A NEW PROOF OF EXISTENCE OF A BOUND STATE IN THE QUANTUM COULOMB FIELD III*

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This is a sequel to the paper published under the same title [*Acta Phys. Pol. B* **35**, 2249 (2004)] in which the integral representation of the matrix element $\langle u|\exp(-\sigma C_1)|u\rangle$, where $|u\rangle = \exp(-iS(u))|0\rangle$ is the quantum Coulomb field and $C_1 = -(1/2)\, M_{\mu\nu}M^{\mu\nu}$ is the first Casimir operator of the proper, orthochronous Lorentz group, was given. In this paper another integral representation of the same matrix element is given. In this new representation contributions from the bound state, which belongs to the supplementary series, and from the continuous spectrum, which belongs to the main series, are separated. This allows to calculate the asymptotic behaviour of the matrix element for $\sigma \to \infty$. The matrix element $\langle u|\exp(-\sigma C_1)|u\rangle$ is a non-analytic function of σ at $\sigma = 0$. The nature of this non-analyticity is clarified by means of a representation of the relevant integrals with the help of the function $g(x) = \sum_{n=-\infty}^{+\infty}\exp(-\pi n^2 x)$ which satisfies the well known functional equation $g(x) = x^{-1/2}g(1/x), x > 0$.

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1. Introduction

The notation used in this paper is the same as that used in [1]. In particular, $|u\rangle$ denotes the quantum Coulomb field *i.e.* the eigenstate of the total charge Q, $Q|u\rangle = e|u\rangle$, which does not contain transversal photons. $C_1 = -(1/2)\, M_{\mu\nu}M^{\mu\nu}$, $M_{\mu\nu}$ being generators of the proper, orthochronous Lorentz group, is the first Casimir operator of the proper, orthochronous Lorentz group. The object of our study is the matrix element

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$\langle u|\exp(-\sigma C_1)|u\rangle$, $\sigma > 0$. It was shown in [1] that

$$\langle u|e^{-\sigma C_1}|u\rangle = \frac{1}{2\sqrt{\pi}}\frac{e^{-\sigma}}{\sigma^{3/2}}\int\limits_0^\infty d\psi \sinh\psi e^{-z(\psi\coth\psi-1)}\cdot\psi e^{-\psi^2/(4\sigma)}\,, \qquad \text{(I.25)}$$

where $z = e^2/\pi$. Multiplying this equation by $\exp(\lambda\sigma)$, where λ is smaller then the lowest eigenvalue of the Casimir operator C_1 present in the spectral decomposition of the state $|u\rangle$, and integrating both sides over σ, $0 < \sigma < \infty$, we obtain

$$\left\langle u\left|\frac{1}{C_1-\lambda}\right|u\right\rangle = \int\limits_0^\infty d\psi \sinh\psi e^{-z(\psi\coth\psi-1)-\psi\sqrt{1-\lambda}}\,. \qquad \text{(I.33)}$$

It was shown in [1] that this is equal to

$$\left\langle u\left|\frac{1}{C_1-\lambda}\right|u\right\rangle = \frac{(1-z)e^z}{z(2-z)-\lambda} + 2z^2e^z\sum_{n=0}^{\infty}\frac{(\sqrt{1-\lambda}+2n+1-z)^{n-1}}{(\sqrt{1-\lambda}+2n+1+z)^{n+2}}\,, \qquad \text{(I.42)}$$

which shows that the quantum Coulomb field $|u\rangle$ contains a bound state of the first Casimir operator C_1 with the eigenvalue $z(2-z)$ and probability of occurence $(1-z)e^z$. This probability cannot be negative, which means that the bound state exists only for $0 < z < 1$.

2. A new integral representation of the matrix element $\langle u|\exp(-\sigma C_1)|u\rangle$

Since the average $\langle u|1/(C_1-\lambda)|u\rangle$ is the Laplace transform of the average $\langle u|\exp(-\sigma C_1)|u\rangle$, the latter can be obtained as the inverse Laplace transform of the average $\langle u|1/(C_1-\lambda)|u\rangle$. Let us apply the inverse Laplace transform to both sides of Eq. (I.42). One has to take into account that our Laplace transform has unconventional sign, because we have to have the conventional sign of λ in the resolvent $1/(C_1-\lambda)$. Therefore the contour of integration in the inverse Laplace transform is also unconventional, namely closed on the right hand side. Calculating this contour integral in the standard way we obtain

$$\langle u|\exp(-\sigma C_1)|u\rangle = (1-z)e^z e^{-\sigma z(2-z)} + z^2e^z\frac{2}{\pi}\int\limits_0^\infty d\nu\nu e^{-\sigma(1+\nu^2)}\times$$

$$(-)\sum_{n=-\infty}^{+\infty}\frac{[\nu+i(2n+1-z)]^{n-1}}{[\nu+i(2n+1+z)]^{n+2}}\,. \qquad \text{(II.1)}$$

The right hand side of Eq. (II.1) is equal identically to the right hand side of Eq. (I.25), for $0 < z < 1$ of course. The first term represents the contribution of the bound state of the first Casimir operator C_1. This state exists for $0 < z < 1$ and belongs to the supplementary series [2] since for $0 < z < 1$, $0 < z(2-z) < 1$. The integral in Eq. (II.1) represents the contribution from the main series of unitary irreducible representations of the proper orthochronous Lorentz group. For the main series and for spherically symmetric states annihilated by $C_2 = M_{01}M_{23} + M_{02}M_{31} + M_{03}M_{12}$ $1 < C_1 < \infty$ [2].

3. The nature of the singularity of the average $\langle u|\exp(-\sigma C_1)|u\rangle$ at $\sigma = 0$

We stated in [1] that the average $\langle u|\exp(-\sigma C_1)|u\rangle$ cannot have a convergent Taylor series in σ at $\sigma = 0$. The averages $\langle u|(C_1)^n|u\rangle$, $n = 1, 2, 3, \ldots$, calculated from first principles by Professor Wosiek and dr Rostworowski grow too quickly with n to allow for convergence of the Taylor series of $\langle u|\exp(-\sigma C_1)|u\rangle$ at $\sigma = 0$. In this section we investigate this matter in some detail.

It was shown in [1] that the average $c(\sigma, z) = \langle u|\exp(-\sigma C_1)|u\rangle$ satisfies the partial differential equation

$$z\frac{\partial^2 c}{\partial z^2} - (z+1)\frac{\partial c}{\partial z} - 2\sigma(1+2\sigma z)\frac{\partial c}{\partial \sigma} - 2\sigma(1+3z+2\sigma z)c = 0\,. \qquad \text{(I.29)}$$

It is clear from the integral representation (I.25) that $c(\sigma, z)$ does have the Taylor series in z at $z = 0$. Therefore

$$\begin{aligned} c(\sigma,z) &= \langle u|e^{-\sigma C_1}|u\rangle \\ &= \frac{1}{2\sqrt{\pi}}\frac{e^{-\sigma}}{\sigma^{3/2}}\int_0^\infty d\psi \sinh\psi \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n(\psi\coth\psi - 1)^n \cdot \psi e^{-\psi^2/(4\sigma)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n \frac{1}{2\sqrt{\pi}}\frac{e^{-\sigma}}{\sigma^{3/2}}\int_0^\infty d\psi \sinh\psi(\psi\coth\psi - 1)^n \cdot \psi e^{-\psi^2/(4\sigma)} \\ &= 1 - 2\sigma z + z^2\left\{2\sigma^2 + \sigma - \frac{1}{2} + \frac{1}{4\sqrt{\pi}}\frac{e^{-\sigma}}{\sigma^{3/2}}\int_0^\infty d\psi \frac{\psi^3}{\sinh\psi}e^{-\psi^2/(4\sigma)}\right\} + \ldots\,, \end{aligned} \qquad \text{(II.2)}$$

where dots denote terms with higher powers of z which are calculable by recurrence form those written down and the differential equation (I.29). Thus all terms nonanalytic in σ, if they are there, are generated recurrently from those present in the last integral in (II.2) and it is enough to investigate the analytic properties of the function (we put $\nu = 1/4\sigma$)

$$F(\nu) = \int_{-\infty}^{+\infty} d\psi \frac{\psi}{\sinh\psi} e^{-\nu\psi^2} \tag{II.3}$$

from which the last integral in (II.2) can be obtained by differentiation. Introducing another function

$$K(\nu) = \int_{-\infty}^{+\infty} d\psi\, \psi \coth\psi e^{-\nu\psi^2} \tag{II.4}$$

we have

$$F(\nu) = 4K(4\nu) - K(\nu) \tag{II.5}$$

so that properties of $F(\nu)$ are reduced to those of $K(\nu)$. We have, using the Lerch identity [3],

$$\int_0^\infty \frac{e^{-x^2}dx}{\left(1+\frac{w^2}{4x^2}\right)^{\frac{1}{2}s}} = \frac{\pi^{1/2}}{\Gamma(\frac{1}{2}s)} \int_0^\infty e^{-x^2-wx} x^{s-1} dx\,, \tag{II.6}$$

that

$$\begin{aligned} K(\nu) &= \int_{-\infty}^{+\infty} d\psi\, \psi \coth\psi e^{-\nu\psi^2} = 2\int_0^\infty d\psi \sum_{n=-\infty}^{+\infty} \frac{1}{1+\frac{n^2\pi^2}{\psi^2}} e^{-\nu\psi^2} \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\nu}} \int_0^\infty \sum_{n=-\infty}^{+\infty} e^{-n^2\pi^2\nu t}(1+t)^{-3/2}dt \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\nu}} \int_0^\infty g(\pi\nu t)(1+t)^{-3/2}dt\,, \end{aligned} \tag{II.7}$$

where

$$g(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} = (x)^{-1/2} g(1/x)\,. \tag{II.8}$$

Using the functional equation for $g(x)$ stated above we have for $K(\nu)$:

$$K(\nu) = (\pi\nu)^{-3/2} K\left(\frac{1}{\pi^2\nu}\right). \tag{II.9}$$

The above can also be written as

$$K(\nu) = \frac{\pi}{2} \int_0^\infty g(x)(x + \pi\nu)^{-3/2} dx \tag{II.10}$$

with $g(x)$ given by Eq. (II.8). The point $\nu = 1/\pi^2\nu$ *i.e.* $\nu = 1/\pi$ is the fixed point of the inversion (II.9). This means that there is a distinguished "time" of diffusion in Lobachevsky space equal to $\sigma = 1/4\nu = \pi/4$, at least for the special initial value equal to $\exp[-z(\lambda \coth \lambda - 1)]$, where λ is the distance from a certain fixed point.

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