connected* if any two vertices v and v' of G can be connected by a path, *i.e.* they are end vertices of a path. The *graph distance* between v and v' is then defined as the minimal number of edges in a path connecting them. A connected graph is called a *tree* if it has no circuits.

A *planar graph* is a graph together with an embedding $\phi : V(G) \to \mathbb{R}^2$ and an association to each edge $(v, v') \in E(G)$ of an arc $\psi(v, v')$ in $\mathbb{R}^2$ connecting $\phi(v)$ and $\phi(v')$ such that arcs corresponding to different edges are disjoint except possibly for endpoints. Two planar graphs are considered identical if one can be continuously deformed into the other in $\mathbb{R}^2$.

2.2. Simple random walk

A *walk* on a graph G is a sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ of (not necessarily different) edges in G. We shall denote such a walk by $\omega : v_0 \to v_k$ and call v_0 the *origin* and v_k the *end* of the walk. Moreover, the number k of edges in ω will be denoted by $|\omega|$. To each such walk ω we associate a weight

$$p_G(\omega) = \prod_{i=0}^{|\omega|-1} \sigma_{\omega(i)}^{-1} \,,$$

where $\omega(i)$ is the i-th vertex in ω. Denoting by $\Pi_n(G, v_0)$ the set of walks of length n originating at vertex v_0 we have

$$\sum_{\omega \in \Pi_n(G, v_0)} p_G(\omega) = 1 \,,$$

i.e. p_G defines a probability distribution on $\Pi_n(G, v_0)$. We call p_G the *simple random walk* on G.

2.3. Hausdorff dimension

Given a connected graph G and $R \geq 0$ and $v \in V(G)$ we denote by $B_R(G, v)$ the closed ball of radius R centered at v, *i.e.* the subgraph of G spanned by vertices at graph distance at most R from v. If G is connected and the limit

$$d_{\mathrm{h}} = \lim_{R \to \infty} \frac{\ln |B_R(G, v)|}{\ln R} \tag{1}$$

exists, we call d_{h} the *Hausdorff dimension* of G. If G is a finite graph we clearly have $d_{\mathrm{h}} = 0$, a case we leave out of consideration in the following. It is easily seen that the existence of the limit as well as its value do not depend on the vertex v.

2.4. Spectral dimension

For a connected graph G and $v \in V(G)$ we denote by $p_t(G,v)$ the *return probability* of the simple random walk to v at time t, that is

$$p_t(G,v) = \sum_{\substack{\omega: v \to v \\ |\omega| = t}} p_G(\omega)\,.$$

One can in a standard manner relate this quantity to the discrete heat kernel on G, but we shall not need this interpretation in the following. If the limit

$$d_{\mathrm{s}} = -2 \lim_{t \to \infty} \frac{\ln p_t(G,v)}{\ln t} \tag{2}$$

exists, we call d_{s} the *spectral dimension* of G. Again in this case, the existence and value of the of the limit are independent of v. Moreover, $d_{\mathrm{s}} = 0$ if G is finite since $p_t(G,v) \to (\sharp V(G))^{-1}$ for $t \to \infty$. If G is infinite one has

$$d_{\mathrm{h}} \geq 1 \qquad \text{and} \qquad d_{\mathrm{s}} \geq 1\,.$$

If G is the hyper-cubic lattice $\mathbb{Z}^d$ it is clear that $d_{\mathrm{h}} = d$ and by Fourier analysis it is straight-forward to see that also $d_{\mathrm{s}} = d$. However, examples of graphs with $d_{\mathrm{h}} \neq d_{\mathrm{s}}$ are abundant, see *e.g.* [13].

2.5. Random graphs

An ensemble of graphs, or a *random graph*, is a set $\mathcal{G}$ of graphs equipped with a probability measure μ. By $\langle \cdot \rangle_\mu$ we denote the expectation w.r.t. μ. In the following it is assumed that graphs in $\mathcal{G}$ are rooted, *i.e.* each graph has a marked vertex r, called the *root*. We shall then use the notation $B_R(G)$ for $B_R(G,r)$ and $p_t(G)$ for $p_t(G,r)$.

The *annealed* Hausdorff dimension $\bar{d}_{\mathrm{h}}$ and spectral dimension $\bar{d}_{\mathrm{s}}$ of an ensemble $(\mathcal{G},\mu)$ are defined by

$$\bar{d}_{\mathrm{h}} = \lim_{R \to \infty} \frac{\ln \langle\, |B_R(G)|\, \rangle_\mu}{\ln R} \tag{3}$$

and

$$\bar{d}_{\mathrm{s}} = -2 \lim_{t \to \infty} \frac{\ln \langle\, p_t(G)\, \rangle_\mu}{\ln t}\,, \tag{4}$$

respectively, provided the limits exist. If there exists a subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $\mu(\mathcal{G}_0) = 1$ and such that every $G \in \mathcal{G}_0$ has Hausdorff dimension d_{h} we say that the Hausdorff dimension of $(\mathcal{G},\mu)$ is almost surely d_{h}, and similarly for the spectral dimension.

2.6. Random infinite graphs

As indicated above we shall mainly be interested in infinite graphs and ensembles of such graphs. In the latter case the measure μ will be obtained as a limit of measures $\mu_N\,, N \in \mathbb{N}$, defined on sets of finite graphs. It is then appropriate to consider $\mathcal{G}$ as a metric space with distance between two rooted graphs $G, G' \in \mathcal{G}$ defined by

$$d_{\mathcal{G}}\left(G, G'\right) = \inf\left\{(R+1)^{-1} :\, B_R(G) = B_R\left(G'\right)\right\} \tag{5}$$

and the measures under consideration will be assumed to be Borel measures. We say that a sequence of probability measures (μ_N) converges to a probability measure μ on $\mathcal{G}$ if

$$\int_{\mathcal{G}} f\, d\mu_N \overset{N\to\infty}{\longrightarrow} \int_{\mathcal{G}} f\, d\mu\,,$$

for all bounded continuous functions f on $\mathcal{G}$. This notion of convergence is called *weak convergence* by probabilists [9] and can be shown [12] to be equivalent to the requirement

$$\mu_N(\{G :\, B_R(G) = G_0\}) \overset{N\to\infty}{\longrightarrow} \mu(\{G :\, B_R(G) = G_0\})\,, \tag{6}$$

for all finite rooted graphs G_0 and all $R \geq 1$. A sufficient condition for the existence of a limiting measure μ is, in addition to the existence of the limit in (6), a so-called *tightness requirement* ensuring that the total probability is conserved in the limit [9, 12].

3. The incipient infinite percolation cluster

Let G be an infinite, connected, rooted graph with root r and let $0 \leq q \leq 1$. For an edge $e \in E(G)$ let η_e be the probability distribution on $\{0,1\}$ such that

$$\eta_e(\zeta) = \begin{cases} q & \text{if } \zeta = 1 \\ 1-q & \text{if } \zeta = 0 \end{cases}\,.$$

Bond percolation on G is defined by the product measure

$$\rho_q = \prod_{e\in E(G)} \eta_e$$

on the compact space $\{0,1\}^{E(G)}$ of configurations assigning the value 0 or 1 to each edge in G.

Given such a configuration c we define the percolation cluster c_r containing r as the maximal connected rooted subgraph of G with root r such that c has value 1 for all edges in c_r. If no edge emerging from r has value 1 the cluster c_r consists of the root r alone.

3.1. Percolation on a hyper-cubic lattice

The most extensively studied case of bond percolation is when G is a hyper-cubic lattice $\mathbb{Z}^d$ of dimension $d \geq 2$ with $r = \underline{0}$. It can be shown ([18] Sec. 1.4) that there exists a critical probability $q_{\rm cr} \in]0,1[$ with the property that

$$\rho_q(\{c \mid |c_r| = \infty\}) \begin{cases} = 0 & \text{if } q < q_{\rm cr} \\ > 0 & \text{if } q > q_{\rm cr} \end{cases} ,$$

where $|c_r|$ denotes the number of edges in the cluster c_r. Moreover, it is known [18] for $d = 2$ or $d \geq 19$ that $\rho_{q_{\rm cr}}(\{c \mid |c_r| = \infty\}) = 0$, *i.e.* the percolation cluster c_r is almost surely finite at the percolation threshold $q_{\rm cr}$. For $3 \leq d \leq 18$ it is an open question whether c_r is infinite or not. For $q < q_{\rm cr}$ it is known that the distribution of $|c_r|$ has an exponential tail

$$\rho_q(\{c \mid |c_r| = n\}) \sim \text{const.}\, e^{-\alpha(q)n} , \qquad n \to \infty ,$$

such that the mean cluster size

$$\chi(q) = \langle\, |c_r|\, \rangle_{\rho_q}$$

is well defined and finite for $q < q_{\rm cr}$. It is expected that $\chi(p)$ diverges at $q_{\rm cr}$ with an associated critical exponent γ,

$$\chi(q) \sim (q_{\rm cr} - q)^{-\gamma} , \qquad q \nearrow q_{\rm cr} .$$

This entails that clusters of arbitrarily large size become increasingly frequent as q approaches $q_{\rm cr}$.

In order to describe the structure of c_r close to $q_{\rm cr}$ it is therefore reasonable to attempt to define the probability measure μ on the space $\mathcal{C}\ell$ of clusters c_r, viewed as rooted graphs embedded in $\mathbb{Z}^d$, by

$$\mu = \lim_{N\to\infty} \mu_N , \tag{7}$$

where μ_N is the measure $\rho_{q_{\rm cr}}$ conditioned on clusters of size N,

$$\mu_N(A) = \rho_{q_{\rm cr}}(A \,|\, |c_r| = N) .$$

Existence of a limit similar to (7) for $d = 2$, where conditioning on non-empty intersection with the boundary of a box of side length N centered at $(0,0)$ is used, has been proven in [22]. Similarly, for large dimension the existence of a limit has been established in [20].

The ensemble $(\mathcal{C}\ell, \mu)$ is called the *d-dimensional incipient infinite cluster* assuming it exists. It was originally conjectured by Alexander and Orbach [1] that $(\mathcal{C}\ell, \mu)$ has spectral dimension $d_{\rm s} = \frac{4}{3}$ for all dimensions $d \geq 2$, but it is now generally believed to hold only for d sufficiently large. For $d \geq 19$ a proof has been announced in [23]. Not much seems to be known about the Hausdorff dimension of $(\mathcal{C}\ell, \mu)$, except that clearly $d_{\rm h} \leq d$.

3.2. Percolation on a Cayley tree

Let Γ be a fixed planar Cayley tree with all vertices of degree n except for the root r, which is assumed for convenience to be of degree 1.

It is seen that in this case a cluster c_r is a subtree of Γ and for a given finite, rooted, planar tree T with root of degree 1 we have

$$\rho_q(\{c \mid c_r = T\}) = q \prod_{v \in V(T) \setminus r} \binom{n-1}{\sigma_v - 1} q^{\sigma_v - 1}(1-q)^{n-\sigma_v} ,$$

where the binomial coefficient is by definition equal to 0 if $\sigma_v > n$. Conditioning on the set of configurations where the root link has value 1 the pre-factor q drops out and we get the distribution ρ on the set $\mathcal{T}_{\mathrm{fin}}$ of finite, rooted, planar trees with root of degree 1 given by

$$\rho(T) = \prod_{v \in V(T) \setminus r} p_{\sigma_v - 1} , \tag{8}$$

where

$$p_k = \binom{n-1}{k} q^k (1-q)^{n-1-k} , \qquad k = 0, 1, 2, \ldots . \tag{9}$$

Note that

$$\sum_{k=0}^{\infty} p_k = 1 . \tag{10}$$

A distribution ρ of the form (8) on $\mathcal{T}_{\mathrm{fin}}$ where $(p_k)_{k\geq 0}$ satisfies (10) and $p_k \geq 0$ is called a *Galton–Watson* process [19], and the numbers p_k are called the *off-spring probabilities* of the process.

In order to determine p_{cr} let

$$Z_N = \sum_{T:|T|=N} \rho(T) = \rho(\{T \mid |T| = N\}) \tag{11}$$

and define the corresponding generating function by

$$Z(t) = \sum_{N=1}^{\infty} Z_N t^N . \tag{12}$$

Letting $f(t)$ be the generating function for $(p_k)_{k\geq 0}$

$$f(t) = \sum_{k=0}^{\infty} p_k t^k \tag{13}$$

one finds that $Z(t)$ is determined by

$$Z(t) = tf(Z(t)) \,. \tag{14}$$

With p_k given by (9) we have

$$f(t) \equiv f_q(t) = (1 - q + tq)^{n-1} \,.$$

The total probability that c_r is finite is

$$\rho(\{T \mid |T| < \infty\}) = \sum_{N=1}^{\infty} \rho(\{T \mid |T| = N\}) = Z(1) \,,$$

so that by (14) we have that $Z(1) = 1$ if q is small enough such that $f_q'(1) \leq 1$. Since

$$f_q'(1) = (n-1)q$$

it follows that this is the case if $q \leq \frac{1}{n-1}$ whereas $Z(1) < 1$ if $q > \frac{1}{n-1}$, that is

$$q_{\mathrm{cr}} = \frac{1}{n-1} \,.$$

Hence we have found that q_{cr} is determined as the value of q fulfilling

$$\sum_{k=1}^{\infty} kp_k = 1 \,. \tag{15}$$

A Galton–Watson process fulfilling (15), *i.e.* the expected number of offspring equals 1 is called *critical* [19]. In this sense critical percolation on Γ is described by a critical Galton–Watson process.

Conditioning on trees of fixed size N we obtain from ρ the measure

$$\mu_N(T) = \rho(T \,|\, |T| = N) = Z_N^{-1} \rho(T) \,. \tag{16}$$

As will be explained in Section 5 the incipient infinite cluster, *i.e.* the limit $\lim_{N\to\infty} \mu_N$, exists as a special case of a more general result for so-called generic critical Galton–Watson processes, and is a probability distribution defined on the set $\mathcal{T}$ of infinite planar rooted trees. In order to explain the idea of those arguments we shall consider in the next section some particularly simple examples of random trees, obtained by restricting the trees to be so-called combs. Besides serving as a useful illustration those models are of some interest in their own right; in particular, the resulting limiting measures can capture the behaviour of certain N-particle dynamical systems known as balls in boxes at large N as will be explained further below. Related models have been considered from a different point of view in [21].

4. Random combs

As it turns out the basic arguments that apply to establish existence of the infinite incipient percolation cluster on a Cayley tree as well as to generic, critical Galton–Watson processes can be effectively illustrated in the case of comb ensembles.

A *comb* consists of a linear chain s_i, $0 \leq i < L+1$, called the *spine* (or backbone) of the comb, and to each spine vertex s_i, $i \geq 1$, is attached a linear chain $s_i, t_{i1}, t_{i2}, \ldots$ with an arbitrary number n_i of vertices, where $1 \leq n_i \leq \infty$. Likewise the spine length L fulfills $1 \leq L \leq \infty$. We consider combs as planar rooted trees whose root is $r = s_0$ (which by convention has degree 1) and whose teeth all emerge to the left of the spine when the latter is oriented from the root and outwards, see Fig. 1.

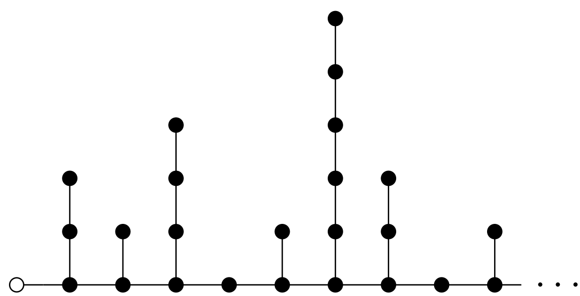

Fig. 1. A comb.

Let $\mathcal{C}^{(L)}$ denote the set of combs with spine of length $L < \infty$. Since vertices in a comb have degree at most 3 it follows that $\mathcal{C}^{(L)}$ is a compact subset of the metric space $(\mathcal{T}, d_{\mathcal{T}})$ of all planar rooted trees with distance defined by (5). Indeed, it is easily seen that $\mathcal{C}^{(L)}$ is homeomorphic to the product space $(\mathbb{N}\cup\{\infty\})^L$, where $\mathbb{N}\cup\{\infty\}$ is the standard one-point compactification of $\mathbb{N}$.

4.1. The case L is fixed

Let us denote by $\mathcal{C}_N^{(L)}$ the subset of $\mathcal{C}^{(L)}$ consisting of combs with exactly N edges

$$\mathcal{C}_N^{(L)} = \{C \in \mathcal{C}^{(L)} \,:\, n_1(C) + \ldots + n_L(C) = N\}\,.$$

We consider probability distributions μ_N on the finite set $\mathcal{C}_N^{(L)}$ of the form

$$\mu_N(C) = \frac{1}{Z_N} \prod_{i=1}^{L} q(n_i(C))\,, \tag{17}$$

where the weight function $q(n)$ is assumed to be positive with asymptotic behaviour

$$q(n) \sim c n^{-b}, \qquad n \to \infty, \tag{18}$$

where $b \in \mathbb{R}$ and $c > 0$ are constants. Here Z_N, the *partition function*, is a normalisation factor

$$Z_N = \sum_{n_1+\ldots+n_L=N} \prod_{i=1}^{L} q(n_i)\,.$$

Let $\zeta(t)$ denote the generating function for $q(n)$

$$\zeta(t) = \sum_{n=1}^{\infty} q(n)\, t^n\,.$$

By (18) it is clear that ζ is analytic in the open unit disc with a singularity at $t = 1$ such that for $t \nearrow 1$

$$\zeta(t) \sim \begin{cases} kc\,(1-t)^{b-1} & \text{if } b < 1 \\ -c\,\ln(1-t) & \text{if } b = 1 \\ a(t) + kc\,(1-t)^{b-1} & \text{if } b > 1 \end{cases}, \tag{19}$$

where $k \neq 0$ is a constant depending on b and $a(t)$ is analytic at $t = 1$. We shall assume, in addition, that $\zeta(t)$ has the required analyticity properties in a slightly larger so-called Δ-domain containing the unit disk such that the generating functions for partition functions encountered below are amenable to singularity analysis. Essentially, this means that (19) is assumed to hold for $t \to 1$ inside the larger domain. We refer to [17] Sec. VI. 3 for precise statements.

We then have the following convergence result.

Theorem 4.1 *For fixed $L < \infty$ the measures $(\mu_N)_{N\geq 1}$ defined by (17) converge to a probability distribution μ on the set $\mathcal{C}_\infty^{(L)}$ of infinite combs in $\mathcal{C}^{(L)}$, characterised as follows:*

If $b \leq 1$ then $\mu = \delta_{C_{L,\infty}}$, the Dirac measure concentrated at the comb $C_{L,\infty}$ all of whose teeth are infinite.

If $b > 1$ then μ is supported on the set of combs with a single infinite tooth and for each such comb C we have

$$\mu(C) = \frac{1}{L}\left(\sum_{n=1}^{\infty} q(n)\right)^{1-L} \prod_{i:n_i(C)\neq\infty} q(n_i(C))\,.$$

Proof. Define the generating function

$$Z(t) = \sum_{N=1}^{\infty} Z_N\, t^N$$

so that

$$Z(t) = \zeta(t)^L$$

and hence by (19)

$$Z(t) \sim \begin{cases} (kc)^L\,(1-t)^{(b-1)L} & \text{if } b < 1 \\ (-c)^L\,(\ln(1-t))^L & \text{if } b = 1 \\ a(t)^L + L\,\zeta(1)^{L-1}\,kc\,(1-t)^{b-1} & \text{if } b > 1 \end{cases}.$$

Using standard transfer theorems [17] we conclude that for $N \to \infty$

$$Z_N \sim \begin{cases} k_1\,N^{(1-b)L-1} & \text{if } b < 1 \\ k_1\,N^{-1}(\ln N)^{L-1} & \text{if } b = 1 \\ L\,\zeta(1)^{L-1}\,c\,N^{-b} & \text{if } b > 1 \end{cases}, \tag{20}$$

where $k_1 > 0$ is a constant depending on b and c.

Compactness of $\mathcal{C}^{(L)}$ implies that in order to prove convergence of $(\mu_N)_{N\geq 1}$ it suffices (see [12]) to establish (6) (with G replaced by C). It is, however, equivalent and more convenient to establish convergence of $\mu_N(A_R(C_0))$, $N \geq 1$, where C_0 is an arbitrary finite comb in $\mathcal{C}^{(L)}$ and $A_R(C_0)$ is the set of combs coinciding with C_0 up to a fixed tooth-height $R \geq 0$. Clearly, we can without loss of generality assume that R equals the maximal tooth-height of C_0 since, if R is larger, $A_R(C_0)$ is empty for N large enough and, if it is smaller, we can replace C_0 by the comb obtained by shortening the teeth by the amount exceeding height R.

For such a comb C_0 we let I denote the set of $i \in \{1, \ldots, L\}$ such that $n_i(C_0) = R+1$ and set $n_i(C_0) = n_i^0$ and $N_0 = \sum_{i\notin I} n_i^0$. For N large enough we then have

$$\mu_N(A_R(C_0)) = \frac{1}{Z_N} \prod_{i\notin I} q(n_i^0) \sum_{\substack{\sum_{i\in I} n_i = N - N_0 \\ n_i > R}} \prod_{i\in I} q(n_i)\,. \tag{21}$$

<u>$b < 1$</u>: In this case, repeating the arguments yielding (20) shows that the sum in (21) has asymptotic behaviour given by

$$S_N \;\sim\; k_2\,N^{(1-b)(\sharp I)-1} \tag{22}$$

and hence

$$\mu_N(A_R(C_0)) = \prod_{i\notin I} q(n_i^0)\frac{S_N}{Z_N} \;\sim\; k_2' N^{(b-1)(L-\sharp I)}\,,$$

where $k_2, k_2' > 0$ are constants. For $\sharp I < L$ the last expression vanishes for $N \to \infty$ while for $\sharp I = L$ we have

$$\mu_N(A_R(C_0)) = \frac{1}{Z_N} \sum_{\substack{\sum n_1+\ldots+n_L=N\\ n_i>R}} \prod_{i=1}^{L} q(n_i) \;\to\; 1\,, \qquad N\to\infty\,,$$

since the last sum deviates from Z_N by a sum of terms of the same form where at least one of the summation variables is fixed at a value less than R and hence is of lower order in N by the preceding estimate. This being valid for any $R \geq 0$ we conclude that the limit $\mu = \lim_{N\to\infty}\mu_N$ exists and that $\mu(C_{L,\infty}) = 1$, which proves the first part of the theorem.

$\underline{b=1}$: In this case, (22) is replaced by

$$S_N \;\sim\; k_2\, N^{-1}(\ln N)^{\sharp I-1}$$

and the conclusion is the same as above.

$\underline{b>1}$: First, consider for fixed $i \neq j$ in $\{1,\ldots,L\}$ and $K \geq 0$ the set $A(i,j;K)$ of combs C such that $n_i(C), n_j(C) > K$. If we remove from a comb C in $A(i,j;K)$ the i-th and j-th tooth as well as the edges (s_{i-1}, s_i) and (s_{j-1}, s_j) and then identify s_{i-1} with s_i and s_{j-1} with s_j we obtain a new comb C' in $\mathcal{C}^{(L-2)}$. Using this observation it follows from (17) that

$$\mu_N(A(i,j;K)) \;\leq\; \sum_{\substack{n,m>K\\ n+m<N}} q(n)q(m)\frac{Z_{N-n-m}}{Z_N} \;\leq\; \text{const.}\sum_{n>K} q(n) \tag{23}$$

since, in the first sum, at least one of the arguments n, m, $N-n-m$ must be $\geq N/3$ and the corresponding factor then cancels Z_N up to a constant by (18) and (20). Since the last expression in (23) is $O(K^{1-b})$ and vanishes uniformly in N as $K\to\infty$ we can evaluate the limit $\lim_{N\to\infty}\mu_N(A_{R,L}(C_0))$ by first imposing the constraint $n_i(C) \leq K$ for all $i \in \{1,\ldots,L\}$ except one and letting $K\to\infty$ in the last step.

Consider for definiteness the case $L = 3$ and $I = \{2,3\}$. By (21) we then have for $K > R$

$$\mu_N(A_R(C_0)) =$$
$$q\left(n_0^1\right)\left(\sum_{\substack{n_2+n_3=N-n_1^0\\ R<n_2\leq K}} q(n_2)\frac{q(n_3)}{Z_N} + \sum_{\substack{n_2+n_3=N-n_1^0\\ R<n_3\leq K}} \frac{q(n_2)}{Z_N}\,q(n_3)\right) + O\left(K^{1-b}\right)\,,$$

where the quotients in the (finite) sums converge to $L^{-1}\zeta(1)^{1-L} = \frac{1}{3}\zeta(1)^{-2}$ for $N \to \infty$ according to (18) and (20). Hence, the two sums have identical limits and we obtain after letting $K \to \infty$

$$\lim_{N\to\infty} \mu_N(A_R(C_0)) \;=\; \tfrac{2}{3}\zeta(1)^{-2} q(n_1^0) \sum_{n>R} q(n)\,.$$

For a general finite comb C_0 this argument immediately generalises and establishes the existence of a limiting measure μ such that

$$\mu(A_R(C_0)) \;=\; \frac{\sharp I}{L}\zeta(1)^{1-L}\left(\sum_{n>R} q(n)\right)^{\sharp I-1} \prod_{i\notin I} q(n_i^0)\,. \qquad (24)$$

If $\mathcal{A}_{i_0}$ denotes the set of combs with tooth at i_0 infinite and all other teeth finite, we have for each $C \in \mathcal{A}_{i_0}$ that

$$\mu(C) = \lim_{R\to\infty} \mu(A_R(C)) = \frac{1}{L}\zeta(1)^{1-L} \prod_{i\neq i_0} q(n_i(C))$$

proving the claimed formula. Since summing this expression over all combs in $\mathcal{C}^{(L)}$ with a single infinite tooth gives 1 the theorem is proven. $\square$

Remark 4.2 Stationary measures for certain simple dynamical systems of balls in boxes fall within the class of finite size measures considered in this subsection. We can see this as follows. Interprete $s_1, s_2, \ldots, s_L$ as labelling a sequence of boxes and think of s_0 as being identified with s_L, *i.e.* we impose periodic boundary conditions. Inside box s_i we assume there is a pile of n_i balls that we identify with the height of the i-th tooth in a comb in $\mathcal{C}^{(L)}$. Given a configuration of balls we assume there is a transition rate $u_i(n_i)$ for a ball in box s_i to move one step in counterclockwise direction to box s_{i-1}. Since the process preserves the total number N of balls this defines a dynamical system with finite state space $\mathcal{C}_N^{(L)}$ that clearly is an indecomposable Markov chain if all u_i are strictly positive. Hence it has a unique stationary state, *i.e.* a probability distribution μ_N that is preserved under the process. Using the Chapman–Kolmogoroff equations it can be found in explicit form [16]

$$\mu_N(C) \;=\; \frac{1}{Z'_N} \prod_{i=1}^{L} \left(\prod_{m=1}^{n_i(C)} \frac{1}{u_i(m)} \right)\,,$$

where Z'_N is a normalisation factor. Choosing $u_i = u$ independent of i we see that this is a measure of the form (17) where

$$q(n) = \prod_{m=1}^{n} \frac{1}{u(m)}\,.$$

In particular, choosing

$$u(m) = 1 + \frac{b}{m} + O(m^{-(1+\epsilon)}), \qquad m \in \mathbb{N},$$

where $\epsilon > 0$, we have

$$q(n) \sim \frac{c}{n^b}, \qquad n \to \infty.$$

Theorem 4.1 implies in this case that increasing the total number of particles has the effect, in the stationary limit, that all boxes become crowded with balls if $b \leq 1$ whereas particles mainly will concentrate in one box at a time if $b > 1$. This phenomenon has been discussed previously in [8, 16] and references given there.

4.2. The case L is arbitrary

We now relax the restriction to combs of fixed spine length and let $\mathcal{C}_N$ denote the set of all combs with N edges and define μ_N on $\mathcal{C}_N$ by

$$\mu_N(C) = \frac{1}{Z_N} \prod_{i=1}^{L(C)} q(n_i(C)), \tag{25}$$

where q obeys the same conditions as previously and Z_N is now given by

$$Z_N = \sum_{C \in \mathcal{C}_N} \prod_{i=1}^{L(C)} q(n_i(C)),$$

with generating function

$$Z(t) = \frac{\zeta(t)}{1 - \zeta(t)}. \tag{26}$$

It is worthwhile noting that whereas the finite size measures μ_N for fixed L were invariant under scaling of the function q by a constant this is no longer true for μ_N given by (25). In fact, the constant c in (18) can be regarded as a fugacity associated with the spine length.

The general result for this case can be formulated as follows.

Theorem 4.3 *The measures $(\mu_N)_{N\geq 1}$ defined by (25) converge to a probability measure μ on $\mathcal{C}_\infty$ that can be characterised as follows:*

If $\zeta(1) \geq 1$ then μ is supported on the set of combs with an infinite spine all of whose teeth are finite and their lengths are identically and independently distributed according to the density ν on $\mathbb{N}_0$ given by

$$\nu(n) = q(n+1)\, t_0^{n+1}, \qquad n \geq 0,$$

where $t_0 > 0$ is determined by $\zeta(t_0) = 1$.

If $\zeta(1) < 1$ then μ is supported on the set of combs with finite spine length and a single infinite tooth such that if $C \in \mathcal{C}^{(L)}$ is such a comb then

$$\mu(C) = (1-\zeta(1))^2 \prod_{i:n_i(C)<\infty} q(n_i(C))\,.$$

Proof. Instead of establishing (6) it is again slightly more convenient, but equivalent, to show that for each finite comb C_0 and all $R \geq 0$, $L \geq 1$, the sequence $\mu_N(A_{R,L}(C_0))$, $N \geq 1$, converges, where $A_{R,L}(C_0)$ is the set of all combs coinciding with C_0 up to tooth height R of the first L teeth. By the same argument as in the proof of Theorem 4.1 we can assume that $R \geq H_0$ and $L \geq L_0$, where H_0 is the maximal height of teeth in C_0 and L_0 is the spine length of C_0, and furthermore that $R = H_0$ or $L = L_0$.

$\underline{\zeta(1) > 1}$: Since the radius of convergence of the generating series for $q(n)$, $n \geq 1$, equals 1 by (18) and $q(n) \geq 0$ there exists $t_0 \in]0,1[$ such that $\zeta(t_0) = 1$. Since ζ is analytic at t_0 we get from (26) that

$$Z(t) \;\sim\; \zeta'(t_0)^{-1}(t_0-t)^{-1}\,, \qquad t \to t_0\,,$$

and hence

$$Z_N \;\sim\; \zeta'(t_0)^{-1}t_0^{-N}\,, \qquad N \to \infty\,.$$

Assume first that $R > H_0$ and let $n_1^0, \dots, n_{L_0}^0$ be the number of vertices in the teeth of C_0, respectively. In this case $L = L_0$ and the combs in $A_{R,L}(C_0)$ coincide with C_0 on the first L_0 teeth and the remaining part of the comb is an arbitrary comb with root s_{L_0}. Hence, for $N \to \infty$,

$$\mu_N(A_{R,L}(C_0)) \;=\; \frac{Z_{N-n_1^0-\dots-n_{L_0}^0}}{Z_N}\prod_{i=1}^{L_0} q(n_i^0) \;\longrightarrow\; \prod_{i=1}^{L_0} q(n_i^0)\,t_0^{n_i^0}\,. \tag{27}$$

Next, assume $R = H_0$ and let $A(j;K) = \{C \in \mathcal{C}:\; n_j(C) > K\}$ for fixed $j \in \mathbb{N}$ and $K \geq 0$. By the same argument as for $A(i,j;K)$ above we have

$$\mu_N(A(j;K)) \leq \sum_{n=K+1}^{N-1}\frac{Z_{N-n}}{Z_N}\,q(n) \;\leq\; \text{const.}\sum_{n>K} q(n)t_0^n\,, \tag{28}$$

where the last expression tends to 0 uniformly in N for $K \to \infty$. This implies that we can obtain the limit $\lim_{N\to\infty}\mu_N(A_{R,L}(C_0))$ by first inserting the constraint $n_j(C) \leq K$, $1 \leq j \leq L$, take the limit $N \to \infty$ and finally let $K \to \infty$.

Using (27) this gives in case $L = L_0$ the result

$$\lim_{N\to\infty}\mu_N(A_{H_0,L}(C_0)) = \prod_{i\notin I} q(n_i^0)\,t_0^{n_i^0}\left(\sum_{n>R} q(n)\,t_0^n\right)^{\sharp I}\,, \tag{29}$$

where I is the set of $i \in \{1, \ldots, L_0\}$ such that $n_i^0 = H_0$. On the other hand, if $L > L_0$ the limit vanishes since in this case $A_{H_0,L}(C_0) \subseteq \mathcal{C}^{(L_0)}$ and hence all combs contributing to $\mu_N(A_{R,L}(C_0))$ have at least one tooth of length $\geq N/L_0 - 1$.

This proves the existence of the limiting measure μ. Letting $R \to \infty$ in (29) the limit vanishes unless $I = \emptyset$ which shows that μ is supported on the set of combs with finite teeth and it also follows from (29) that μ equals the product measure as stated.

$\underline{\zeta(1) < 1}$: The assumption implies $b > 1$ such that (19) and (26) give

$$Z(t) \sim A(t) + \frac{kc}{(1-\zeta(1))^2}(1-t)^{b-1}$$

for $t \to 1$, where $A(t)$ is analytic at $t = 1$. We conclude that for $N \to \infty$

$$Z_N \sim \frac{c\,N^{-b}}{(1-\zeta(1))^2}\,. \tag{30}$$

Assume first $R > H_0$. By the argument given above for (27) we obtain from (30) that

$$\mu_N(A_{R,L}(C_0)) \to \prod_{i=1}^{L_0} q(n_i^0)\,, \qquad N \to \infty\,.$$

Next assume $R = H_0$ and $L > L_0$. In this case $A_{R,L}(C_0)$ is independent of L and contained in $\mathcal{C}^{(L_0)}$. By identical arguments to those yielding (24) we get with notation as above

$$\mu_N(A_{R,L}(C_0)) \to (\sharp I)(1-\zeta(1))^2 \prod_{i \notin I} q(n_i^0)) \left(\sum_{n>R} q(n)\right)^{\sharp I - 1}\,. \tag{31}$$

Finally, to deal with the case $R = H_0$ and $L = L_0$ we first note that the estimate (23) for $\mu_N(A(i,j;K)$ still holds with the same argument. On the other hand, given $L' \geq 1$ let $B(K, L')$ denote the set of combs with spine length at least L' and such that at most one of its first L' teeth has length $\geq K$. It is then easy to see that

$$\mu_N(B(K,L')) \leq \text{const.}\, L'\zeta(1)^{L'}\,,$$

where the last expression clearly vanishes uniformly in N for $K, L' \to \infty$. These two observations allow us to restrict attention to combs with spine length $\leq L'$ and one tooth of length $\geq K$ and ultimately letting $K, L' \to \infty$.

Using the previous result (31) this gives

$$\mu_N(A_{R,L}(C)) \;\to\; \prod_{i\notin I} q(n_i^0)\left(\sum_{n>R} q(n)\right)^{\sharp I-1}\left((\sharp I)(1-\zeta(1)) + \sum_{n>R} q(n)\right)$$

for $N\to\infty$. This proves the existence of the limiting measure μ.

Taking the limit $R\to\infty$ of the expression found in (31) we see that it vanishes unless $\sharp I = 1$ for R large enough, which shows that in $\mathcal{C}^{(L)}$ the measure μ is supported on the set of combs with a single infinite tooth and that for each such comb C in $\mathcal{C}^{(L)}$

$$\mu(C) = (1-\zeta(1))^2 \prod_{i:n_i(C)<\infty} q(n_i(C)) \tag{32}$$

as stated in the theorem. Summing over C gives

$$\mu(\mathcal{C}^{(L)}) = L(1-\zeta(1))^2\zeta(1)^{L-1}$$

and in turn summing this expression over $L\geq 1$ gives 1 proving that μ is supported on the set of combs with finite spine and a single infinite tooth.

$\underline{\zeta(1)=1}$: In this case we necessarily have $b>1$. Assuming first $1<b<2$ we have from (19) and (26) that

$$Z(t) \;\sim\; -\frac{1}{kc}(1-t)^{1-b}$$

and therefore

$$Z_N \;\sim\; k_2\, N^{b-2}\,,$$

where $k_2>0$ is a constant. Going back to the case $\zeta(1)>1$ above we obviously have that (27) still holds, and one easily verifies that corresponding to (28) we have

$$\mu_N(A(j;K)) \;\leq\; \text{const.}\, K^{1-b}\,,$$

where the constant is independent of N and j. It follows that the rest of the argument in the case $\zeta(1)>1$ above carries through unchanged proving the claim for $1<b<2$.

For $b\geq 2$ the argument is similar but the details depend on the order of the zero at $t=1$ of the analytic part $a(t)$ of $\zeta(t)$ and are left to the reader. □

Considering a family of distributions q fulfilling (18) and parametrised by b and c the theorem asserts that for $b\leq 1$ the limiting measure is always a product measure with exponentially distributed teeth, and for $b>1$ there is a critical curve $c=c_0(b)$ such that the same holds if $c>c_0(b)$ while for

$c = c_0(b)$ the exponential distribution of tooth lengths is replaced by a power law distribution

$$\nu(n) \sim c_0(b)\, n^{-b}\,. \tag{33}$$

Finally, for $c < c_0(b)$ the combs have finite spine and one infinite tooth almost surely.

Random combs with independent and identical tooth length distributions of the form (33), were extensively studied in [13] and shown to have Hausdorff and spectral dimensions given by

$$\bar{d}_{\rm h} \;=\; 3-b \qquad \text{and} \qquad \bar{d}_{\rm s} \;=\; 2-\frac{b}{2}\,, \qquad 1<b<2\,,$$

whereas $\bar{d}_{\rm h} = \bar{d}_{\rm s} = 1$ for $b \geq 2$. This latter statement consequently holds also for the exponential distribution in the case $\zeta(1) > 1$ of Theorem 4.3. It is a trivial matter to verify that for the remaining cases of Theorems 4.1 and 4.3 we have $d_{\rm h} = d_{\rm s} = 1$ almost surely.

Note that except for the degenerate first case in Theorem 4.1 the limiting measures found in this section are concentrated on trees with a single infinite path. In the last case of Theorem 4.1 and of Theorem 4.3 this path consists of a finite piece of the spine together with the unique infinite tooth and in the first two cases of Theorem 4.3 it coincides with the infinite spine of the combs. In the next section we consider cases of random trees exhibiting more complicated behaviour. However, the feature that the limiting measures are concentrated on trees with a single infinite path persists.

5. Generic random trees

We consider infinite planar random trees whose distribution is obtained as a limit of finite size distributions that are defined by conditioning critical Galton–Watson processes on fixed size N. More precisely, the finite size measures μ_N are defined by (16) where ρ and Z_N are given by (8) and (11) and the off-spring probabilities $(p_n)_{n\geq 0}$ define a critical Galton–Watson process in the sense explained in Section 3.2. Such a process is called *generic* if the generating function f for $(p_n)_{n\geq 0}$ is analytic at $t = 1$, that is if the generating series (13) has radius of convergence strictly larger than 1. As is well known [2, 17] the generating function (12) for $(Z_N)_{N\geq 1}$ fulfills the functional relation

$$Z(t) = tf(Z(t))\,.$$

By the assumption of criticality this equation determines $Z(t)$ uniquely as an analytic function of t for $|t| < 1$ such that $Z(t) \to 1$ for $t \to 1$. Assuming the Galton–Watson process is generic it follows by Taylor expanding f around

$t = 1$ that

$$Z(t) \sim 1 + \sqrt{\frac{2}{f''(1)}}\,(1-t)^{\frac{1}{2}}\,, \qquad t \nearrow 1\,.$$

Applying standard transfer theorems [17] we get

$$Z_N \sim \frac{1}{\sqrt{2\pi f''(1)}}\,N^{-\frac{3}{2}}\,, \qquad N \to \infty\,.$$

By applying essentially the same arguments as those used in the last case of Theorems 4.1 and 4.3 although technically more elaborate (see [12, 14] for details) we have the following theorem establishing the existence of the desired limiting measure μ. By a slight abuse of notation an infinite path originating from the root in a tree is called a spine in the following and, given a spine $s_0, s_1, s_2 \ldots$ of a tree T, a *branch* of T at s_i is a subtree of T sharing only the vertex s_i with the spine and such that s_i has degree 1 in T. Hence the number of branches at s_i equals the degree of s_i in T minus 2 and we consider s_i to be the root of the corresponding branches.

Theorem 5.1 *The measures $(\mu_N)_{N\geq 1}$ defined by a generic critical Galton–Watson process converge to a probability distribution μ on the space of all rooted planar trees $\mathcal{T}$. It is concentrated on the subset of trees $\mathcal{S}$ with a single spine $s_1, s_2, \ldots$, and can be characterized as follows:*
(i) The probability that s_i has degree $k+1$ is kp_k, $k \geq 1$.
(ii) The branches of the tree are independently and identically distributed according to ρ given by (8).

The infinite random trees defined by the preceding theorem will be called *generic random trees*. It turns out that the information on the measure μ supplied by Theorem 5.1 is sufficient to determine both the Hausdorff and the spectral dimension of generic random trees [6, 14, 15].

Theorem 5.2 *For a generic random tree the annealed Hausdorff and spectral dimensions are given by*

$$\bar{d}_{\mathrm{h}} = 2 \quad \textit{and} \qquad \bar{d}_{\mathrm{s}} = \tfrac{4}{3}\,.$$

It also holds that $d_{\mathrm{h}} = 2$ almost surely, and that $d_{\mathrm{s}} = \frac{4}{3}$ almost surely.

We shall not at this place give a detailed proof of this result, but rather supply some comments on the ideas that go into it and relevant references.

The statements concerning the Hausdorff dimension can be obtained by rather standard generating function techniques. Thus the generating function f_R for the average size $\langle\, |B_R|\,\rangle_\rho$ of the ball of radius R w.r.t. the Galton–Watson measure ρ fulfills the recursion relation

$$f_{R+1}(t) = tf(f_R(t)) \;\text{ for }\; R \geq 1\,, \qquad \text{and} \qquad f_1(t) = t\,,$$

from which one obtains

$$\langle\, |B_R|\,\rangle_\rho \;=\; R\,.$$

Combining this with Theorem 5.1 yields

$$\langle\, |B_R|\,\rangle_\mu \;=\; \frac{f''(1)}{2} R(R-1) + R$$

which shows that $\bar{d}_{\rm h} = 2$. By a closer analysis of f_R in the vicinity of $t = 1$ using the genericity assumption and applying the Borel–Cantelli lemma (see [15] for details) one can show that there exists a constant $c_1 > 0$ such that for μ-almost all trees $T \in \mathcal{T}$ we have

$$|B_R(T)| \;\leq\; c_1\, R^2 \ln R\,, \qquad R \geq R_T\,,$$

for some finite R_T. This proves that $d_{\rm h} \leq 2$ almost surely. To get the reverse inequality one combines a classical result by Kolmogoroff stating that

$$\rho\left(\{T \mid\; |B_R(T)| > |B_{R-1}(T)|\}\right) \;\sim\; \frac{2}{f''(1)} \frac{1}{R}$$

for $R \to \infty$ [19] with some rather simple probability estimates [6,14] to show that there exists a constant $c_2 > 0$ such that

$$|B_R(T)| \;\geq\; c_2\, R^2 (\ln R)^{-2}\,, \qquad R \geq R_T\,, \tag{34}$$

for μ-almost all T. This proves the statements on the Hausdorff dimension.

Introducing the generating function $Q_G(x)$ for return probabilities to the root in a graph G by

$$Q_G(x) \;=\; \sum_{t=0}^{\infty} p_t(G)(1-x)^{t/2} \tag{35}$$

the spectral dimension of G may be expressed in terms of the singular behaviour of Q_G at $x = 0$. More precisely we have, assuming transfer theorems are applicable, that if

$$Q_G(x) \;\sim\; c\, x^{-\alpha}\,, \qquad x \to 0\,,$$

where $0 < \alpha < 1$ then

$$d_{\rm s} = 2 - 2\alpha\,, \tag{36}$$

and similarly $\bar{d}_{\mathrm{s}}$ can be expressed in terms of the singular behaviour of $\langle\, Q_G(x)\,\rangle_\mu$. In the formulation of the theorem the definition (36) of d_{s} is assumed, where

$$\alpha = \lim_{x\to 0} \frac{\ln Q_G(x)}{|\ln x|}\,,$$

and similarly for $\bar{d}_{\mathrm{s}}$.

It is proven in [14] that for generic random trees the estimates

$$\underline{c}\, x^{-\frac{1}{3}} \;\leq\; \langle\, Q_T x)\,\rangle_\mu \;\leq\; \overline{c}\, x^{-\frac{1}{3}}$$

hold for $0 < x < \frac{1}{2}$, where $\overline{c}$, $\overline{c} > 0$ are constants. Hence $\bar{d}_{\mathrm{s}} = \frac{4}{3}$ follows.

The almost sure lower bound $d_{\mathrm{s}} \geq \frac{4}{3}$ is a consequence of the inequality

$$Q_T(x) \;\leq\; c_1'\, R \;+\; \frac{c_2'}{x|B_R(T,r)|} \tag{37}$$

holding for all $R \geq 1$, where the constants c_1', $c_2' > 0$ are independent of R and T. There are various ways of obtaining this inequality one of which, especially tailored for trees, is given in [14]. A more general version can be obtained by different methods commented further on below. Using (34) in (37) we obtain

$$Q_T(x) \;\leq\; c_1'\, R \;+\; \frac{c_2''\,(\ln R)^2}{x\, R^2}\,, \qquad R \geq R_T\,, \tag{38}$$

for μ-almost all T. For $x > 0$ small enough we then get by choosing R to be the integer part of $x^{-\frac{1}{3}}$ that

$$Q_T(x) \;\leq\; \text{const.}\, x^{-\frac{1}{3}} (\ln x)^2$$

which proves that $\alpha \leq \frac{1}{3}$ and hence $d_{\mathrm{s}} \geq \frac{4}{3}$ almost surely.

Having established this almost sure lower bound it follows, up to technical details, that $d_{\mathrm{s}} < \frac{4}{3}$ on a set of positive μ-measure would contradict $\bar{d}_{\mathrm{s}} = \frac{4}{3}$. Thus $d_{\mathrm{s}} = \frac{4}{3}$ almost surely. An alternative proof of this fact using the definition (2) of d_{s} and continuous time random walk on T is given in [6].

6. Concluding remarks

A notable common feature of the random graph models considered in Sections 4 and 5 is the absence of circuits. Introducing circuits by adding links to a tree in such a way that distances of vertices to the root are preserved does not change the Hausdorff dimension but in general will affect the spectral dimension. It remains a challenging problem to develop general methods for evaluating or estimating the spectral dimension beyond models of random trees. An attractive and interesting class of models involves planar maps or planar surfaces. For such models there is a general result by Benjamini and Schramm [7] stating that under the assumption that vertex degrees are bounded in addition to a homogeneity condition planar random surfaces have spectral dimension at most 2. Bounding the order of vertices is not a natural constraint to introduce on models of planar maps of relevance to 2-dimensional quantum gravity [2]. For such models the construction of a limiting measure on infinite surfaces can be implemented along the lines described above for random combs and generic random trees. For triangulated planar surfaces this was accomplished in [4] and the resulting ensemble called the *uniform infinite planar triangulation.* It has Hausdorff dimension $d_{\mathrm{h}} = 4$ almost surely [5]. For the simpler case of so-called *causal planar triangulations* [3] the proof can be based on a particular case of Theorem 5.1 for which one as $d_{\mathrm{h}} = 2$ almost surely as well as $\bar{d}_{\mathrm{h}} = 2$ [15]. The *uniform infinite planar quadrangulation* has been constructed in [10] and it was shown that $\bar{d}_{\mathrm{h}} = 4$. The spectral dimension is not known for these cases. In [15] it is proven that for causal planar triangulations $d_{\mathrm{s}} \leq 2$ almost surely and it seems reasonable to conjecture that $d_{\mathrm{s}} = 2$ almost surely, but strong lower bounds on d_{s} seem hard to obtain. The general lower bound

$$d_{\mathrm{s}} \;\geq\; \frac{2d_{\mathrm{h}}}{d_{\mathrm{h}}+1} \tag{39}$$

valid for general connected rooted graphs such that d_{s} and d_{h} exist is a consequence of the inequality

$$Q_G(x) \;\leq\; c_1'\,R \;+\; \frac{c_2'}{x|B_R(G)|} \tag{40}$$

for $0 < x < 1$, $R \geq 1$ of which (35) is a special case. The derivation of this latter inequality can be based on the fact that the function

$$Q_T(v;x) \;=\; \sum_{\omega:r\to v}(1-x)^{|\omega|/2}\prod_{i=0}^{|\omega|}\sigma_{\omega(i)}^{-1}\,, \qquad v\in V(G)\,,$$

is a fundamental solution of a discrete form of the Laplace equation on G; details of the argument will appear elsewhere, see also [11]. The constants c_1', $c_2' > 0$ only depend on the order of the root of G such that if one chooses R to be the integer part of $x^{-\frac{1}{d_{\mathrm{h}}+1}}$ in (40) one obtains $\alpha \leq \frac{1}{d_{\mathrm{h}}+1}$ and the inequality (39) follows. Using that $d_{\mathrm{h}} = 2$ almost surely for the infinite causal planar triangulation we obtain

$$d_{\mathrm{s}} \;\geq\; \tfrac{4}{3}$$

almost surely in this case, while for the uniform infinite planar triangulation we get

$$d_{\mathrm{s}} \;\geq\; \tfrac{8}{5}$$

almost surely by using the result of [5] that $d_{\mathrm{h}} = 4$ almost surely. We know of no good reason to expect either of these inequalities to be sharp and it remains a challenge to develop methods to improve them.

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