HOLOGRAPHY FOR NON-RELATIVISTIC CFTs*

Mukund Rangamani

Centre for Particle Theory and Department of Mathematical Sciences
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
mukund.rangamani@durham.ac.uk

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We discuss the holographic duals of non-relativistic conformal field theories and their realisation in string theories. Based on lectures given at the XLIX Cracow School of Theoretical Physics.

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1. Introduction

The AdS/CFT correspondence [1] is a remarkable framework for studying the dynamics of quantum field theories at strong coupling, in addition to providing an in-principle answer to the question: "What is quantum gravity?" The correspondence was originally proposed based on studies of D-brane dynamics in flat space. By taking an appropriate decoupling limit to focus on the dynamics of the light open string degrees of freedom, one ends up with an isomorphism between the Hilbert space of open string degrees of freedom and a closed string system on a curved background. One natural way to motivate the correspondence is to start from the open string description and 'integrate out' the holes in the world-sheet. Over the years the correspondence has been generalized and has yielded many interesting insights into both the dynamics of field theories at strong coupling and into the nature of quantum gravity.

One way to view the AdS/CFT correspondence is entirely at a kinematical level based on symmetries. While this does not capture the entire beauty and the details of the correspondence, it nonetheless provides an entry point towards discussing generalizations. In order to do this one has to distill the essentials of the correspondence to asking the following question: "What is the geometric description of a fixed point of the Wilsonian renormalization group flow?" For instance, one could ask for a geometry that has the symmetries of a conformally invariant fixed point. In $d > 1$ spatial dimensions

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for relativistic field theories this is just $SO(d+1,2)$. The unique geometry with this isometry group is AdS_{d+2} which is a maximally symmetric spacetime. Thus one might conclude that the holographic dual for a CFT in $d+1$ dimensional spacetime is provided by AdS_{d+2}. Of course, this is certainly far from adequate in string theory, for one would still have to complete the AdS_{d+2} to a full solution of the world-sheet beta function equations. Nevertheless, from a phenomenological point of view this provides an elementary perspective on the correspondence. Moreover, this viewpoint makes sense in hindsight: from the study of the AdS/CFT correspondence we know that the energy scale of the field theory should be treated as an extra dimension (the radial direction of AdS).

Given field theories with different symmetries one could try to use this type of heuristic reasoning to write down an effective geometry that could potentially capture the dynamics of the field theory in some dual sense holographically, *à la* AdS/CFT. Of interest to us will be the case of non-relativistic conformal field theories which are invariant under the so-called Schrödinger symmetry. For this case in [2,3] the phenomenological approach to AdS/CFT outlined above was used to construct a dual spacetime which we will discuss in detail in Sec. 4.

The basic philosophy behind these constructions and other recent developments in the AdS/Condensed Matter applications, (see [4–6] for recent reviews) is to exploit the power of the holographic duality to learn about strongly interacting quantum field theories, where conventional field theoretic methods have failed to provide detailed insight. The main advantage of the AdS/CFT approach is the computational simplicity: strong coupling dynamics in the field theory is mapped to the dynamics of some classical fields interacting gravitationally. This allows one to extract interesting observables in a strongly coupled theory by performing calculations in the effective gravity description. In most of the known examples of AdS/CFT correspondence one could view the gravity description as providing a classical master field configuration capturing the planar sector dynamics of the field theory.

One disadvantage of the phenomenological construction of holographic duals, is that there is no control on the microscopic dynamics of the field theory. We cannot *a priori* answer questions regarding the nature of the microscopic degrees of freedom, the regime of validity of the holographic description, and how to incorporate corrections beyond supergravity. Of course, this would not be an issue if we were in a position to derive the holographic correspondence given the field theory path integral. The determination of this gravitational lore even more generally a stringy master field for an arbitrary field theory remains to date a hard problem.

Motivated by this, we will explore recent attempts to embed the constructions of [2, 3] into string theory. This was achieved soon after the original proposal in three concurrent papers [7–9] by exploiting certain solution generating techniques in string theory, *viz.*, the Null Melvin Twist [10] and the TsT transformation [11]. This particular embedding allows one to understand better the nature of the field theories which have as their holographic duals the geometries proposed by [2,3]. This article will explore the construction of such holograms and their implications.

Part of the interest in the non-relativistic conformal field theories is because they are expected to be realized in real-world systems. More specifically, the dynamics of cold fermionic atoms such as lithium or potassium with fine tuned interactions to achieve critical behaviour is predicted to be one such system. Over the last decade or so, the development of precision techniques to study many-body atomic systems has unearthed interesting physical properties. Of interest in the current context is the dynamics of Fermi superfluids which can be made to undergo a cross-over between a Bose condensate and a standard BCS superconductor. This is often referred to as the system of fermions at unitarity, since at this cross-over point the s-wave scattering cross-section saturates the unitarity bound. Apart from intrinsic interest as a many-body system, one also finds this system attractive for particle physics reasons. These cold fermionic atoms provide another naturally realized example of a system with low viscosity similar to the Quark Gluon Plasma (QGP). One expects that the origins of the low viscosity are due to the fact that the cross-over physics happens in the non-perturbative regime; having theoretical tools to study such strongly coupled systems is certainly of great interest.

The organization of this article is as follows: we will begin with a brief review of the symmetry algebras involved in non-relativistic conformal systems in Sec. 2. Following this we will explore the experimental systems which are said to realize such symmetries in Sec. 3. In Sec. 4 we will take the first steps towards constructing a holographic dual spacetime using a purely phenomenological approach and finally in Sec. 5 embed these constructions into string theory. In Sec. 6 we explore some basic applications of the hologram in computing observables of interest such as the equation of state, transport coefficients, *etc.* and conclude with a brief discussion in Sec. 7.

2. Conformal symmetry in non-relativistic systems

We will begin our discussion with a review of conformal symmetries for non-relativistic systems. We will in fact see that in the non-relativistic case there are two distinct ways of enlarging the Galilean algebra to include scale invariance. As in relativistic field theories, scale invariance is expected to arise in a non-relativistic quantum system when we fine tune the parameters to achieve criticality. As we will see it is certainly possible to encounter non-relativistic systems which are scale invariant without being conformally invariant. However, it is believed based on theoretic studies of the Wilsonian RG flow that the physics of fermionic cold atoms is described in terms of conformally invariant Galilean theories.

For starters let us consider the simplest Lorentz violating scaling symmetry: we consider a dynamical system in d-spatial dimensions[1] (the spatial coordinates will be denoted as $\boldsymbol{x}$) which respects the following spatio-temporal anisotropic scaling

$$t \to \lambda^z\, t\,, \qquad \boldsymbol{x} \to \lambda\, \boldsymbol{x}\,, \tag{2.1}$$

z here is often referred to as the dynamical critical exponent. This scaling symmetry forms part of the so-called Lifshitz algebra which has in addition, spatio-temporal translations and spatial rotations. In terms of the generators for these symmetries: momenta P_i, Hamiltonian H, spatial rotations M_{ij} and dilatation D, we have the algebra (henceforth $\mathrm{Lif}_z(d)$)

$$\begin{aligned}
[M_{ij}, M_{kl}] &= i\, \left(\delta_{ik}\, M_{jl} - \delta_{jk}\, M_{il} + \delta_{il}\, M_{kj} - \delta_{jl}\, M_{ki}\right)\,, \\
[M_{ij}, P_k] &= i\, \left(\delta_{ik}\, P_j - \delta_{jk}\, P_i\right)\,, \qquad [M_{ij}, H] = 0\,, \\
[H, P_i] &= [P_i, P_j] = 0\,, \\
[D, H] &= i\, z\, H\,, \qquad [D, P_i] = i\, P_i\,, \qquad [D, M_{ij}] = 0\,,
\end{aligned} \tag{2.2}$$

where the only unusual commutator is $[D, H]$ which implies that the Hamiltonian has scaling dimension z, commensurate with the scaling symmetry (2.1).

An important element which is missing in the Lifshitz algebras are the Galilean boosts K_i: it is of course possible to include them and extend the algebra. In fact, it is possible to go further and incorporate another scalar operator N which can be viewed as the particle number. We will call this enlarged algebra the Galilean Scaling Algebra, (henceforth $\mathrm{GSA}_z(d)$, in d spatial dimensions). In addition to the commutators given in (2.2) we have to supplement the details for the generators K_i and N:

$$\begin{aligned}
[M_{ij}, K_l] &= i\, \left(\delta_{il}\, K_j - \delta_{jl}\, K_i\right)\,, \qquad [M_{ij}, N] = 0\,, \\
[P_i, K_j] &= -i\, N\,, \qquad [H, K_i] = -i\, P_i\,, \qquad [D, K_i] = i\,(1-z)\, K_i\,, \\
[D, N] &= -i\,(2-z)\, N\,, \qquad [H, N] = [P_i, N] = [K_i, N] = 0\,.
\end{aligned} \tag{2.3}$$

The algebras $\mathrm{Lif}_z(d)$ and $\mathrm{GSA}_z(d)$ are just involve non-trivial scale transformations, but no obvious analog of special conformal symmetries. For general values of z these are in fact, the best one can do. However, for special

[1] We will adhere to the convention of stating the number of spatial dimensions for non-relativistic systems, and will typically denote this number by d.

values of z one can extend the algebra to allow for what would be non-trivial special conformal transformations. One obvious value for z, where the special conformal generators enter is $z = 1$, where the spatio-temporal anisotropy disappears (and one can envisage extending the algebra all the way to the Lorentz algebra).

There is however, another value of z, *viz.*, $z = 2$ where things are more interesting. In this case we will focus on and it goes by the name of the Schrödinger algebra. In fact in this case, we encounter only a single special conformal generator (which we denote by C). Furthermore, for this special value of z, the generator N corresponding to particle number becomes a central term; it only appears in the commutation relation in the $[P_i, K_j]$.

Having seen the algebra somewhat abstractly, it is worthwhile noting some physical aspects which motivate one's interest in it. The Schrödinger algebra is the symmetry algebra of the free Schrödinger operator in $d+1$ dimensions, *i.e.*, is generated by operators that commute with

$$S = i\,\partial_t + \frac{1}{2\,m}\,\partial_i^2\,. \tag{2.4}$$

It is analog of the conformal algebra for relativistic systems — we will see how to relate the two shortly. It is believed that the system of cold atoms at unitarity is an example of an interacting QFT which realizes this symmetry [12]. One can write down the Schrödinger group as the following set of transformations:

$$\begin{aligned} \boldsymbol{x} \;&\to\; \boldsymbol{x}' = \frac{\mathfrak{R}\,\boldsymbol{x} + \mathfrak{v}\,t + \mathfrak{a}}{\gamma\,t + \delta}\,, \\ t \;&\to\; t' = \frac{\alpha\,t + \beta}{\gamma\,t + \delta} \end{aligned}$$

with $\alpha\delta - \beta\gamma = 1$. The group includes, spatial translations indicated by $\mathfrak{a}$, rotations captured by $\mathfrak{R}$, Galilean boosts with velocity $\mathfrak{v}$, a scale transformation and a special conformal generator. We will now re-derive the Schrödinger algebra (2.6) for $z = 2$ by employing a useful trick.

Recall that one can get the Galilean algebra in d dimensions by reducing the Poincaré algebra SO$(d+1, 1)$ on light-cone

$$x^{\pm} = x^0 \pm x^{d+1}\,, \tag{2.5}$$

where x^0 is the time direction in the relativistic theory. It is a well known fact that propagation in light-cone time x^+ respects Galilean invariance. We can similarly reduce the conformal algebra SO$(d+2, 2)$ in $d+2$ dimensions on a light-cone to obtain the Schrödinger algebra in d-spatial dimensions.

Starting from the conformal algebra we keep all generators which commute with the particle number: $\{H,\ M_{ij},\ P_i,\ K_i,\ D,\ C,\ N\}$ where the particle number operator is simply the momentum in the light-cone direction P_-.

Generator	Galilean	Conformal
Particle number	N	P_-
Hamiltonian	H	P_+
Momenta	P_i	P_i
Angular momenta	M_{ij}	M_{ij}
Galilean boost	K_i	M_{i-}
Dilatation	D	$D + M_{+-}$
Special conformal	C	K_-

By virtue of the fact that we are only allowing generators that commute with P_- we lose some of the relativistic conformal generators. In particular, all the spatial special conformal generators are projected out. The full algebra is then given as

$$
\begin{aligned}
[M_{ij}, M_{kl}] &= i\,(\delta_{ik}\,M_{jl} - \delta_{jk}\,M_{il} + \delta_{il}\,M_{kj} - \delta_{jl}\,M_{ki})\ , \\
[M_{ij}, P_k] &= i\,(\delta_{ik}\,P_j - \delta_{jk}\,P_i)\ , \\
[M_{ij}, K_k] &= i\,(\delta_{ik}\,K_j - \delta_{jk}\,K_i)\ , \\
[M_{ij}, H] &= [M_{ij}, D] = [M_{ij}, C] = 0\,, \\
[P_i, P_j] &= [K_i, K_j] = 0\,, \\
[K_i, P_j] &= i\,\delta_{ij}\,N\,, \\
[H, P_i] &= 0\,, \qquad [H, K_i] = -i\,P_i\,, \\
[D, P_i] &= i\,P_i\,, \qquad [D, K_i] = -i\,K_i\,, \\
[C, P_i] &= i\,K_i\,, \qquad [C, K_i] = 0\,, \\
[D, H] &= 2i\,H\,, \qquad [D, C] = -2i\,C\,, \\
[H, C] &= -i\,D\,.
\end{aligned}
\tag{2.6}
$$

The Schrödinger algebra unfortunately is not the only conformal algebra that one can write down for non-relativistic systems. As explained recently in [13] one can have in addition the Galilean Conformal Algebra (GCA(d) in d-spatial dimensions).This algebra can be obtained by a suitable contraction of the relativistic conformal algebra in $d+1$ dimensions. A major difference between the GCA(d) and the Schrödinger algebra is the fact that the former has many more generators. The GCA(d) retains all the generators of the $d+1$ dimensional relativistic conformal algebra, and in particular contains the special conformal generators in the spatial directions (recall that the Schrödinger algebra only has the temporal special conformal generator). Furthermore, the GCA does not allow for the central extension of the momentum-boost commutator. For further details we refer the reader to the excellent account in [13].

Let us now return to the Schrödinger algebra which will be our primary focus for the rest of the article. One interesting question we would like to understand is the nature of the representation theory for this algebra and in particular, whether the notion of the state-operator correspondence which exists for relativistic conformal systems has an analog in this case.

It turns out that the representation theory of Schrödinger algebra can be described in terms of highest weight states as usual, see [12]. In particular, we will talk about 2 quantum numbers

- The scaling dimension:

$$[D, \mathcal{O}] = i\, \Delta_{\mathcal{O}}\, \mathcal{O}\,. \tag{2.7}$$

- The particle number:

$$[N, \mathcal{O}] = N_{\mathcal{O}}\, \mathcal{O}\,. \tag{2.8}$$

From the commutation relations it is clear that we have $\Delta_H = 2$ and $\Delta_P = 1$.

We will realize highest weight representations in terms of quasi-primary operators which have a given conformal dimension $\Delta_{\mathcal{O}}$ and particle number $N_{\mathcal{O}}$. As usual, the spacetime dependence of the operator can be inferred via translation:

$$\mathcal{O}(t, \boldsymbol{x}) = e^{i\,H\,t - i\,P_i\,x_i}\, \mathcal{O}(0)\, e^{-i\,H\,t + i\,P_i\,x_i}\,. \tag{2.9}$$

The quasi-primary operators are defined so that lowering operators K and C (which have scaling dimensions -1 and -2, respectively annihilate it *i.e.*,

$$[K_i, \mathcal{O}] = [C, \mathcal{O}] = 0\,. \tag{2.10}$$

Given a quasi-primary operator we can then construct descendants by acting with the raising generators of the algebra H and P_i.

One can give a simple representation of the algebra in terms using the usual expressions for the operators in terms of derivations. For an operator $\mathcal{O}(t, \boldsymbol{x})$ we have:

$$\begin{aligned}
[H, \mathcal{O}] &= -i\, \partial_t \mathcal{O}\,, \\
[P_i, \mathcal{O}] &= i\, \partial_i \mathcal{O}\,, \\
[D, \mathcal{O}] &= i\, \left(2\,t\,\partial_t + x_i\,\partial_i + \Delta_{\mathcal{O}}\right) \mathcal{O}\,, \\
[K_i, \mathcal{O}] &= \left(-i\,t\,\partial_i + N_{\mathcal{O}}\,x_i\right) \mathcal{O}\,, \\
[C, \mathcal{O}] &= -i\left(t^2\,p_t + t\,x_i\,\partial_i + t\,\Delta_{\mathcal{O}}\right) \mathcal{O}
\end{aligned} \tag{2.11}$$

which in particular implies that the quasi-primary operators satisfy

$$e^{-i\,\lambda\,D}\,\mathcal{O}(t,\boldsymbol{x})\,e^{i\,\lambda\,D} = e^{\lambda\,\Delta_{\mathcal{O}}}\,\mathcal{O}\left(e^{2\,\lambda}\,t, e^{\lambda}\,\boldsymbol{x}\right)\,. \tag{2.12}$$

In relativistic conformal field theories we have a state-operator correspondence. States of the CFT on the cylinder $\boldsymbol{R}\times\mathbf{S}^d$ are in one-to-one correspondence with local operators inserted at the origin of $\boldsymbol{R}^{d,1}$. This is familiar from the usual radial quantization methods. For the non-relativistic conformal symmetry, there is an analogous state-operator correspondence. In particular, the quasi-primary operators are in one–one correspondence with the eigenstates of a quantum system in a harmonic trap.

Given an operator $\mathcal{O}$ with conformal weight $\Delta_{\mathcal{O}}$ we can consider a state built by acting this operator on the vacuum state:

$$|\,\psi_{\mathcal{O}}\rangle = e^{-H}\,\mathcal{O}^{\dagger}\,|\,0\rangle\,, \tag{2.13}$$

where we have dressed the operator by e^{-H}. It is not hard to check that is an eigenstate of the Hamiltonian $H_{\rm osc} = H + C$ with eigenvalue $\Delta_{\mathcal{O}}$. In fact, the Schrödinger algebra has a $\mathrm{SL}(2,\boldsymbol{R})$ sub-algebra generated by $\{D, H, C\}$:

$$\begin{aligned} H_{\rm osc} &= \tfrac{1}{2}\,(H+C)\;,\\ a^{\dagger} &= \tfrac{1}{2}\,(H-C+i\,D)\;,\\ a &= \tfrac{1}{2}\,(H-C-i\,D) \end{aligned} \tag{2.14}$$

which makes is clear how the state-operator correspondence works.

3. Real world non-relativistic CFTs

It is worthwhile to understand the physical motivation to study non-relativistic CFTs before we enter into a discussion of the holographic description of such theories. Part of the motivation will be to contrast the class of examples we are interested from experimental stand-point, to those that are amenable to a holographic treatment.

The experimental results from studies of cold atoms, specifically, fermionic atoms like Li^6 or K^{40} at the so-called Feshbach-resonance indicate that the system is described by non-relativistic CFT. For a detailed account of the experimental techniques and evidence we refer the reader to [14] (see also [15] for a theoretical account). Basically, this phase of the atomic system is a cross-over phase between two regimes: one where the fermionic atoms pair up into bosonic molecules (the BCS phase), and another where the binding mechanism is very strong and the bosonic molecules are tightly bound enough to Bose condense (the BEC phase). The critical point is achieved by tuning the s-wave scattering length by an external field.

In terms of the basic physics, fermionic atoms exhibit no vibrational losses near the Feshbach resonance, which is achieved by pairing the hyperfine levels in the atoms to an auxiliary molecular excitational level with high vibrational damping. In the regime of interest the atoms achieve fermionic superfluidity, with a large triplet scattering length.

To understand the critical behaviour as we saturate the s-wave scattering cross-section, it is useful to study a model in $2+\epsilon$ dimensions following [12]. One considers a fermionic systems with hyperfine states $\psi_\uparrow$ and $\psi_\downarrow$ with a characteristic four-fermi interaction. By standard renormalization group arguments [12] were able to show that the system achieves a critical point in $2+\epsilon$ dimensions. In physical terms the four-fermi interaction is tuned so that the s-wave scattering length a diverges. For $a < 0$ we have the BCS phase, while for $a > 0$ we end up in the BEC phase.

The excitement about this system is that it seems to provide another natural example of a nearly ideal fluid. Experiments with cigar shaped clouds of fermionic atoms, show that the system evolves hydrodynamically with $\eta/s \sim 0.5$ which is close to the famous bound proposed in [33]. Further, evidence for strong coupling in these systems comes from the notion of the Bertsch parameter, which captures the deviation of the average energy per particle from the free Fermi value $\xi = \epsilon/\epsilon_{\rm free}$ which from Monte Carlo simulations is expected to around 0.4. The parameter ξ is best thought of as the analog of the famous 3/4 in the ratio of entropy density at strong coupling to the entropy density at weak coupling for $\mathcal{N} = 4$ SYM.

Given that there are systems which are strongly coupled and exhibit non-relativistic conformal symmetry, it is interesting to ask whether one can bring to bear the machinery of the holographic AdS/CFT correspondence to shed some light on such systems. In the rest of this article we will explore this possibility.

4. Phenomenological construction of the Galilean hologram

Having seen the symmetry algebra for non-relativistic conformal field theories, we are now in a position to motivate the study of a particular class of geometries. First of all let us consider the geometry of AdS_{d+3} whose metric is given as

$$ds^2 = -r^2\, \eta_{\mu\nu}\, dx^\mu\, dx^\nu + \frac{dr^2}{r^2}\,, \tag{4.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric on flat $\boldsymbol{R}^{d+1,1}$ parameterized by x^μ, with $\mu, \nu \in \{0, \cdots, d{+}1\}$. This metric covers the Poincaré patch of AdS spacetime and its isometry group is $\mathrm{SO}(d+2, 2)$, which is the relativistic conformal group in $d+2$ spacetime dimensions. Of this large set of symmetries, let us look at the scaling symmetry

$$x^\mu \to \lambda x^\mu \,, \qquad r \to \frac{1}{\lambda}\, r \tag{4.2}$$

which clearly leaves (4.1) invariant. These are the familiar scale transformations for the relativistic CFT on $\boldsymbol{R}^{d+1,1}$ which is the timelike boundary of (4.1) and illustrate clearly that the radial direction of the AdS spacetime can be viewed as the energy scale of the dual field theory along the lines of the holographic renormalization group [16].

This scaling symmetry respects spatio-temporal isotropy, but it is easy to conjure up a version that breaks this. Consider introducing light-cone coordinates (2.5) into the metric (4.1). It is easy to see that one can define a non-trivial scaling as follows:

$$x^+ \to \lambda^z \, x^+ \,, \qquad x^- \to \lambda^{2-z} \, x^- \,, \qquad x^i \to \lambda \, x^i \,, \qquad r \to \frac{1}{\lambda}\, r \,, \tag{4.3}$$

where now $i = 1, \cdots, d$. This unconventional scaling is a symmetry of the AdS_{d+3} metric and if one applied AdS/CFT with regarding x^+ as time, then one would have realized a theory with anisotropic spatio-temporal scaling. In fact, one obtains a bit more than the scaling, for the symmetries transverse to the light-cone is just the Galilean symmetry. This way one could hope to realize a version of $\text{GSA}_z(d)$. In fact, shortly after the proposal of [2, 3] it was proposed in [17, 18] that one might achieve the required hologram for non-relativistic CFTs simply by this light-cone reduction of AdS/CFT.

This is, however, a bit too quick: we are essentially exploiting the fact that the light-cone reduction of any relativistic field theory gives rise to a Galilean invariant system in a sector with fixed light-cone momentum. However, we have to deal carefully with the zero mode under light-cone reduction. This is familiar from the standard discussion of the Discrete Light Cone Quantization (DLCQ). We will return to this issue shortly after discussing another geometry that actually has the full $\text{GSA}_z(d)$ symmetry.

Consider the metric

$$ds^2 = r^2 \left(-2\, dx^+ \, dx^- - \beta^2 \, r^{2\nu} \, (dx^+)^2 + d\boldsymbol{x}^2\right) + \frac{dr^2}{r^2} \tag{4.4}$$

with $z \equiv 1+\nu$, which naturally has the required scaling (4.3). The geometry with $\nu = 0$ corresponds to pure AdS_{d+3} as does the case $\beta = 0$. However, the other values of ν lead to spacetimes with non-trivial asymptopia; for $\nu > 0$ the causal structure of such spacetimes is non-distinguishing [19, 21]. These spacetimes were proposed by [2, 3] as holographic duals for non-relativistic field theories which have the $\text{GSA}_z(d)$ as a global symmetry algebra. We will call these spacetimes Sch^ν_{d+3}.

It is worthwhile understanding the distinction between the light-cone compactification of AdS_{d+3} and the spacetime with metric Sch^{ν}_{d+3} in some detail before we proceed to analyze the applications of (4.4). The main argument is that the causal structure of (4.4) naturally reproduces the Galilean light-cone of the field theory. The causal structure of a non-relativistic field theory is degenerate — interactions can propagate instantaneously. This could sound like a problem for holography; a bulk geometry with decent causal structure cannot be holographically dual to a non-relativistic field theory. However, the spacetime geometry (4.4) evades this issue beautifully — its causal structure is also degenerate and precisely in a manner to be consistent with the boundary Galilean invariance.

The spacetime (4.4) is conformal (with an overall conformal factor r^2) to a *pp*-wave spacetime[2]. The class of *pp*-wave spacetimes which this belongs to are known to be non-distinguishing [19]. This essentially means that while the spacetime (4.4) is causal (in the sense of not having closed causal curves), different points in the spacetime share the same past and future, thereby preventing us from distinguishing spacetime events by their causal sets. Note that the timelike future $\mathscr{I}^+(p)$ for a point p is the set of points which can be reached from p by future-directed timelike curves; the timelike past $\mathscr{I}^-(p)$ is defined similarly in terms of past-directed timelike curves. Causal future/past are defined likewise in terms of causal (*i.e.*, timelike or null) curves.

In fact, in (4.4), all points on a surface with $u = u_0$ (and arbitrary values of other coordinates) have an identical causal future/past [19]. But this is precisely the causal structure of a Galilean CFT; all spatial points on an equal time surface can influence any abritrary spatial point at an infinitesimal time later. More pertinently, the $\nu = 2$ geometry has previously been studied in the context of its causal properties in [21, 22]; it arises as the holographic dual of non-commutative $\mathcal{N} = 4$ Super Yang–Mills with light-like non-commutativity [20].

The fact the bulk spacetime causal structure is consistent with the boundary causal structure is a crucial ingredient in the AdS/CFT correspondence. If this were not true we would have easily been able to set up gedanken experiments wherein bulk physics would give drastically different results from boundary physics. A pure AdS spacetime with boundary conditions engineered to give Galilean invariance does not posses a bulk light-cone which agrees with the light-cone of the relativistic field theory.

Before proceeding with the holographic duals for theories enjoying the $\mathrm{GSA}_z(d)$ symmetry, it is useful to realize that these phenomenological techniques can be employed to construct duals for theories with Lifshitz symme-

[2] The special case $\nu = 0$ is of course pure AdS with a well-behaved causal structure.

try as described originally in [23]. Since the $\mathrm{Lif}_z(d)$ algebra does not have Galilean boosts one can simply write down a geometry

$$ds^2 = -r^{2\,z}\, dt^2 + r^2\, d\boldsymbol{x}^2 + \frac{dr^2}{r^2}\,. \tag{4.5}$$

These have been investigated in detail over the last year but so far this class of spacetimes has not been embedded into string theory.

5. Non-relativistic CFTs in string theory

The geometry (4.4) can be consistently embedded in a solution to string theory. Indeed, geometries of this type have previously been studied, in investigations of the application of solution generating transformations to construct geometries corresponding to twisted versions of the D3-brane world-volume theory [20,21]. In this section, we first review this solution generating transformation, and use it to construct a string theory solution which reduces to (4.4) in five dimensions. We then apply the same transformation to obtain a non-extremal generalization, and construct a five-dimensional theory for which the non-extremal geometry is a solution.

5.1. Generating the geometry dual to the vacuum state

To begin with, consider the geometry of $\mathrm{AdS}_5 \times \mathbf{S}^5$ in Poincaré coordinates, which is the near-horizon geometry of D3-branes in flat space:

$$\begin{aligned} ds^2 &= r^2\,\left(-dt^2 + d\boldsymbol{x}^2 + dy^2\right) + \frac{dr^2}{r^2} + (d\psi + A)^2 + d\Sigma_4^2\,, \\ F_{(5)} &= dC_{(4)} = 2\,(1+\star)\, d\psi \wedge J \wedge J\,, \end{aligned} \tag{5.1}$$

where we have written the metric on the unit $\mathbf{S}^5$ as a fibration over a $\mathbf{CP}^2$ base and now $\boldsymbol{x} = \{x_1, x_2\}$. The five-form is given explicitly in terms of the volume form of $\mathbf{S}^5$, which has been decomposed into quantities related to the fibration. J is the Kähler form on $\mathbf{CP}^2$ and A is the associated potential. Our conventions are

$$dA = 2\,J\ , \qquad \mathrm{Vol}\left(\mathbf{CP}^2\right) = \tfrac{1}{2} J \wedge J\,. \tag{5.2}$$

We apply a Null Melvin Twist to this geometry, as described in [10]; the idea is to generate light-like NS–NS flux by a series of boosts and twisted T-dualities. Algorithmically we proceed as follows[3]:

[3] The D3-brane geometry above has a full SO(1,1) symmetry in the (t, y) plane which renders the first step inconsequential here, but it will be meaningful for the non-extremal solution which follows.

1. Pick a translationally invariant direction (say y) and boost by amount γ along y.
2. T-dualize along y.
3. Twist some one-form σ: $\sigma \to \sigma + \alpha\, dy$.
4. T-dualize along y again.
5. Boost by $-\gamma$ along y.
6. Scale the boost and twist: $\gamma \to \infty$ and $\alpha \to 0$, keeping

$$\beta = \tfrac{1}{2}\alpha\, e^{\gamma} = \text{fixed}\,. \tag{5.3}$$

The only data needed to describe the construction is the choice of the one-form σ. We can choose σ to be along the world-volume directions (linear combination of dx_1 and dx_2) or transverse to the D-brane. The former leads to turning on constant electric and magnetic fields on the D-brane world-volume leading to a light-like non-commutative field theory [20, 21].

Twisting along the R-symmetry direction is more interesting. A natural choice is to take the one-form σ to be along the fiber direction: $\sigma = d\psi$. The Null Melvin Twist leads to the geometry [21]

$$\begin{aligned} ds^2 &= r^2\left(-2\,dx^+\,dx^- - r^2\,du^2 + d\boldsymbol{x}^2\right) + \frac{dr^2}{r^2} + (d\psi + A)^2 + d\Sigma_4^2\,,\\ F_{(5)} &= 2\,(1+\star)\,d\psi\wedge J\wedge J\,,\\ B_{(2)} &= r^2\,dx^+\wedge(d\psi + A)\,, \end{aligned} \tag{5.4}$$

where the light-cone coordinates are

$$x^+ = \beta\,(t+y)\,,\qquad x^- = \frac{1}{2\,\beta}\,(t-y)\,. \tag{5.5}$$

Note that our boosted $x^{\pm}$ coordinate frame scales β out not only from the metric but also from the field strengths. The five-dimensional part of this metric is precisely the geometry (4.4), with $\nu = 1$ and $d = 2$. This geometry will correspond to the vacuum state of the dual non-relativistic field theory.

One can equivalently obtain the solution by a different solution generating technique [8] known as the TsT transformation [11]. This transformation involves a T-duality along the ψ direction followed by a shift $x^- \to x^- + \psi$ and then a T-duality again on the resulting ψ direction. More pithily, given the $\mathbf{T}^2$ spanned by the U(1) isometries ∂_{x^-} and ∂_ψ we can perform a twisted T-duality on this $\mathbf{T}^2$ which generates a non-trivial magnetic field (NS–NS B-field flux). It is this flux that supports the deformation of the geometry in (4.4).

5.2. Kaluza–Klein reductions and effective actions

The solutions we have discussed above (5.4) and (6.2) satisfy the 10-dimensional Type IIB equations of motion. In [2], the vacuum geometry (4.4) was considered as a solution to Einstein–Proca theory with negative cosmological constant, which has the action

$$\mathcal{S}_{\rm EP} = \int d^{d+2}x\, dr\, \sqrt{-g}\, \left(R - 2\,\Lambda - \tfrac{1}{4} F_{\mu\nu}\, F^{\mu\nu} - \tfrac{1}{2} m^2\, A^\mu\, A_\mu\right) , \qquad (5.6)$$

with $F_{\mu\nu} = 2\,\nabla_{[\mu} A_{\nu]}$. The metric (4.4) with $A^- = 1$ satisfies the field equations for the choice

$$\Lambda = -\tfrac{1}{2}(d+1)(d+2)\,, \qquad m^2 = 2\,(d+2)\,. \qquad (5.7)$$

We would now like to understand the relation between this phenomenological Lagrangian and the ten-dimensional IIB theory. Starting from Type IIB supergravity, we can KK reduce the solution (5.4) on the $\mathbf{S}^5$ (which is undeformed). The reduction of the metric is straightforward, and gives (4.4) in five dimensions. The NS–NS two-form, however, depends on the $\mathbf{S}^5$ coordinates. In a linear analysis [24], such a mode of the two-form produces a massive vector transforming in the **15** of SO(6): in AdS units (set here to 1) its mass is $m^2 = 8$. This is precisely the value of the mass required according to (5.7) (with $\Lambda = -6$ as necessary to get AdS radius equal to 1).

While, the above argument shows that the phenomenological Lagrangian introduced in [2] can be embedded into string theory, it does not fully explicate whether all solutions of (5.6) can be realized in Type IIB string theory. For this we need some more information — we require that (5.6) be a consistent truncation of Type IIB supergravity. By considering a suitable ansatz for compactification of Type IIB supergravity down to five dimensions, which generalizes the Freund–Rubin compactifications [8] was able to show that there exists a consistent truncation of Type IIB supergravity involving massive vectors. However, this theory has in addition three scalar fields as well and the low energy effective action takes the form:

$$\begin{aligned}
\mathcal{S}_{\rm bulk} &= \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big(R + V(\phi_i) - 5(\partial\phi_1)^2 - \tfrac{15}{2}(\partial\phi_2)^2 - \tfrac{1}{2}(\partial\phi_3)^2 \\
&\quad -\tfrac{1}{4} g(\phi_i) F_{\mu\nu}F^{\mu\nu} - 4e^{-2\phi_1-3\phi_2-\phi_3} A_\mu A^\mu\Big)\,, \\
V(\phi_i) &= 24\, e^{-\phi_1 - 4\,\phi_2} - 4\, e^{-6\,\phi_1 - 4\,\phi_2} - 8\, e^{-10\,\phi_2}\,, \\
g(\phi_i) &= e^{4\phi_1+\phi_2-\phi_3}\,.
\end{aligned} \qquad (5.8)$$

The special case where all the scalars are set to constant values results in (5.6). In fact, we will be interested in a different sub-case of this system, one where the scalars are linearly related via

$$\{\phi_1,\phi_2,\phi_3\} = \left\{-\tfrac{2}{5}, -\tfrac{1}{15}, 1\right\} \phi \tag{5.9}$$

which results in a simpler Lagrangian than (5.8):

$$\begin{aligned}\mathcal{S} &= \frac{1}{16\pi G_5}\int d^5x\sqrt{-g} \\ &\quad\times\left(R - \tfrac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4A_\mu A^\mu - V(\phi)\right), \end{aligned}\tag{5.10}$$

where the scalar potential is

$$V(\phi) = 4\,e^{2\phi/3}\left(e^{2\phi} - 4\right). \tag{5.11}$$

The scalar here appears from two sources: *(i)* the black hole geometry involves a non-vanishing dilaton and *(ii)* the twist now causes the fibration over $\mathbf{CP}^2$ to be squashed. Squashing is a common feature of solutions generated by the Null Melvin Twist [10] and intuitively can be ascribed to the distortion of the asymptotics of the spacetime. It should be borne in mind that (5.10) is not a consistent truncation of Type IIB supergravity, but will suffice for our purposes when we turn to applications of the holographic correspondence.

5.3. The dual field theory

The Null Melvin Twist construction makes the interpretation of the dual field theory clean: it is nothing but $\mathcal{N} = 4$ Super Yang–Mills twisted by an R-charge. The U(1) isometry generating the R-charge is generated in the spacetime by $\frac{\partial}{\partial\psi}$. This twist breaks the SU(4) symmetry of $\mathcal{N} = 4$ down to an SU(3) × U(1) (the isometry group of $\mathbf{CP}^2$) through the non-vanishing NS–NS potential $B_{(2)}$ (the metric (5.4) of course enjoys full SU(4) invariance).

From the CFT point of view, the twist by R-symmetry corresponds to a deformation by an irrelevant operator of dimension 5 transforming in the antisymmetric tensor representation of SU(4). The operator in question [25] is $\mathcal{O}^{IJ}_\mu = \mathrm{Tr}\left(F_\mu{}^\nu\,\Phi^{[I}\,D_\nu\Phi^{J]} + \sum_K D_\mu\,\Phi^K\,\Phi^{[K}\,\Phi^I\,\Phi^{J]}\right) + \text{fermions}$, where Φ^I are the adjoint scalars of $\mathcal{N} = 4$ SYM transforming in the vector **6** of SU(4) and $F_{\mu\nu}$ is the gauge field strength. The Lorentz symmetry is broken by adding $\mathcal{O}^{IJ}_+$ to the field theory Lagrangian. This field theory realization makes it clear that the massive vector used in the construction of [2] oxidises to NS–NS flux in ten dimensions.

The TsT transformation, however, gives a better insight into the dual field theory. Recall that generating a world-volume magnetic field results in spatially non-commuting field theories (in an appropriate decoupling limit) [26]. Likewise, a non-trivial B-field obtained by twisted T-duality in the transverse directions will result in an analog of β-deformed field theories [11]. Both of these are special examples of the TsT transformation, where the U(1) isometries are either entirely in the world-volume directions (non-commutative field theories) or completely transverse (beta-deformation). We are interested in situations where the TsT transformation acts on a $\mathbf{T}^2$ spanned by one U(1) isometry in the directions transverse to the D3-brane and another along the world-volume, which is a hybrid of the two situations described above. The resulting field theory can be described by deforming the original relativistic CFT by a (heterotic) star product

$$f \star g = e^{i\,\beta\left(\mathcal{V}^f\, R^g - \mathcal{V}^g\, R^f\right)}\, f\, g\,, \tag{5.12}$$

where $\mathcal{V}$ is the x^--momentum of the field and R refers to a global $\mathrm{U}(1)_R$ charge.

So for the special case where we take the internal space to be $\mathbf{S}^5$ we obtain a non-relativistic deformation of $\mathcal{N} = 4$ SYM via the star-product (5.12). Since the deformation only depends upon the existence of a U(1) isometry in the space transverse to the D3-branes one can immediately engineer examples starting from the infinite class of $\mathcal{N} = 1$ SCFTs obtained from D3-branes probing Calabi–Yau singularities.

6. Applications of the Schr/NRCFT correspondence

Having described the basic features of the correspondence we are now in a position to apply the same to derive some physical observables of interest. While the choice of observables we calculate is governed by the interest from the experimental systems of cold atoms, one should, however, take the results with a grain of salt, for the field theories whose duals we are describing here are not quite fermions at unitarity.

6.1. Thermal non-relativistic CFTs

As we have generated (5.4) by a solution generating technique, we can just as well generate the non-extremal version of the solution. To do so, rather than starting with the near horizon geometry of extremal D3-branes, we start with non-extremal D3-branes and repeat the Null Melvin Twist. Consider then the planar Schwarzschild–AdS black hole (times $\mathbf{S}^5$, with the geometry supported by the five-form flux $F_{(5)}$)

$$ds^2 = r^2\left(-f(r)\,dt^2 + dy^2 + d\boldsymbol{x}^2\right) + \frac{1}{r^2}\left(\frac{dr^2}{f(r)} + r^2\,d\Omega_5^2\right), \qquad (6.1)$$

where as before we will write the $\mathbf{S}^5$ as a $\mathbf{S}^1$ fibration over $\mathbf{CP}^2$. The Null Melvin Twist leads to the string frame metric [27]:

$$\begin{aligned}
ds^2_{\rm str} &= r^2\left(-\frac{\beta^2\,r^2\,f(r)}{k(r)}\,(dt+dy)^2 - \frac{f(r)}{k(r)}\,dt^2 + \frac{dy^2}{k(r)} + d\boldsymbol{x}^2\right)\\
&\quad + \frac{dr^2}{r^2\,f(r)} + \frac{(d\psi + A)^2}{k(r)} + d\Sigma_4^2\,,\\
e^{\varphi} &= \frac{1}{\sqrt{k(r)}}\,,\\
F_{(5)} &= dC_{(4)} = 2\,(1+\star)\,d\psi\wedge J\wedge J\,,\\
B_{(2)} &= \frac{r^2\,\beta}{k(r)}\,\left(f(r)\,dt + dy\right)\wedge(d\psi + A)\,,
\end{aligned} \qquad (6.2)$$

with

$$f(r) = 1 - \frac{r_+^4}{r^4}\,, \qquad k(r) = 1 + \beta^2\,r^2\,(1-f(r)) = 1 + \frac{\beta^2\,r_+^4}{r^2}\,. \qquad (6.3)$$

The solution has a horizon at $r = r_+$. Note that the parameter β appearing in this metric is an independent physical parameter; in the extremal case, we could set it to one by boosting in the t–y plane, but non-extremality has broken this boost symmetry. The remainder of the paper will be devoted to an exploration of the physics of this non-extremal solution.

If we perform the same Kaluza–Klein reduction for the non-extremal solution (6.2), we obtain

$$\begin{aligned}
ds^2_E &= r^2\,k(r)^{-2/3}\left(-\beta^2 r^2 f(r)(dt+dy)^2 - f dt^2 + dy^2 + k d\boldsymbol{x}^2\right)\\
&\quad + k(r)^{1/3}\frac{dr^2}{r^2 f(r)}\,,\\
&= r^2 k(r)^{-2/3}\Bigg(\left[\frac{1-f(r)}{4\beta^2} - r^2\,f(r)\right](dx^+)^2 + \frac{\beta^2 r_+^4}{r^4}(dx^-)^2\\
&\quad - [1+f(r)]\,dx^+dx^-\Bigg) + k(r)^{1/3}\left(r^2 d\boldsymbol{x}^2 + \frac{dr^2}{r^2\,f(r)}\right),
\end{aligned} \qquad (6.4)$$

where we have introduced the light-cone coordinates (5.5) in the second line for future convenience, with the massive vector and scalar

$$\begin{aligned} A &= \frac{r^2\beta}{k(r)}(f(r)\,dt + dy) = \frac{r^2}{k(r)}\left(\frac{1+f(r)}{2}\,dx^+ - \frac{\beta^2 r_+^4}{r^4}\,dx^-\right), \\ e^{\phi} &= \frac{1}{\sqrt{k(r)}}\,, \end{aligned} \tag{6.5}$$

where $f(r)$ and $k(r)$ are given in (6.3). Note that in these light-cone coordinates, the solution asymptotically approaches the extremal solution (4.4), but β remains a physical parameter, as the full metric depends on β. We will henceforth work with the solution (6.4).

The quantities of interest are the thermodynamic variables associated with the black hole solution (6.2). We first note that the NMT/TsT does not change the entropy of the black hole and hence:

$$S = \frac{r_+^3\,\beta}{4\,G_5}\Delta x^-\,\boldsymbol{V}\,, \tag{6.6}$$

where $\boldsymbol{V}$ is the spatial volume in two dimensions and Δx^- the radius of the null circle. In order to obtain finite charges we are going to have to compactify the direction x^-. *A priori* this sounds like a problem, for we are once again going to have to figure out how to deal with the zero modes of this compactification since it sounds like we are once again back to doing DLCQ. Thankfully, the presence of the black hole makes these issues much more benign; for instance, the x^- circle in the presence of a horizon has become space-like and avoids the issues associated with DLCQ.

Furthermore, using the canonically normalized Killing generator of the horizon:

$$\xi^a = \left(\frac{\partial}{\partial x^+}\right)^a + \frac{1}{2\,\beta^2}\left(\frac{\partial}{\partial x^-}\right)^a \tag{6.7}$$

one concludes that the black hole corresponds to the system in a grand canonical ensemble at temperature:

$$T = \frac{r_+}{\pi\,\beta} \tag{6.8}$$

and particle number chemical potential:

$$\mu = \frac{1}{2\,\beta^2}\,. \tag{6.9}$$

One can go further and determine the Gibbs potential of this grand canonical ensemble, via an "Euclidean action" computation. To do so we analytically continue the t coordinate and end up with a complex geometry, which nevertheless has a real Euclidean action. This is the correct saddle point of the Euclidean quantum gravity path integral and can be used to compute the Gibbs potential by evaluating the on-shell action. We want to interpret this action as the saddle-point approximation to the grand canonical partition function,

$$\Xi(T,\mu) = e^{-\mathcal{Q}(T,\mu)/T} = \mathrm{Tr}\left(\exp\left(-\frac{\hat{H}}{T} - \frac{\mu\,\hat{P}_v}{T}\right)\right) \approx e^{-I}\,, \tag{6.10}$$

with temperature T and chemical potential μ given as in (6.8) and (6.9), respectively. Note that the Euclidean action (6.10) is always negative, so the black hole solution makes the dominant contribution to this partition function for any non-zero temperature.

This however, is complicated by the fact that a naive computation leads to a divergent answer. The issue is that in writing down the effective actions in five dimensions (5.8) or (5.10) we have not worried about the boundary terms. By a careful analysis of the asymptotics of the black hole solution (large r fall-offs) and including a complete set of counter-terms allowed by symmetry, in [7] it was found that the following action satisfies the variation principle $\delta\mathcal{S} = 0$

$$\begin{aligned}\mathcal{S} \;=\; & \frac{1}{16\pi\, G_5}\int d^5x\sqrt{-g} \\ & \times\left(R - \frac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4\,A_\mu A^\mu - V(\phi)\right) \\ & + \frac{1}{16\pi G_5}\int d^4\xi\,\sqrt{-h} \\ & \times\left(2\,K - 6 + A_\mu A^\mu + c_4\,A_\mu A^\mu\phi + c_5\,(A_\mu A^\mu)^2 + (2\,c_4 - 4\,c_5 + 3)\phi^2\right)\end{aligned} \tag{6.11}$$

for some arbitrary constants c_4 and c_5. It should be noted that the analysis was carried out only for a restricted class of variations where the sub-leading fall-offs in the metric, massive vector and scalar field were taken to be linearly related (for details we refer the reader to [7]). For the record, it is trivial to extend the analysis to the consistent truncation action of [8]; one finds that the appropriate set of counter-terms are

$$\begin{aligned}\mathcal{S}_{bdy} \;=\; & \frac{1}{16\pi\, G_5}\int d^4\xi\sqrt{-h}\Big(2\,K - 2\,c_0 + c_1^I\,\phi_I + c_2^{IJ}\,\phi_I\,\phi_J \\ & + \left(c_3 + c_4^I\,\phi_I\right)A_\alpha\,A^\alpha + c_5\,(A_\alpha\,A^\alpha)^2\Big)\end{aligned} \tag{6.12}$$

with

$$c_0 = 3\,, \qquad c_1^I = 0\,, \qquad c_3 = 1\,, \tag{6.13}$$

and

$$\begin{aligned} -&\ 1350 + 72\,c_2^{11} + 24\,c_2^{12} - 360\,c_2^{13} + 2\,c_2^{22} - 60\,c_2^{23} + 450\,c_2^{33} \\ +&\ 360\,c_4^1 + 60\,c_4^2 - 900\,c_4^3 = 0\,. \end{aligned} \tag{6.14}$$

Putting the pieces together one can directly compute the on-shell action with the prescribed set of counter-terms to find

$$I = -\frac{\beta\, r_+^3}{16\, G_5} \Delta x^- \boldsymbol{V} \tag{6.15}$$

which leads to

$$S = \frac{\pi^3\, T^3}{16\, G_5\, \mu^2}\, V\, \Delta x^- \ , \tag{6.16}$$

which is the same as the result (6.6) we obtained earlier by direct calculation. That S is given both in terms of the horizon area and by (6.16) is a consistency check of our calculation: in general, by foliating the region outside the horizon by surfaces of constant time, we can always rewrite the Euclidean action as $I = \frac{1}{T}(E + \mu\, N) - S$, which implies the assumed relation between entropy and action. Moreover, this action is the identical to the on-shell action (regulated) for the Schwarzschild–AdS black hole. This is another consistency check for the NMT/TsT does not change the leading large N thermodynamic properties (follows from star product) [8].

We then obtain the conserved charges

$$\langle N \rangle = \langle P_- \rangle \frac{\Delta v}{2\pi} = \frac{\pi^2\, T^4}{64\, G_5\, \mu^3}\, \boldsymbol{V}\, (\Delta x^-)^2\,, \tag{6.17}$$

and

$$\langle E \rangle = \frac{\pi^3\, T^4}{64\, G_5\, \mu^2}\, \boldsymbol{V}\, (\Delta x^-)\,. \tag{6.18}$$

Furthermore, the pressure is given in the grand canonical ensemble directly in terms of the Gibbs potential $\mathcal{Q}(T, \mu, V)$:

$$P\, V = -\mathcal{Q}(T, \mu, V) = \frac{\pi^3\, T^4}{64\, G_5\, \mu^2}\, \boldsymbol{V}\, \Delta x^-\,, \tag{6.19}$$

leading thus to an equation of state

$$P\, \boldsymbol{V} = E\,. \tag{6.20}$$

A non-relativistic system with Galilean conformal invariance has different scalings for temporal and spatial directions as given in (2.1) for $z = 2$. This feature leads to an equation of state $d\, P\, \boldsymbol{V} = 2\, E$ in d-spatial dimensions [2], which is satisfied by (6.20). So indeed, the black hole solution constructed describes a state in the grand-canonical ensemble at temperature T and chemical potential μ for a non-relativistic conformal field theory.

A few comments about this computation are in order:

- The result we have can be readily generalized to higher dimensions, by invoking the scaling symmetry. In fact, it was anticipated in [7] that the Euclidean action in d-spatial dimensions should take the form

$$I = -\Gamma'\, \beta\, r_+^{d+1} = -\Gamma\, \frac{T^{d+1}}{\mu^{\frac{d}{2}+1}} \tag{6.21}$$

  from which one can derive the rest of the thermodynamics. This was later independently derived by [28] who constructed black hole spacetimes in a phenomenological model extending (5.10) to higher dimensions.
- A curious fact with the Gibbs free energy or indeed the other thermodynamic variables is the divergence as $\mu \to 0$. This was contrasted with the expected behaviour in fermions at unitarity in [28] who showed that the real-world systems do not show such power-law divergence in thermodynamic quantities.
- In fact, the underlying cause for this divergence can be traced to the fact that despite the non-trivial background (4.4) for the vacuum state, these field theories behave like DLCQ of conventional relativistic CFTs. By considering a mode sum over light-cone states, in a beautiful piece of work, [29] was able to show that the origin of the small μ divergence is related to the light-cone momentum states.
- We have derived the thermodynamic quantities by first computing the Gibbs potential and then using standard thermodynamic relations to extract the physical quantities. One could alternately have directly used a holographic prescription to compute a full stress-tensor complex for the dual field theory. This was achieved recently in [30] by using the boundary counter-terms proposed above (6.11).

6.2. Hydrodynamic properties of non-relativistic CFTs

Stationary black hole solutions such as (6.4) correspond to equilibrium configurations of the dual field theory in an appropriate ensemble. By perturbing these black holes one can derive physics of linear response in the near-equilibrium regime. This fact has been well explored in the AdS/CFT context starting from the seminal work of [31,32] and resulted in the famous conjecture for the ratio of shear viscosity to entropy density [33]. An excellent review of these works can be found in [34]. More recently the framework of the fluid-gravity correspondence [35] has provided an useful way to think about the hydrodynamic limit of strongly coupled field theories and in particular construct inhomogeneous, dynamical black hole solutions which are dual to arbitrary fluid flows (for a review see [36]).

As one might anticipate the general analysis of black hole perturbations and their response can be carried out for different black hole spacetimes, appropriately taking care of boundary conditions. In particular, there is nothing preventing us from studying fluctuations of the solution (6.4). The fluctuations of the metric degrees of freedom which are polarized along the spatial directions $\boldsymbol{x}$ are especially interesting as they correspond to shear-driven diffusion in the dual field theory [34]. Furthermore, for symmetry reasons these fluctuations decouple from the rest of the fluctuations, *e.g.*, the fluctuations of the scalar field and massive vector field sourcing the background (6.4). This follows from the fact these modes are the only modes transforming in the symmetric traceless representation of the spatial rotation group. By studying their dynamics one expects to learn about the shear mode diffusion in the dual field theory.

In particular, one can study the two point function of the spatial stress tensor $\Pi_{ij}(u,\boldsymbol{x})$ to learn about η for one has via a linear-response Kubo formula a relation between the two-point function of the stress tensor and the shear-viscosity. Specifically, consider the two-point function

$$G_{12,12}(\omega,0) = -i \int dx^+ \, d^2x \, e^{i\omega\, x^+} \, \theta(x^+) \langle \, [\Pi_{x_1x_2}(x^+,\boldsymbol{x}), \Pi_{x_1x_2}(0,\mathbf{0})] \, \rangle . \tag{6.22}$$

The shear viscosity is given by the zero-frequency limit of this two point function,

$$\eta = -\lim_{\omega\to 0} \frac{1}{\omega} \, \mathrm{Im}\left(G_{12,12}(\omega)\right) . \tag{6.23}$$

As explained above, generically the fluctuations δg, δA and $\delta\phi$ give a coupled system: however, the shear mode $\delta g_{x_1x_2}$ which is involved in the computation of (6.22) decouples. Not only that, in fact $\delta g_{x_1x_2}$ satisfies massless, minimally coupled wave equation (for zero spatial momentum), which turns out to reduce to a familiar wave equation (*viz.*, the wave equation on Schwarzschild–AdS background upto a rescaling of the frequency ω). This in fact follows from the fact that the stress tensor has zero particle number $P_- = 0$, and the general arguments given in [18]. Putting the pieces together one can easily compute $\langle \, \Pi_{x_1x_2} \, \Pi_{x_1x_2} \, \rangle$ at zero spatial momentum and read off η using a Kubo formula. One finds [7,9]:

$$\frac{\eta}{s} = \frac{1}{4\,\pi}\,. \tag{6.24}$$

Finally, note that non-relativistic conformal invariance requires that the bulk viscosity vanish; $\zeta = 0$. This follows essentially from a conformal Ward identity [12].

We have thus far described the basic derivation of transport properties of non-relativistic CFTs. One can of course, use the fluid-gravity correspondence [35] to understand the gravity dual to a generic fluid flow in the boundary non-relativistic CFTs. To do so one must take the asymptotically Schr_{d+3} black hole and generalize it to a $d+2$ parameter solution which is achieved by performing a Galilean boost in the spatial directions to make the dependence on the d parameters corresponding to the Galilean velocities $\mathfrak{v}_i$ explicit. Then one should promote the background parameters r_+, β and $\mathfrak{v}_i$ to fields depending on $\{x^+, \boldsymbol{x}\}$ and solve the bulk gravity equations order by order in derivatives of $\{x, \boldsymbol{x}\}$ for asymptotically Schr_{d+3} solutions. The result of this exercise would give the gravity dual of non-relativistic conformal Navier–Stokes equations. In particular, the radial constraint equations (on a fixed r hypersurface) will result in the constraint equations which are the boundary Navier–Stokes equations. One can use the prescription of [30] to extract the boundary stress tensor (really a stress tensor complex) which would exhibit the correct dependence of the fluid dynamical variables on the velocity field, temperature, chemical potential, *etc.*

However, one can use a trick to simplify the analysis. As noted in [8] the leading planar physics of the non-relativistic theory is the same as the parent relativistic theory even with the non-trivial deformation that converts an asymptotically AdS spacetime to an asymptotically Schrödinger spacetime. One can use this fact and obtain the stress tensor complex for the non-relativistic theory by reducing the corresponding relativistic stress tensor on the light-cone (along x^-). The bulk metric is obtained by TsT transformation of the asymptotically AdS fluid black hole solutions constructed in [35], with ∂_{x^-} being the null Killing vector. This procedure was carried out in [37] to which we refer the interested reader. A brief account can also be found in [36].

This light-cone reduction makes it clear that the transport coefficients of the non-relativistic theory are inherited from the parent relativistic hydrodynamics. In this description it is clear that non-relativistic fluids with holographic duals will saturate the conjectured viscosity bound $\eta/s = 1/4\pi$, which was verified explicitly in [7, 9]. Furthermore, it is also possible to use this light-cone reduction to infer the heat conductivity of the non-relativistic fluid:

$$\kappa = 2\,\eta\,\frac{\epsilon + P}{\rho\, T} \tag{6.25}$$

which can be rephrased as the statement that the Prandtl number of the fluid is unity. We recall that the Prandtl number is defined as the ratio of the kinematic viscosity ν and the thermal diffusivity χ,

$$\mathrm{Pr} = \frac{\nu}{\chi}\,, \tag{6.26}$$

where

$$\nu = \frac{\eta}{\rho}\,, \qquad \chi = \frac{\kappa}{\rho\, c_p}\,, \tag{6.27}$$

where c_p is the specific heat at constant pressure.

7. Dicsussion

In this article we have described how the holographic AdS/CFT correspondence may be extended to systems with more exotic symmetries, such as non-relativistic conformal symmetries. We have discussed the spacetimes that have the correct causal structure and allied properties to capture the dynamics of the field theory holographically. However, as we have seen the simplest embeddings of such geometries in string theory have not quite realized the hope of finding duals for real-world systems such as fermions at unitarity. To some extent, this could be anticipated as the following argument reveals.

Consider, the known class of AdS/CFT examples where one has a large hierarchy between the conformal dimensions of operators dual to supergravity fields and those dual to string oscillators. This is familiar from the prototype examples of the correspondence such as $\mathcal{N} = 4$ SYM and strings on $\mathrm{AdS}_5 \times \mathbf{S}^5$. The supergravity states correspond to operators whose conformal dimensions are of $\mathcal{O}\,(1)$ while the stringy states have conformal dimensions of $\mathcal{O}\left(\lambda^{1/4}\right)$ in the regime where $\lambda \gg 1$. Typical condensed matter and atomic systems do not exhibit such a large hierarchy of operator dimensions. Perhaps one should learn to do string theory in backgrounds such as (4.4) in order to realize physical effects more closer to the real world. In the present case one should also try to understand whether it is possible to achieve geometries such as (4.4) without recourse to simple deformations of known supersymmetric field theories.

This article has considered only the basics of the correspondence for non-relativistic CFTs. Over the past year, many interesting results have been derived in this context and we refer the reader to the growing literature on the subject for further details.

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